## Normal Bases

This section summarizes the Wallman-Frink construction of compactifications via normal bases. Detailed proofs and references to original sources for this approach can be found in [1] or [2].

**Definition P3.1** Let $(X, \tau)$ be an arbitrary topological space and let $\mathcal{Z}$ be a family of subsets of $X$. $\mathcal{Z}$ is a normal basis for $X$ if and only if

i) $\mathcal{Z}$ is a base for the closed subsets of $X$.

ii) $\mathcal{Z}$ is closed under finite unions and intersections.

iii) If $E$ is closed and $x \notin E$, then there exists $Z \in \mathcal{Z}$ such that $x \in Z$ and $Z \cap E = \emptyset$.

iv) If $Z_1, Z_2 \in \mathcal{Z}$ with $Z_1 \cap Z_2 = \emptyset$, then there exist $C_1, C_2 \in \mathcal{Z}$ such that $X \subseteq C_1 \cup C_2$ and $Z_i \subseteq X - C_i$ for $i = 1, 2$.

It is easy to check that, for a normal basis $\mathcal{Z}$, if $X \neq \emptyset$ then $X \in \mathcal{Z}$. On this website $X \neq \emptyset$ is always assumed for a space $(X, \tau)$.

**Definition P3.2** Let $(X, \tau)$ be a topological space with normal basis $\mathcal{Z}$. Let $\mathcal{F}$ be a family of subsets of $X$. $\mathcal{F}$ is a $\mathcal{Z}$-filter if and only if

i)$\emptyset \neq \mathcal{F} \subseteq \mathcal{Z}$ and $\emptyset \notin \mathcal{F}$.

ii)$\mathcal{F}$ is closed under finite intersections.

iii)For $A, B \in \mathcal{Z}$, if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.

Convergence of a $\mathcal{Z}$-filter is defined as expected: $\mathcal{F} \to x$ provided every neighborhood of $x$ contains an element of $\mathcal{F}$. A maximal $\mathcal{Z}$-filter is called a $\mathcal{Z}$-ultrafilter. A routine Zorn's Lemma argument show that every $\mathcal{Z}$-filter is contained in a $\mathcal{Z}$-ultrafilter. The point-filters, defined for $x \in X$ as $\mathcal{F}_x = \{Z \in \mathcal{Z} : x \in Z\}$, are examples of $\mathcal{Z}$-ultafilters.

**Lemma P3.3** Let $(X, \tau)$ be a topological space with normal basis $\mathcal{Z}$. Let $\mathcal{F}$ be a $\mathcal{Z}$-filter. Then the following are equivalent:

i)$\mathcal{F}$ is a $\mathcal{Z}$-ultrafilter.

ii) If $A \in \mathcal{Z}$ and $A \cap F \neq \emptyset$ for all $F \in \mathcal{F}$, then $A \in \mathcal{F}$.

iii) For every $A \in \mathcal{Z}$, either $A \in \mathcal{F}$ or there exists $Z \in \mathcal{F}$ with $Z \subseteq X - A$.

A $\mathcal{Z}$-filter $\mathcal{F}$ is called free if $\cap\mathcal{F} = \emptyset$ and fixed if $\cap\mathcal{F} \neq \emptyset$. Clearly point-filters are fixed. In fact, a $\mathcal{Z}$-ultrafilter is fixed if and only if it is a point-filter. Also, if $X$ is $T_1$, then $\cap\mathcal{F}_x = \{x\}$.

**Proposition P3.4** Let $(X, \tau)$ be a topological space with normal basis $\mathcal{Z}$. Then the following are equivalent:

i) $(X, \tau)$ is compact.

ii) Every $\mathcal{Z}$-filter is fixed.

iii) Every $\mathcal{Z}$-ultrafilter is fixed.

