Some Metric Compactifications of $\mathbb{N}$

This section studies properties of certain metric compactifications of the natural numbers, which have the discrete topology throughout. These compactifications are constructed from normal bases by the technique presented in [6].

Let $k, n \in \mathbb{N}$. Throughout this section $E_n(k)$ will denote equivalence mod k^n. The corresponding equivalence classes are $C_n^i(k)$ for $i = 1 \ldots k^n$. Definitions, notation, and facts from [1], [4], and [5] will be used as needed.

The Spaces $\mathbb{N}_k$ and $\mathbb{N}_\infty$

Lemma R10.1.1 Let E, F be equivalence relations on an infinite discrete space X with E l-compatible and F m-compatible. If $E \subseteq F$, then $\mathcal{Z}(F) \subseteq \mathcal{Z}(E)$.

Proof: Let $C_1, \ldots, C_l$ be the distinct infinite equivalence classes of E and $D_1, \ldots, D_m$ the distinct infinite equivalence classes of F. Let $Z \in \mathcal{Z}(F)$ be associated with $\Delta \subseteq \{1, \ldots, m\}$ relative to F. Since $E \subseteq F$, each C_i must be contained in a unique D_j. Let $\Gamma = \{i : C_i \subseteq D_j \text{ for some } j \in \Delta\}$. If $i \in \Gamma$, then $Z \cap C_i$ is contained in the corresponding $Z \cap D_j$ which is finite. If $i \in \{1, \ldots, l\} - \Gamma$, then $C_i \subseteq D_j$ for some $j \notin \Delta$ and $(X - Z) \cap C_i$ must be contained in the finite $(X - Z) \cap D_j$. Thus Z is associated with Γ relative to E and $Z \in \mathcal{Z}(E)$.

Lemma R10.1.2 Let $k, n \in \mathbb{N}$. Then

i)$E_{n+1}(k) \subseteq E_n(k)$.

ii)$C_n^i(k) = \bigcup\{C_{n+1}^{i+jk^n}(k) : j = 0, \ldots k-1\}$.

iii) $\mathcal{Z}(E_n(k)) \subseteq \mathcal{Z}(E_{n+1}(k))$.

Proof: The first assertion is clear and the second is a routine application of the division algorithm. The third is immediate from the first and R10.1.1.

Definition R10.1.3 Let $k \in \mathbb{N}$. $\mathcal{Z}_k = \bigcup\{\mathcal{Z}(E_n(k)) : n \in \mathbb{N}\}$.

Proposition R10.1.4 Let $k \in \mathbb{N}$ .Then

i) $\mathcal{Z}_k$ is a normal basis for $\mathbb{N}$.

ii) $(\omega(\mathcal{Z}_k), \iota_{\mathcal{Z}_k}) = \bigvee\{(\omega(\mathcal{Z}(E_n(k))), \iota_{\mathcal{Z}(E_n(k))}) : n \in \mathbb{N}\}$.

Proof: R10.1.2.iii shows that $\{\mathcal{Z}(E_n(k)) : n \in \mathbb{N}\}$ is a chain and so the first conclusion follows from R9.2.1 and the second from R9.2.4.

From now on, $\mathbb{N}_k$ will denote the T_2 compactification $(\omega(\mathcal{Z}_k), \iota_{\mathcal{Z}_k})$. Note that $E_n(1) = \mathbb{N} \times \mathbb{N}$ for all n so that $\mathcal{Z}(E_n(1))$ is always the collection of all subsets of $\mathbb{N}$ which are either finite or co-finite and $\mathbb{N}_1$ is the one-point compactification.

Lemma R10.1.5 The collection $\{\mathcal{Z}(E_n(k)) : n, k \in \mathbb{N}\}$ is a directed set under containment.

Proof: Only the directed set property needs to be verified. Given $\mathcal{Z}(E_n(k))$ and $\mathcal{Z}(E_m(l))$, let $r = mn$ and $q = kl$. Since k^n and l^m both divide q^r, $E_r(q) \subseteq E_n(k) \cap E_m(l)$. By R10.1.1 $\mathcal{Z}(E_n(k)) \cup \mathcal{Z}(E_m(l)) \subseteq \mathcal{Z}(E_r(q))$.

