Existence of the Supremum via Uniform Space Theory

The existence of suprema for collections of compactifications of a given space was first proved by Lubben [3] in 1941. The method used in this section to show that existence was likely known by the early 1950s, but I have no reference.

Since the closure of a totally bounded subspace is also totally bounded, it has often been noted that a Hausdorff completion of a totally bounded separated uniform space leads to a T_2 compactification of the underlying topological space, e.g. Isbell [1 pp 21-24]. Examining this observation carefully leads to this section's main result.

Given a $T_{3\frac{1}{2}}$ space (X, τ), $\mathcal{TB}(X)$ will denote the set of totally bounded uniformities that generate τ. Facts and definitions mentioned in [4], [5], and [6] will be used without reference.

The first lemma, which can be found in Kelley [2 p.195], is restated without proof for convenient reference.

Lemma R1.1 Let $(Y_1, \mathcal{V}_1)$ and $(Y_2, \mathcal{V}_2)$ be uniform spaces with $(Y_2, \mathcal{V}_2)$ complete and separated. Let X be a dense subspace of Y_1 and let $f : X \to Y_2$ be uniformly continuous. Then f has a unique uniformly continuous extension $F : Y_1 \to Y_2$.

Lemma R1.2 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{U}$ be a totally bounded uniformity such that $\tau(\mathcal{U}) = \tau$. Let $(X_1, \mathcal{V}_1)$ and $(X_2, \mathcal{V}_2)$ be separated completions of $(X,\mathcal{U})$ with uniform embeddings $f_1 : X \to X_1$ and $f_2 : X \to X_2$. Then (X_1, f_1) and (X_2, f_2) are equivalent T_2 compactifications of X.

Proof: $(X_1, \mathcal{V}_1)$ and $(X_2, \mathcal{V}_2)$ must be unimorphic, and, as an easy application of R1.1, a unimorphism $h : X_1 \to X_2$ can be chosen so that $h \circ f_1 = f_2$. At the topological level h is the required homeomorphism between the T_2 compactifications X_1 and X_2.

Lemma R1.2 makes possible the next definition.

Definition R1.3 For a $T_{3\frac{1}{2}}$ space (X, τ) and $\mathcal{U} \in \mathcal{TB}(X)$, let $\Psi_0(\mathcal{U}) = [(X_1, f_1)]$, where $(X_1,\mathcal{U}_1)$ is a separated completion of $(X,\mathcal{U})$, $f_1 : X \to X_1$ is a uniform embedding, and the topology on X_1 is $\tau(\mathcal{U}_1)$.

Proposition R1.4 For any $T_{3\frac{1}{2}}$ space (X, τ), if (Y, g) is a T_2 compactification of X, then there exists $\mathcal{U} \in \mathcal{TB}(X)$ such that $\Psi_0(\mathcal{U}) = [(Y, g)]$.

Proof: Let $\mathcal{V}$ be the unique uniformity generating the topology of Y. Then $(Y, \mathcal{V})$ is complete and totally bounded, and the subspace uniformity on $g[X]$ inherits total boundedness. Let $\mathcal{U}$ be the uniformity on X induced by the subspace uniformity on $g[X]$, i.e., $\mathcal{U} = \{U : U \supseteq (gxg)^{-1}[V] \text{ for some } V \in \mathcal{V}\}$. Then $\mathcal{U}$ is totally bounded, $\tau(\mathcal{U}) = \tau$, and g is a uniform embedding. Thus $(Y, \mathcal{V})$ is a separated completion of $(X,\mathcal{U})$ with embedding g, and so by definition $\Psi_0(\mathcal{U}) = [(Y, g)]$.

Proposition R1.5 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{U}_1,\mathcal{U}_2 \in \mathcal{TB}(X)$. Then $\mathcal{U}_2 \supseteq \mathcal{U}_1$ if and only if $\Psi_0(\mathcal{U}_2) \geq \Psi_0(\mathcal{U}_1)$.