**Definition P3.5** Let $(X, \tau)$ be a topological space with normal basis $\mathcal{Z}$. $\omega(\mathcal{Z})$ denotes the set of all $\mathcal{Z}$-ultrafilters. For $Z \in \mathcal{Z}$, $Z^\omega = \{\mathcal{F} \in \omega(\mathcal{Z}) : Z \in \mathcal{F}\}$ and $\mathcal{Z}^\omega = \{Z^\omega : Z \in \mathcal{Z}\}$.

**Proposition P3.6** Let $(X, \tau)$ be a topological space with normal basis $\mathcal{Z}$. For $\mathcal{S} \subseteq \omega(\mathcal{Z})$, call $\mathcal{S}$ closed provided for every $\mathcal{F} \notin \mathcal{S}$ there is $Z^\omega \in \mathcal{Z}^\omega$ such that $\mathcal{S} \subseteq Z^\omega$ and $\mathcal{F} \notin Z^\omega$. Then the closed sets so defined determine a topology on $\omega(\mathcal{Z})$ for which $\mathcal{Z}^\omega$ is a normal basis.

In what follows $\omega(\mathcal{Z})$ will always have the topology described in P3.6.

**Proposition P3.7** Let $(X,\tau)$ be a topological space with normal basis $\mathcal{Z}$. Then $\omega(\mathcal{Z})$ is compact and $T_2$.

**Definition P3.8** Let $(X,\tau)$ be a topological space with normal basis $\mathcal{Z}$. The map $\iota_{\mathcal{Z}} : X \to \omega(\mathcal{Z})$ is defined by $\iota_{\mathcal{Z}}(x) = \mathcal{F}_x$.

**Proposition P3.9** Let $(X,\tau)$ be a topological space with normal basis $\mathcal{Z}$. Then

i) $\iota_{\mathcal{Z}}$ is continous and a closed map into its image $\iota_{\mathcal{Z}}[X]$.

ii) $\iota_{\mathcal{Z}}$ is one-to-one if and only if $(X,\tau)$ is $T_1$.

**Theorem P3.10** Let $(X,\tau)$ be a $T_1$ topological space with normal basis $\mathcal{Z}$. Then $(X,\tau)$ is $T_{3\frac{1}{2}}$ and $(\omega(\mathcal{Z}),\iota_{\mathcal{Z}})$ is a $T_2$ compactification of $X$.

**Definition P3.11** Let $(X,\tau)$ be a topological space and let $A \subseteq X$. $A$ is a zero-set of $X$ if and only if there is $f : X \to \mathbb{R}$ continous such that $A = f^{-1}[\{0\}]$. $\mathcal{Z}(X)$ will denote the collection of zero-sets of $X$.

**Proposition P3.12** Let $(X,\tau)$ be completely regular. Then $\mathcal{Z}(X)$ is a normal basis for $X$.

**Theorem P3.13** Let $(X,\tau)$ be a $T_1$ space. Then $(X,\tau)$ is $T_{3\frac{1}{2}}$ if and only if there is a normal basis for $X$.

If $(X,\tau)$ is not $T_1$ but has a normal basis $\mathcal{Z}$, it is not hard to verify that $X$ must be stable in the sense of [3], i.e., $x \in O \in \tau \Rightarrow \overline{\{x\}} \subseteq O$. It can also be shown that $(X,\tau)$ has a normal basis if and only if $(X,\tau)$ is completely regular.

**Theorem P3.14** Let $(X,\tau)$ be a $T_{3\frac{1}{2}}$ space. Then $(\omega(\mathcal{Z}(X)),\iota_{\mathcal{Z}(X)})$ is equivalent to $(\beta X,\iota)$.

Albert J. Klein 2003
http://www.susanjkleinart.com/compactification/

## References

1. Alò, R. A. and Shapiro, H.L., Normal Topological Spaces, Cambridge University Press, 1974.

2. Chandler, R.E., Hausdorff Compactifications, Marcel Dekker, Inc., 1976.

3. Friedman, N.A. and Metzler, R.C., On Stable Topologies, Amer. Math. Monthly 75(1968) 493-498.