Definition R10.1.6 $\mathcal{Z}_\infty = \bigcup\{\mathcal{Z}(E_n(k)) : n, k \in \mathbb{N}\}$.

Proposition R10.1.7 $\mathcal{Z}_\infty$ is a normal basis for $\mathbb{N}$ and $(\omega(\mathcal{Z}_\infty), \iota_{\mathcal{Z}_\infty}) = \bigvee\{(\omega(\mathcal{Z}(E_n(k))), \iota_{\mathcal{Z}(E_n(k))}) : n, k \in \mathbb{N}\}$.

Proof: These statements follow immediately from R.10.1.5, R9.2.1, and R9.2.4.

From now on, $\mathbb{N}_\infty$ will denote the T_2 compactification $(\omega(\mathcal{Z}_\infty), \iota_{\mathcal{Z}_\infty})$.

Corollary R10.1.8 $\mathbb{N}_\infty = \bigvee\{\ \mathbb{N}_k : k \in \mathbb{N}\ \}$.

Proof: This is clear from R10.1.7 and R10.1.4.

Lemma R10.1.9 Let E be an n-compatible equivalence relation on $\mathbb{N}$. Then $\omega(\mathcal{Z}(E))$ is 2nd countable.

Proof: Let $C_1, \ldots, C_n$ be the distinct infinite equivalence classes of E. By R5.3.8 $(\omega(\mathcal{Z}(E)), \iota_{\mathcal{Z}(E)})$ is equivalent to (Y, ι_E), where $Y = \mathbb{N} \cup \{p_1, \ldots, p_n\}$, $p_1, \ldots, p_n$ are distinct and not in $\mathbb{N}$, $\tau(E) = \{O \subseteq Y : p_i \in O \Rightarrow (\mathbb{N} - O) \cap C_i$ is finite $\}$ is the topology on Y, and ι_E is the identity map. It is sufficient to show that $(Y, \tau(E))$ is 2nd countable. If $p_i \in O \in \tau(E)$, let F be the finite set $(\mathbb{N} - O) \cap C_i$. It is easy to check that $(C_i - F) \cup \{p_i\}$ is open in Y and $(C_i - F) \cup \{p_i\} \subseteq O$. Thus $\{(C_i - H) \cup \{p_i\} : H$ is a finite subset of $\mathbb{N}$ $\}$ is a countable local basis at p_i. These n local bases together with the singletons for points of $\mathbb{N}$ form the required countable basis in Y.

Proposition R10.1.10 The compatifications $\mathbb{N}_\infty$ and $\mathbb{N}_k$ for any $k \in \mathbb{N}$ have the following properties:

i) They are suprema of finite-point compactifications.
ii) They are 2nd countable.
iii) They are metrizable.
iv) They are zero-dimensional.
v) They are not homeomorphic to $\beta\mathbb{N}$.
vi) They are not extremely disconnected.

Proof: By R5.3.8 each $\omega(\mathcal{Z}(E_n(k)))$ is a finite-point compactification and so i) follows immediately from the definitions. Property ii) follows from the preceding lemma and R3.2.8 in [3]. The third conclusion is immediate from Urysohn's Theorem: a 2nd countable T_3 space must be metrizable. The fourth follows from i) and R9.3.3. Property v) follows because $\beta\mathbb{N}$ is not metrizable. The last holds because $\beta\mathbb{N}$ is the only extremely disconnected compactification of $\mathbb{N}$.

Filters and Convergence

Lemma R10.2.1 Let $k \in \mathbb{N}$. $\mathcal{F} \in \mathbb{N}_k$ if and only if $\mathcal{F} = \cup\{\mathcal{F}^n : n \in \mathbb{N}\ \}$, where, for all n, $\mathcal{F}^n \in \omega(\mathcal{Z}(E_n(k))$ and $\mathcal{F}^n \subseteq \mathcal{F}^{n+1}$.

