

## Compactifications and their Ordering

Given a  $T_{3\frac{1}{2}}$  space  $(X, \tau)$ ,  $(Y, f)$  is a  $T_2$  compactification of  $X$  provided  $Y$  is a compact  $T_2$  space and  $f : X \rightarrow Y$  is an embedding with  $f[X]$  dense in  $Y$ . Two compactifications  $(Y_1, f_1)$  and  $(Y_2, f_2)$  are related provided there is a homeomorphism  $h : Y_1 \rightarrow Y_2$  such that  $h \circ f_1 = f_2$ . This is known to be an equivalence relation; a partial ordering of the equivalence classes is defined as follows:  $[(Y_1, f_1)] \leq [(Y_2, f_2)]$  provided there is a continuous onto map  $g : Y_2 \rightarrow Y_1$  such that  $g \circ f_2 = f_1$ . In this ordering the Stone-Ćech compactification is known to be the largest element and, if  $X$  is a non-compact locally compact space, that of the one-point compactification the smallest. The Stone-Ćech compactification and the one-point compactification are unique up to homeomorphism and will be denoted  $(\beta X, \iota)$  and  $(X^+, \iota^+)$ . The Stone-Ćech compactification has an important extension property: Every continuous map from  $X$  into a compact,  $T_2$  space  $Y$  has a unique extension to  $\beta X$ . If  $f : X \rightarrow Y$  is the continuous map, the extension will be denoted  $\beta f : \beta X \rightarrow Y$ , where  $\beta f \circ \iota = f$ .

Notice that the equivalence classes of compactifications are only classes but not sets. For example, any set of the appropriate cardinality can be topologized and supplied with an imbedding to make a compactification equivalent to the Stone-Ćech compactification. Since the collection of all sets of a fixed cardinality is a class but not a set, the equivalence class of the Stone-Ćech compactification is also not a set. When necessary, representatives of classes can be chosen to avoid this difficulty.

Detailed presentations of the above can be found in many introductory topology texts, e.g. Wilansky [1].

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## References

1. Wilansky, A., Topology for Analysis ,Ginn and Co., 1970.