Proof: Let $\Psi_0(\mathcal{U}_i) = [(Y_i, f_i)]$, where Y_i is a completion of $(X,\mathcal{U}_i)$ and $f_i : (X,\mathcal{U}_i) \to Y_i$ is a uniform embedding.

First assume $\mathcal{U}_2 \supseteq \mathcal{U}_1$. Then the identity map, $I : (X,\mathcal{U}_2) \to (X,\mathcal{U}_1)$, is uniformly continuous as is the map $f_1 \circ I \circ (f_2)^{-1} : f_2[X] \to f_1[X]$. By 1.1 the latter has a unique uniformly continuous extension $h : Y_2 \to Y_1$. Since h extends $f_1 \circ I \circ (f_2)^{-1}$, the equation

$h \circ f_2 = f_1$ follows immediately. Since the dense $f_1[X]$ is contained in $h[Y_2]$, h is onto. The existence of the continuous surjection h shows $\Psi_0(\mathcal{U}_2) \geq \Psi_0(\mathcal{U}_1)$.

Next assume $\Psi_0(\mathcal{U}_2) \geq \Psi_0(\mathcal{U}_1)$. By definition there is a continuous onto map h from Y_2 to Y_1 such that $h \circ f_2 = f_1$. Since Y_2 is compact, h is also uniformly continuous and so is $f_1^{-1} \circ h \circ f_2$, which is the identity map from $(X,\mathcal{U}_2)$ to $(X,\mathcal{U}_1)$. Thus $\mathcal{U}_2 \supseteq \mathcal{U}_1$.

Corollory R1.6 For any $T_{3\frac{1}{2}}$ space (X,τ), let $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{TB}(X)$. If $\Psi_0(\mathcal{U}_1) = \Psi_0(\mathcal{U}_2)$, then $\mathcal{U}_1 = \mathcal{U}_2$.

Proof: Immediate from R1.5.

The next corollary is not needed immediately but will be useful later in identifying the uniformity associated with a given compactification.

Corollary R1.6a Let (X,τ) be a $T_{3\frac{1}{2}}$ space, let (Y,g) be a T_2 compactification of X, and let $\mathcal{W} \in \mathcal{TB}(X)$. Then $\Psi_0(\mathcal{W}) = [(Y,g)]$ if and only if g is a uniform embedding from $(X,\mathcal{W})$.

Proof: Let $\mathcal{U}$ be the uniformity in $\mathcal{TB}(X)$ constructed as in the proof of R1.4. As shown there, $\Psi_0(\mathcal{U}) = [(Y,g)]$ and g is a uniform embedding from $(X,\mathcal{U})$. If $\Psi_0(\mathcal{W}) = [(Y,g)]$, then $\mathcal{U} = \mathcal{W}$ by R1.6 and g is a uniform embedding from $(X,\mathcal{W})$ as required. For the converse, the hypothesis implies that $g \circ g^{-1}$ is a unimorphism from $(X,\mathcal{W})$ to $(X,\mathcal{U})$. Since $g \circ g^{-1}$ is the identity map on X, $\mathcal{W} = \mathcal{U}$ and $\Psi_0(\mathcal{W}) = [(Y,g)]$.

Theorem R1.7 [Lubben] Let (X,τ) be a $T_{3\frac{1}{2}}$ space , and let $\{(Y_\alpha, g_\alpha) : \alpha \in \Delta\}$ be a non-empty set of T_2 compactifications of X. Then there exists a T_2 compactification (Y,g) such that $[(Y,g)] \geq [(Y_\alpha, g_\alpha]\ \forall \alpha \in \Delta$ and, if $[(Z,h)] \geq [(Y_\alpha, g_\alpha]\ \forall \alpha \in \Delta$, then $[(Z,h)] \geq [(Y,g)]$.