Proof: Given $\mathcal{F} \in \mathbb{N}_k$, let $\mathcal{F}^n = \mathcal{F} \cap \mathcal{Z}(E_n(k))$. By R9.2.2 each $\mathcal{F}^n \in \omega(\mathcal{Z}(E_n(k))$. The union equation is immediate from the definition of $\mathcal{Z}_k$ and $\mathcal{F}^n \subseteq \mathcal{F}^{n+1}$ since $\mathcal{Z}(E_n(k)) \subseteq \mathcal{Z}(E_{n+1}(k))$. The converse follows easily from R9.2.3.

Lemma R10.2.2 Let $k \in \mathbb{N}$, let $\mathcal{F} \in \mathbb{N}_k$, and let $\mathcal{F}^n = \mathcal{F} \cap \mathcal{Z}(E_n(k))$. Then $\mathcal{F}$ is a non-point $\mathcal{Z}_k$-ultrafilter if and only if each $\mathcal{F}^n$ is a non-point element of $\omega(\mathcal{Z}(E_n(k)))$.

Proof: Since $\mathcal{Z}(E_n(k))$ contains all singletons and $\mathcal{F}$ is the point $\mathcal{Z}_k$-ultrafilter generated by x if and only if $\{x\} \in \mathcal{F}$, the conclusion follows easily.

By R5.3.7 each non-point element of $\omega(\mathcal{Z}(E_n(k)))$ can be described uniquely as $\{Z \in \mathcal{Z}(E_n(k)) : Z$ is associated with $\Delta \subseteq \{1, \ldots, k^n\} - \{i\}\}$, where $1 \le i \le k^n$. That fact allows the following definition.

Definition R10.2.3 Let $k \in \mathbb{N}$ and let $\mathcal{F} \in \mathbb{N}_k$ be a non-point $\mathcal{Z}_k$-ultrafilter. The sequence $\{x_n\}$ associated with $\mathcal{F}$ is defined by $x_n = i$ where $\mathcal{F} \cap \mathcal{Z}(E_n(k)) = \{Z \in \mathcal{Z}(E_n(k)) : Z$ is associated with $\Delta \subseteq \{1, \ldots, k^n\} - \{i\}\}$.

Since i is unique at each level by R5.3.7, this definition can be interpreted as defining a function from the non-point elements of $\mathbb{N}_k$ to the set of sequences of natural numbers.

The next lemma shows that the function is one-to-one.

Lemma R10.2.4 Let $k \in \mathbb{N}$ and let $\mathcal{F}, \mathcal{G} \in \mathbb{N}_k$ be non-point $\mathcal{Z}_k$-ultrafilters associated with the sequences $\{x_n\}$ and $\{y_n\}$ respectively. If $x_n = y_n$ for all n, then $\mathcal{F} = \mathcal{G}$.

Proof: Let $\mathcal{F}^n = \mathcal{F} \cap \mathcal{Z}(E_n(k))$ and $\mathcal{G}^n = \mathcal{G} \cap \mathcal{Z}(E_n(k))$. If $x_n = y_n$ for all n, then by R5.3.7 $\mathcal{F}^n = \mathcal{G}^n$ for all n and so $\mathcal{F} = \mathcal{G}$.

Lemma R10.2.5 Let $k \in \mathbb{N}$ and let $\mathcal{F} \in \mathbb{N}_k$ be a non-point $\mathcal{Z}_k$-ultrafilter. Assume the sequence $\{x_n\}$ is associated with $\mathcal{F}$. Then

i)For all $n \in \mathbb{N}$, $x_{n+1} \in \{x_n + jk^n : 0 \leq j \leq (k-1)\}$.

ii) For all $n, m \in \mathbb{N}$, k^n divides $x_{n+m} - x_n$.

