

Existence of the Supremum via Uniform Space Theory

The existence of suprema for collections of compactifications of a given space was first proved by Lubben [3] in 1941. The method used in this section to show that existence was reportedly known by the late 40s, but I have no reference.

Since the closure of a totally bounded subspace is also totally bounded, it has often been noted that a Hausdorff completion of a totally bounded separated uniform space leads to a T_2 compactification of the underlying topological space, e.g. Isbell [1 pp 21-24]. Examining this observation carefully leads to this section's main result.

Given a $T_{3\frac{1}{2}}$ space (X, τ) , $\mathcal{TB}(X)$ will denote the set of totally bounded uniformities that generate τ . Facts and definitions mentioned in [4] and [5] will be used without reference.

The first lemma, which can be found in Kelley [2 p.195], is restated without proof for convenient reference.

Lemma R1.1 Let (Y_1, \mathcal{V}_1) and (Y_2, \mathcal{V}_2) be uniform spaces with (Y_2, \mathcal{V}_2) complete and separated. Let X be a dense subspace of Y_1 and let $f : X \rightarrow Y_2$ be uniformly continuous. Then f has a unique uniformly continuous extension $F : Y_1 \rightarrow Y_2$.

Lemma R1.2 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let \mathcal{U} be a totally bounded uniformity such that $\tau(\mathcal{U}) = \tau$. Let (X_1, \mathcal{V}_1) and (X_2, \mathcal{V}_2) be separated completions of (X, \mathcal{U}) with uniform embeddings $f_1 : X \rightarrow X_1$ and $f_2 : X \rightarrow X_2$. Then (X_1, f_1) and (X_2, f_2) are equivalent T_2 compactifications of X .

Proof: (X_1, \mathcal{V}_1) and (X_2, \mathcal{V}_2) must be unimorphic, and, as an easy application of R1.1, a unimorphism $h : X_1 \rightarrow X_2$ can be chosen so that $h \circ f_1 = f_2$. At the topological level h is the required homeomorphism between the T_2 compactifications X_1 and X_2 .

Lemma R1.2 makes possible the next definition.

Definition R1.3 For a $T_{3\frac{1}{2}}$ space (X, τ) and $\mathcal{U} \in \mathcal{TB}(X)$, let $\Psi_0(\mathcal{U}) = [(X_1, f_1)]$, where (X_1, \mathcal{U}_1) is a separated completion of (X, \mathcal{U}) , $f_1 : X \rightarrow X_1$ is a uniform embedding, and the topology on X_1 is $\tau(\mathcal{U}_1)$.

Proposition R1.4 For any $T_{3\frac{1}{2}}$ space (X, τ) , if (Y, g) is a T_2 compactification of X , then there exists $\mathcal{U} \in \mathcal{TB}(X)$ such that $\Psi_0(\mathcal{U}) = [(Y, g)]$.

Proof: Let \mathcal{V} be the unique uniformity generating the topology of Y . Then (Y, \mathcal{V}) is complete and totally bounded, and the subspace uniformity on $g[X]$ inherits total boundedness. Let \mathcal{U} be the uniformity on X induced by the subspace uniformity on $g[X]$, i.e., $\mathcal{U} = \{U : U \supseteq (gXg)^{-1}[V] \text{ for some } V \in \mathcal{V}\}$. Then \mathcal{U} is totally bounded, $\tau(\mathcal{U}) = \tau$, and g is a uniform embedding. Thus (Y, \mathcal{V}) is a separated completion of (X, \mathcal{U}) with embedding g , and so by definition $\Psi_0(\mathcal{U}) = [(Y, g)]$.

Proposition R1.5 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{TB}(X)$. Then $\mathcal{U}_2 \supseteq \mathcal{U}_1$ if and only if $\Psi_0(\mathcal{U}_2) \geq \Psi_0(\mathcal{U}_1)$.

Proof: Let $\Psi_0(\mathcal{U}_i) = [(Y_i, f_i)]$, where Y_i is a completion of (X, \mathcal{U}_i) and $f_i : (X, \mathcal{U}_i) \rightarrow Y_i$ is a uniform embedding.

First assume $\mathcal{U}_2 \supseteq \mathcal{U}_1$. Then the identity map, $I : (X, \mathcal{U}_2) \rightarrow (X, \mathcal{U}_1)$, is uniformly continuous as is the map $f_1 \circ I \circ (f_2)^{-1} : f_2[X] \rightarrow f_1[X]$. By 1.1 the latter has a unique uniformly continuous extension $h : Y_2 \rightarrow Y_1$. Since h extends $f_1 \circ I \circ (f_2)^{-1}$, the equation

$h \circ f_2 = f_1$ follows immediately. Since the dense $f_1[X]$ is contained in $h[Y_2]$, h is onto. The existence of the continuous surjection h shows $\Psi_0(\mathcal{U}_2) \geq \Psi_0(\mathcal{U}_1)$.

Next assume $\Psi_0(\mathcal{U}_2) \geq \Psi_0(\mathcal{U}_1)$. By definition there is a continuous onto map h from Y_2 to Y_1 such that $h \circ f_2 = f_1$. Since Y_2 is compact, h is also uniformly continuous and so is $f_1^{-1} \circ h \circ f_2$, which is the identity map from (X, \mathcal{U}_2) to (X, \mathcal{U}_1) . Thus $\mathcal{U}_2 \supseteq \mathcal{U}_1$.