Proof: For each α, by R1.4 there is $\mathcal{U}_\alpha \in \mathcal{TB}(X)$ with $\Psi_0(\mathcal{U}_\alpha) = [(Y_\alpha, g_\alpha)]$. Let $\mathcal{U} = \bigvee \mathcal{U}_\alpha$ and let (Y,g) be a T_2 compactification of X such that $\Psi_0(\mathcal{U}) = [(Y,g)]$. Since $\mathcal{U}$ is the supremum of $\{\mathcal{U}_\alpha : \alpha \in \Delta\}$, it is clear from R1.5 that $[(Y,g)]$ will also satisfy the two requirements for a least upper bound.

In the context of R1.7, (Y,g) will be loosely refered to as the supremum of $\{(Y_\alpha, g_\alpha) : \alpha \in \Delta\}$, although clearly uniqueness holds only up to equivalence.

Notation: For any $T_{3\frac{1}{2}}$ space (X,τ), let $\mathcal{U}_M$ denote $\bigvee\{\mathcal{U} : \mathcal{U} \in \mathcal{TB}(X)\}$. If (X,τ) is a locally compact, T_2 space, let $\mathcal{U}_m$ denote $\{U : U \supseteq \bigcup_{i=1}^n O_i \times O_i$ where $O_1, \ldots, O_n$ are an open cover of X and at least one O_i has a compact complement$\}$.

Corollary R1.8 For any $T_{3\frac{1}{2}}$ space (X,τ), $\Psi_0(\mathcal{U}_M) = [(\beta X, \iota)]$, where $(\beta X, \iota)$ is the Stone-Čech compactification of X.

Proof: Since $\mathcal{U}_M$ is the largest element of $\mathcal{TB}(X)$, $\Psi_0(\mathcal{U}_M)$ must be the largest compactification class, i.e. $\Psi_0(\mathcal{U}_M) = [(\beta X, \iota)]$.

Proposition R1.9 Let (X,τ) be a non-compact locally compact, T_2 space. Then $\mathcal{U}_m$ is the smallest element in $\mathcal{TB}(X)$ and $\Psi_0(\mathcal{U}_m)$ is the equivalence class of the one-point compactification.

Proof: Let $X^+ = X \cup \{\infty\}$ have the one-point compactification topology, and let the embedding ι^+ be the inclusion map. Let $\mathcal{V}$ be the element of $\mathcal{TB}(X)$ with $\Psi_0(\mathcal{V}) = [(X^+, \iota^+)]$. It is sufficient to show $\mathcal{U}_m = \mathcal{V}$. First note that $\mathcal{V}$ is simply the subspace uniformity induced from the unique uniformity for X^+, which is the collection of all neighborhoods of the diagonal in $X^+ \times X^+$. The required equality easily follows since arbitrary open covers of X^+ reduce to finite covers and open sets containing ∞ are characterized as having a compact complement in X.

Albert J. Klein 2003
http://www.susanjkleinart.com/compactification/

References

An asterisk indicates a reference not seen by me.

1. Isbell, J. R., Uniform Spaces, Math. Surveys 12, American Mathematical Society, 1964.

2. Kelley, J. L., General Topology, Van Nostrand, 1955.

3* Lubben, R. G., Concerning the decomposition and amalgamation of points, upper semi-continuous collections, and topological extensions, Trans. AMS 49(1941),410-466.

4. This website, P1: Ordering of Compactifications

5. This website, P2: Uniform Spaces

6. This website, P4: Nets

Added 2023

This added subsection provides a proof of R1.1, partly to make the site more self-contained and partly because [2] may not be as widely available as it once was. The proof may or may not differ from Kelley's. I no longer have a copy to check.

Lemma R1.Add.1 Let $(Y_1, \mathcal{V}_1)$ and $(Y_2, \mathcal{V}_2)$ be uniform spaces with $(Y_2, \mathcal{V}_2)$ complete. Let X be a $\tau(\mathcal{V}_1)$-dense subspace of Y_1 and let $f : (X, \mathcal{U}) \to (Y_2, \mathcal{V}_2)$ be uniformly continuous, where $\mathcal{U}$ is the subspace uniformity on X from $\mathcal{V}_1$. Then f extends to a uniformly continuous $F : (Y_1, \mathcal{V}_1) \to (Y_2, \mathcal{V}_2)$.

