Nets

The filter theory of convergence in general topology seems to be most often studied but there is another approach to convergence: nets. Nets are a generalization of sequences and, for those coming to topology after analysis, make possible analogous thinking which can seem more intuitive. Nets have the disadvantage of requiring an external structure, a directed set, which seems extraneous to the topological framework.

This summary lists only definitions and results used on this website. Proofs, examples, and material not used here including the connections between nets and filters can be found in [1] or [2].

Definition P4.1 Let D be a set and $\leq$ a relation on D. $(D, \leq)$ (often just D) is a partially ordered set provided $\leq$ is reflexive and transitive.

Note that the definition does not require the antisymmetric property. Examples which are not antisymmetric sometimes occur.

Definition P4.2 Let $(D, \leq)$ be a non-empty partially ordered set. D is a directed set provided, for any $a, b \in D$, there is $c \in D$ with $a \leq c$ and $b \leq c$.

The property in the previous definition is sometimes referred to as the directed set property.

Definition P4.3 Let X be a set. A net in X is a function from a directed set into X.

Every sequence is a net. Given a net $S : D \to X$, the net may described in quasi-sequential notation as $\{x_d\}$ where $d \in D$ and $x_d = S(d)$.

Definition P4.4 Let (X, τ) be a topological space and let $S : D \to X$ be a net in X. Let $x_0 \in X$. S converges to x_0 (often $S \to x_0$ or $\{x_d\} \to x_0$) if and only if, for every $O \in \tau$ with $x_0 \in O$, there $d_O \in D$ such that, for every $d \in D$, $d \geq d_O$ implies $S(d) \in O$.

The property there is $d_O \in D$ such that, for every $d \in D$, $d \geq d_O$ implies $S(d) \in O$ can be expressed by saying S is eventually in O. Thus $S \to x_0$ means S is eventually in every open set containing x_0, i.e., eventually in every neighborhood of x_0.

If a net S converges to x_0, x_0 is said to be a limit of S.

In calculus the definition of the Reimann integral can be thought of as an example of net convergence, with the directed set being the collection of partitions of an interval ordered by the partition norm. The ordering in this example is not antisymmetric.

Proposition P4.5 Let (X, τ) be a topological space and let $A \subseteq X$. Let $b \in X$. Then $b \in \overline{A}$ (the closure of A in X) if and only if there is S, a net in A, such that $S \to b$.

Note that nets converging to different points of the closure may be maps from different directed sets.

Proposition P4.6 Let (X, τ) be a T_2 topological space and let $S : D \to X$ be a net in X. Assume $S \to x$ and $S \to y$, where $x, y \in X$. Then $x = y$.

The previous fact is often expressed by saying that in a T_2 space a convergent net has a unique limit.

Proposition P4.7 Let (X, τ) and (Y, σ) be topological spaces and let $f : X \to Y$ be continuous. Assume S is a net in X converging to $x_0 \in X$. Then the net $f \circ S$ converges to $f(x_0)$.

Definition P4.8 Let E, D be directed sets and let $T : E \to D$. T is finalizing if and only if for every $d_0 \in D$ there is $e_0 \in E$ such that $e \geq_E e_0$ implies $T(e) \geq_D d_0$.

The defining condition for a finalizing map can be expressed as for every $d_0 \in D$ eventually $T(e) \geq_D d_0$. The subscripts in the definition emphasize that there can be two different ordering relations, but in practice those subscripts are dropped unless necessary for clarity. A finalizing map is also called a map with the subnet property.

Definition P4.9 Let $S : D \to X$ be a net. A subnet of S is a map $S \circ T$ where $T : E \to D$ is a finalizing map.

A subsequence of a sequence is a subnet, but a subnet of a sequence need not be a subsequence. In quasi-sequential notation a subnet of $\{x_d\}$ may be written $\{x_{d_e}\}$ where $e \in E$, $d_e = T(e)$, and $x_{d_e} = S(T(e))$.

Proposition P4.10 Let (X, τ) be a topological space, let $x_0 \in X$, and let $S : D \to X$ be a net with $S \to x_0$. Let $T : E \to D$ be a finalizing map. Then $S \circ T \to x_0$.

Many prefer to express P4.10 with less notation as follows.

Corollary P4.11 Let (X, τ) be a topological space. Every subnet of a convergent net S in X converges to the limit of S.

Definition P4.12 Let (X, τ) be a T_2 topological space and let $S : D \to X$ be a net in X. A point $x_0 \in X$ is a cluster point of S provided, for every $O \in \tau$ with $x_0 \in O$ and for every $d \in D$, there is $d^* \in D$ with $d^* \geq d$ such that $S(d^*) \in O$.

The property that, for every $d \in D$, there is $d^* \in D$ with $d^* \geq d$ such that $S(d^*) \in O$ is also expressed as S is frequently in O.

Proposition P4.13 Let (X, τ) be a topological space, let $S : D \to X$ be a net in X and let $x_0 \in X$ be a cluster point of S. Then S has a subnet converging to x_0.

Proposition P4.14 Let (X, τ) be a compact topological space and let $S : D \to X$ be a net in X. Then S has a cluster point in X.

Corollary P4.15 Let (X, τ) be a compact topological space. Then every net in X has a convergent subnet.

The idea of a Cauchy net in a uniform space is defined in P2.5.

Proposition P4.16 Let $(X, \mathcal{U})$ be a uniform space and let $S : D \to X$ be a $\tau(\mathcal{U})$-convergent net. Then S is a $\mathcal{U}$-Cauchy net.

Proposition P4.17 Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be a uniform spaces. Let $S : D \to X$ be a $\mathcal{U}$-Cauchy net and suppose $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is uniformly continuous. Then $f \circ S$ is a $\mathcal{V}$-Cauchy net.

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References

1. Kelley, J. L., General Topology, Van Nostrand, 1955.
2. Wilansky, A., Topology for Analysis, Ginn and Co., 1970.