Corollary R1.6 For any $T_{3\frac{1}{2}}$ space (X, τ) , let $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{TB}(X)$. If $\Psi_0(\mathcal{U}_1) = \Psi_0(\mathcal{U}_2)$, then $\mathcal{U}_1 = \mathcal{U}_2$.

Proof: Immediate from R1.5.

The next corollary is not needed immediately but will be useful later in identifying the uniformity associated with a given compactification.

Corollary R1.6a Let (X, τ) be a $T_{3\frac{1}{2}}$ space, let (Y, g) be a T_2 compactification of X , and let $\mathcal{W} \in \mathcal{TB}(X)$. Then $\Psi_0(\mathcal{W}) = [(Y, g)]$ if and only if g is a uniform embedding from (X, \mathcal{W}) .

Proof: Let \mathcal{U} be the uniformity in $\mathcal{TB}(X)$ constructed as in the proof of R1.4. As shown there, $\Psi_0(\mathcal{U}) = [(Y, g)]$ and g is a uniform embedding from (X, \mathcal{U}) . If $\Psi_0(\mathcal{W}) = [(Y, g)]$, then $\mathcal{U} = \mathcal{W}$ by R1.6 and g is a uniform embedding from (X, \mathcal{W}) as required. For the converse, the hypothesis implies that $g \circ g^{-1}$ is a unimorphism from (X, \mathcal{W}) to (X, \mathcal{U}) . Since $g \circ g^{-1}$ is the identity map on X , $\mathcal{W} = \mathcal{U}$ and $\Psi_0(\mathcal{W}) = [(Y, g)]$.

Theorem R1.7 [Lubben] Let (X, τ) be a $T_{3\frac{1}{2}}$ space, and let $\{(Y_\alpha, g_\alpha) : \alpha \in \Delta\}$ be a non-empty set of T_2 compactifications of X . Then there exists a T_2 compactification (Y, g) such that $[(Y, g)] \geq [(Y_\alpha, g_\alpha)] \forall \alpha \in \Delta$ and, if $[(Z, h)] \geq [(Y_\alpha, g_\alpha)] \forall \alpha \in \Delta$, then $[(Z, h)] \geq [(Y, g)]$.

Proof: For each α , by R1.4 there is $\mathcal{U}_\alpha \in \mathcal{TB}(X)$ with $\Psi_0(\mathcal{U}_\alpha) = [(Y_\alpha, g_\alpha)]$. Let $\mathcal{U} = \bigvee \mathcal{U}_\alpha$ and let (Y, g) be a T_2 compactification of X such that $\Psi_0(\mathcal{U}) = [(Y, g)]$. Since \mathcal{U} is the supremum of $\{\mathcal{U}_\alpha : \alpha \in \Delta\}$, it is clear from R1.5 that $[(Y, g)]$ will also satisfy the two requirements for a least upper bound.

In the context of R1.7, (Y, g) will be loosely referred to as the supremum of $\{(Y_\alpha, g_\alpha) : \alpha \in \Delta\}$, although clearly uniqueness holds only up to equivalence.

Notation: For any $T_{3\frac{1}{2}}$ space (X, τ) , let \mathcal{U}_M denote $\bigvee \{\mathcal{U} \mid \mathcal{U} \in \mathcal{TB}(X)\}$. If (X, τ) is a locally compact, T_2 space, let \mathcal{U}_m denote $\{U : U \supseteq \bigcup_{i=1}^n O_i \times O_i \text{ where } O_1, \dots, O_n \text{ are an open cover of } X \text{ and at least one } O_i \text{ has a compact complement}\}$.

Corollary R1.8 For any $T_{3\frac{1}{2}}$ space (X, τ) , $\Psi_0(\mathcal{U}_M) = [(\beta X, \iota)]$, where $(\beta X, \iota)$ is the Stone-Ćech compactification of X .

Proof: Since \mathcal{U}_M is the largest element of $\mathcal{TB}(X)$, $\Psi_0(\mathcal{U}_M)$ must be the largest compactification class, i.e. $\Psi_0(\mathcal{U}_M) = [(\beta X, \iota)]$.

Proposition R1.9 Let (X, τ) be a non-compact locally compact, T_2 space. Then \mathcal{U}_m is the smallest element in $\mathcal{TB}(X)$ and $\Psi_0(\mathcal{U}_m)$ is the equivalence class of the one-point compactification.

Proof: Let $X^+ = X \cup \{\infty\}$ have the one-point compactification topology, and let the embedding ι^+ be the inclusion map. Let \mathcal{V} be the element of $\mathcal{TB}(X)$ with $\Psi_0(\mathcal{V}) = [(X^+, \iota^+)]$. It is sufficient to show $\mathcal{U}_m = \mathcal{V}$. First note that \mathcal{V} is simply the subspace uniformity induced from the unique uniformity for X^+ , which is the collection of all neighborhoods of the diagonal in $X^+ \times X^+$. The required equality easily follows since arbitrary

open covers of X^+ reduce to finite covers and open sets containing ∞ are characterized as having a compact complement in X .

Albert J. Klein 2003

<http://www.susanjkleinart.com/compactification/>

References

An asterisk indicates a reference not seen by me.

1. Isbell, J. R., Uniform Spaces, Math. Surveys 12, American Mathematical Society, 1964.
2. Kelley, J. L., General Topology, Van Nostrand, 1955.
- 3* Lubben, R. G., Concerning the decomposition and amalgamation of points, upper semi-continuous collections, and topological extensions, Trans. AMS 49(1941),410-466.
4. This website, P1: Ordering of Compactifications
5. This website, P2: Uniform Spaces