Uniform Spaces

This website will use the entourage or Bourbaki approach to uniform spaces as presented in Kelley [2]. The equivalent Tukey approach via uniform coverings, which some find more intuitive, can be found in Isbell [1].

Definition P2.1 Let X be a set. A uniformity for X is $\mathcal{U}$, a non-empty set of relations on X, such that

i) $\forall\ U \in \mathcal{U}, \{(x, x) : x \in X\} \subseteq U$
ii) $U \in \mathcal{U}$ and $U \subseteq W \subseteq X \times X \Rightarrow W \in \mathcal{U}$
iii) $U \in \mathcal{U}$ and $W \in \mathcal{U} \Rightarrow U \cap W \in \mathcal{U}$
iv) $U \in \mathcal{U} \Rightarrow U^{-1} \in \mathcal{U}$
v) $U \in \mathcal{U} \Rightarrow \exists V \in \mathcal{U}$ with $V \circ V \subseteq U$

The set $\{(x, x) : x \in X\}$ may be referred to as the diagonal of X or Δ_X. The pair $(X,\mathcal{U})$ is called a uniform space. $(X,\mathcal{U})$ and the uniformity $\mathcal{U}$ are called separated provided $\bigcap\{U : U \in \mathcal{U}\} = \Delta_X$.

A motivating example for definition P2.1 arises when X has a pseudo-metric ρ. For $\epsilon > 0$ let $V_\epsilon = \{(x, y) : \rho(x, y) < \epsilon\}$. Let $\mathcal{U}_\rho = \{U \subseteq X \times X : \exists \epsilon > 0 \text{ with } V_\epsilon \subseteq U\}$. Then $\mathcal{U}_\rho$ is a uniformity, which is separated if and only if ρ is a metric.

Definition P2.2 Let $(X,\mathcal{U})$ be a uniform space. $\tau(\mathcal{U}) = \{G \subseteq X : x \in G \Rightarrow \exists\ U \in \mathcal{U}$ with $U[x] \subseteq G\}$.

$\tau(\mathcal{U})$ is a topology for X. A topological space (X, τ) is called uniformizable provided there exists a uniformity $\mathcal{U}$ for X with $\tau(\mathcal{U}) = \tau$. When X has a pseudo-metric ρ, $\tau(\mathcal{U}_\rho)$ is the topology generated by the pseudo-metric. In general, there need not be a unique uniformity generating a given uniformizable topology. A uniformity $\mathcal{U}$ is separated if and only if $\tau(\mathcal{U})$ is T_2.

Theorem P2.3 A topological space is uniformizable if and only if it is completely regular. It is uniformizable via a separated uniformity if and only if it is $T_{3\frac{1}{2}}$.

Proposition P2.4 Let (X, τ) be compact and T_2. Then there exists a unique uniformity $\mathcal{U}$ such that $\tau(\mathcal{U}) = \tau$. Moreover, $\mathcal{U}$ is the set of all neighborhoods of the diagonal in $X \times X$.

Definition P2.5 Let $(X,\mathcal{U})$ be a uniform space, and let $S : D \to X$ be a net. S is Cauchy if and only if $\forall\ U \in \mathcal{U}, \exists\ d_0 \in D$ so that $d, e \geq d_0 \Rightarrow (S(d), S(e)) \in U$. $(X,\mathcal{U})$ is complete if and only if every Cauchy net converges in $(X, \tau(\mathcal{U}))$.

Definition P2.6 Let $(X,\mathcal{U})$ be a uniform space. $(X,\mathcal{U})$ is totally bounded if and only if $\forall\ U \in \mathcal{U}, \exists\ x_1, ..., x_n \in X$ such that $X \subseteq \cup_{i=1}^n U[x_i]$.

When X has a pseudo-metric ρ, these definitions of complete and totally bounded for $(X,\mathcal{U}_\rho)$ are equivalent to the usual definitions based on the pseudo-metric. For $A \subseteq X$, A is totally bounded provided A with the subspace uniformity from X is totally bounded. Every subset of a totally bounded set is totally bounded. If A is totally bounded, then $\overline{A}$ is also totally bounded, where $\overline{A}$ denotes the closure of A in $(X, \tau(\mathcal{U}))$.

Theorem P2.7 Let $(X,\mathcal{U})$ be a uniform space. Then $(X, \tau(\mathcal{U}))$ is compact if and only if $(X,\mathcal{U})$ is complete and totally bounded.

Definition P2.8 Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be uniform spaces, and let $f : X \to Y$. f is uniformly continous if and only if $\forall\ V \in \mathcal{V}, (f \times f)^{-1}[V] \in \mathcal{U}$.

When X and Y have pseudo-metrics ρ and σ, uniform continuity relative to $(X,\mathcal{U}_\rho)$ and $(Y,\mathcal{V}_\sigma)$ is equivalent to the usual definition of uniform continuity based on the pseudo-metrics. A map f as in P2.8 is called a unimorphism provided f is a bijection, f is uniformly continuous, and f^{-1} is uniformly continuous. f is called a uniform embedding provided f is a unimorphism from X to $f[X]$, where $f[X]$ has the subspace uniformity.

Proposition P2.9 Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be uniform spaces, and let $f : X \to Y$. If f is uniformly continous, then f is continuous relative to $(X, \tau(\mathcal{U})$ and $(Y, \tau(\mathcal{V}))$. If f is continuous and $(X, \tau(\mathcal{U}))$ is compact, then f is uniformly continuous.

Definition P2.10 Let $(X,\mathcal{U})$ be a uniform space. A completion of $(X,\mathcal{U})$ is a pair $((Y,\mathcal{V}), f)$, where $(Y,\mathcal{V})$ is a complete uniform space, $f : X \to Y$ is a uniform embedding, and $f[X]$ is dense in Y.

Theorem P2.11 Every uniform space has a completion. Every separated uniform space has a separated completion, which is unique up to unimorphism.

Proposition P2.12 Let $\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ be a non empty family of uniformities on X. Then there is a uniformity $\mathcal{U}$ on X such that $\mathcal{U}_\alpha \subseteq \mathcal{U}\ \forall \alpha \in \Delta$ and, if $\mathcal{V}$ is a uniformity for X with $\mathcal{U}_\alpha \subseteq \mathcal{V}\ \forall \alpha \in \Delta$, then $\mathcal{U} \subseteq \mathcal{V}$.

The uniformity $\mathcal{U}$ is the supremum of $\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. The notations used will be $\mathcal{U} = \bigvee\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ or $(X,\mathcal{U}) = \bigvee\{(X,\mathcal{U}_\alpha) : \alpha \in \Delta\}$.

Proposition P2.13 Let $\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ be a non-empty family of uniformities on X. If $(X,\mathcal{U}_\alpha)$ is totally bounded $\forall \alpha \in \Delta$, then $\bigvee\{(X,\mathcal{U}_\alpha) : \alpha \in \Delta\}$ is also totally bounded.

Proposition P2.14 Let $\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ be a non-empty family of uniformities on X. Then $\tau(\bigvee\{\mathcal{U}_\alpha : \alpha \in \Delta\}) = \bigvee\{\tau(\mathcal{U}_\alpha) : \alpha \in \Delta\}$.

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http://www.susanjkleinart.com/compactification/

References

1. Isbell, J. R., Uniform Spaces, Math. Surveys 12, American Mathematical Society, 1964.
2. Kelley, J. L., General Topology, Van Nostrand, 1955.