Proof: $C_n^{x_n}(k)$ is associated with $\{1, \ldots, k^n\} - \{x_n\}$ and so, by definition of x_n, is in $\mathcal{F} \cap \mathcal{Z}(E_n(k))$. By R10.1.2iii it is also in $\mathcal{F} \cap \mathcal{Z}(E_{n+1}(k))$, and at the $(n+1)^{st}$ level it must be associated with some $\Delta \subseteq \{1, \ldots, k^{n+1}\} - \{x_{n+1}\}$ by definition of x_{n+1}. By R10.1.2ii, $C_n^{x_n}(k) \cap C_{n+1}^t(k)$ is infinite for $t \in \{x_n + jk^n : 0 \leq j \leq k-1\}$ and is empty for the other possible values of t. Thus x_{n+1} must be in $\{x_n + jk^n : 0 \leq j \leq k-1\}$, as required for i). Part ii) follows from i) by fixing n and doing a routine induction on m.

Lemma R10.2.6 Let $k \in \mathbb{N}$ and assume $\{x_n\}$ is a sequence of natural numbers such that $x_1 \in \{1, \ldots, k\}$ and, for all $n \in \mathbb{N}$, $x_{n+1} \in \{x_n + jk^n : 0 \leq j \leq (k-1)\}$. Then there is a unique non-point $\mathcal{Z}_k$-ultrafilter associated with $\{x_n\}$.

Proof: Let $\{Z \in \mathcal{Z}(E_n(k)) : Z$ is associated with some $\Delta \subseteq \{1, \ldots, k^n\} - \{x_n\}\}$ be denoted by $\mathcal{F}^n$. By R5.3.7 $\mathcal{F}^n$ is a non-point $\mathcal{Z}(E_n(k))$-ultrafilter. First, it will be shown that $\mathcal{F}^n \subseteq \mathcal{F}^{n+1}$. Let $Z \in \mathcal{F}^n$ be associated with $\delta \subseteq \{1, \ldots, k^n\} - \{x_n\}\}$. By R10.1.2iii Z is also in $\mathcal{Z}(E_{n+1}(k))$ and, as such, must be associated with a unique $\delta^* \subseteq \{1, \ldots, k^{n+1}\}$. Then $(\mathbb{N} - Z) \cap C_n^{x_n}$ is finite and, by R10.1.2ii, $(\mathbb{N} - Z) \cap C_{n+1}^t$ must be finite for $t \in \{x_n + jk^n : 0 \leq j \leq (k-1)\}$. Therefore $x_{n+1} \notin \delta^*$ and so $Z \in \mathcal{F}^{n+1}$. Next R10.2.1 and R10.2.2 apply to show that $\mathcal{F} = \cup\{\mathcal{F}^n : n \in \mathbb{N}\}$ is a non-point $\mathcal{Z}_k$-ultrafilter. By construction $\mathcal{F}$ is associated with $\{x_n\}$. If $\mathcal{G}$ in $\mathbb{N}_k$ is also associated with $\{x_n\}$, then $\mathcal{G} \cap \mathcal{Z}(E_n(k)) = \mathcal{F}^n$ for all n and so $\mathcal{G} = \mathcal{F}$.

Proposition R10.2.7 Let $k \in \mathbb{N}$ with $k \geq 2$. Then the cardinality of $\mathbb{N}_k$ is $2^{\aleph_0}$.

Proof: Let S be the set of sequences associated with some non-point $\mathcal{Z}_k$-ultrafilter. Since the set of all sequences in $\mathbb{N}$ has cardinality $\aleph_0^{\aleph_0} = 2^{\aleph_0}$, $|S| \leq 2^{\aleph_0}$. For any $A \subseteq \mathbb{N}$ let $x_1 = 1$ if $1 \in A$ and $x_1 = 2$ if $1 \notin A$. Let $x_{n+1} = x_n$ if $n+1 \in A$ and $x_{n+1} = x_n + k^n$ if $n+1 \notin A$. By R10.2.6 $\{x_n\}$ must be in S. This defines a map from the power set of $\mathbb{N}$ into S. This map is easily seen to be one-to-one and so $2^{\aleph_0} \leq |S|$. Clearly S is the image of the function defined in R10.2.3 and, since that function is one-to-one by R10.2.4, the set of non-point $\mathcal{Z}_k$-ultrafilters has cardinality $|S| = 2^{\aleph_0}$. Since there are countably many point $\mathcal{Z}_k$-ultrafilters, the cardinality of $\mathbb{N}_k$ is $\aleph_0 + 2^{\aleph_0} = 2^{\aleph_0}$.