Proof: If $X = Y_1$, set $F = f$. Thus assume $Y_1 \neq X$. For each $y \in Y_1 - X$, let S_y be the set of nets in X converging to y. By density each $S_y \neq \emptyset$ and by the axiom of choice $\prod\{S_y : y \in Y_1 - X\}$ is non-empty. Pick $p \in \prod\{S_y : y \in Y_1 - X\}$. For $x \in X$, let $F(x) = f(x)$. For $y \in Y_1 - X$, write the net $p(y)$ as $\{x_\alpha\}$. Since $\{x_\alpha\}$ converges to y, the net is Cauchy by P4.16. Since f is uniformly continuous, the image net, $\{f(x_\alpha)\}$, is also Cauchy by P4.17. Since $(Y_2, \mathcal{V}_2)$ is complete, the image net is convergent. Let $F(y)$ be defined as a limit of the image map. By definition F extends f. It will be shown that F is uniformly continuous. Let $V \in \mathcal{V}_2$ and pick $W \in \mathcal{V}_2$ with $W = W^{-1}$ and $W \circ W \circ W \subseteq V$. By hypothesis $(f \times f)^{-1}[W] \in \mathcal{U}$ and so there is $U \in \mathcal{V}_1$ with $U = U^{-1}$ and $(U \circ U \circ U) \cap (X \times X) \subseteq (f \times f)^{-1}[W]$. It is claimed that $U \subseteq (F \times F)^{-1}[V]$, which would show that $(F \times F)^{-1}[V] \in \mathcal{V}_1$ so that F is uniformly continuous. Let $(a, b) \in U$. Pick nets in X converging to a, respectively b as follows. (Notation: $\{a_\alpha\}$ with directed set D and $\{b_\beta\}$ with directed set E such that $\{a_\alpha\}$ converges to a and $\{b_\beta\}$ converges to b.) If a or b is in X, use a constant net. Otherwise use the net determined by the choice function p. By the defintion of F, $\{f(a_\alpha)\}$ converges to $F(a)$ and $\{f(b_\beta)\}$ converges to $F(b)$. By the directed set property, there is $\alpha_1 \in D$ such that $\alpha \geq \alpha_1$ implies $(a_\alpha, a) \in U$ and $(f(a_\alpha), F(a)) \in W$. Similarly, there is $\beta_1 \in E$ such that $\beta \geq \beta_1$ implies $(b_\beta, b) \in U$ and $(f(b_\beta), F(b)) \in W$. Pick $\alpha \geq \alpha_1$ and $\beta \geq \beta_1$. Since $(a, b) \in U$, $(a_\alpha, b_\beta) \in (U \circ U \circ U) \cap (X \times X)$ and so $(f(a_\alpha), f(b_\beta)) \in W$. It follows that $(F(a), F(b)) \in W \circ W \circ W \subseteq V$. Thus $U \subseteq (F \times F)^{-1}[V]$ as claimed.

The following corollary is R1.1.

Corollary R1.Add.2 Let $(Y_1, \mathcal{V}_1)$ and $(Y_2, \mathcal{V}_2)$ be uniform spaces with $(Y_2, \mathcal{V}_2)$ complete and separated. Let X be a $\tau(\mathcal{V}_1)$-dense subspace of Y_1 and let $f : (X, \mathcal{U}) \to (Y_2, \mathcal{V}_2)$ be uniformly continuous, where $\mathcal{U}$ is the subspace uniformity on X from $\mathcal{V}_1$. Then f has a unique uniformly continuous extension $F : (Y_1, \mathcal{V}_1) \to (Y_2, \mathcal{V}_2)$.

Proof: The lemma above shows there is at least one extension. Since $\mathcal{V}_2$ is separated, $\tau(\mathcal{V}_2)$ is T_2. Two continuous maps into a T_2 space which agree on a dense subset of the domain must be identical. Thus the extension is unique.