Lemma R10.2.8 Let $\mathcal{F}$ be a $\mathcal{Z}_\infty$-ultrafilter. Then $\mathcal{F} \cap \mathcal{Z}_k$ is a $\mathcal{Z}_k$-ultrafilter for all k in $\mathbb{N}$ and $\mathcal{F}$ is a point ultrafilter if and only if $\mathcal{F} \cap \mathcal{Z}_k$ is a point ultrafilter for all k.

Proof: By the definition of $\mathcal{Z}_k$, $\mathcal{F} \cap \mathcal{Z}_k = \cup\{\mathcal{F} \cap \mathcal{Z}(E_n(k)) : n \in \mathbb{N}\}$. By R9.2.2 each $\mathcal{F} \cap \mathcal{Z}(E_n(k))$ is a $\mathcal{Z}(E_n(k))$-ultrafilter. By R9.2.3 $\mathcal{F} \cap \mathcal{Z}_k$ is a $\mathcal{Z}_k$-ultrafilter. The second assertion follows easily because $\{x\} \in \mathcal{Z}_k$ for all k and $\mathcal{F}$ is the point ultrafilter determined by x if and only if $\{x\} \in \mathcal{F}$.

Corollary R10.2.9 The cardinality of $\mathbb{N}_\infty$ is $2^{\aleph_0}$.

Proof: Since $\mathbb{N}_2 \leq \mathbb{N}_\infty$, there is a surjection from $\mathbb{N}_\infty$ to $\mathbb{N}_2$. Therefore

$\mid \mathbb{N}_\infty \mid \geq \mid \mathbb{N}_2 \mid = 2^{\aleph_0}$ by R10.2.7. Given $\mathcal{F}$, a non-point $\mathcal{Z}_\infty$-ultrafilter, the non-point ultrafilter $\mathcal{F} \cap \mathcal{Z}_k$ is associated with a unique sequence $\{x_n^k\}$ as in R10.2.3. This defines a map from the non-point ultrafilters of $\mathbb{N}_\infty$ to the set of sequences of sequences in $\mathbb{N}$, and this map is clearly one-to-one. Since there are only countably many point $\mathcal{Z}_\infty$-ultrafilters and the set of sequences of sequences in $\mathbb{N}$ has cardinality $(\aleph_0^{\aleph_0})^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$,
$\mid \mathbb{N}_\infty \mid \leq 2^{\aleph_0}$.

Notice that R10.2.7 and R10.2.9 yield cardinality proofs that neither $\mathbb{N}_k$ nor $\mathbb{N}_\infty$ is $\beta\mathbb{N}$, since $\beta\mathbb{N}$ is known to have cardinality 2^c where $c = 2^{\aleph_0}$.

For the next three results the following notational simplification will be used: For $k, x, m \in \mathbb{N}$ the point $\mathcal{Z}_k$-ultrafilter $\mathcal{F}_x$ will be written as $\hat{x}$ and $\mathcal{F}_x \cap \mathcal{Z}(E_m(k))$ as $\hat{x}^m$.

Lemma R10.2.10 Let $k \in \mathbb{N}$, let $\{x_n\}$ be a sequence in $\mathbb{N}$, and let $\mathcal{F} \in \mathbb{N}_k$. Then $\hat{x}_n \to \mathcal{F}$ in $\mathbb{N}_k$ if and only if $\hat{x}_n^m \to \mathcal{F} \cap \mathcal{Z}(E_m(k))$ for every $m \in \mathbb{N}$.

Proof: As in the proof of R9.1.1, for any m, the map $h : \mathbb{N}_k \to \omega(\mathcal{Z}(E_m(k)))$ given by $h(\mathcal{G}) = \mathcal{G} \cap \mathcal{Z}(E_m(k))$ is continuous and so $\hat{x}_n \to \mathcal{F}$ in $\mathbb{N}_k$ implies $\hat{x}_n^m \to \mathcal{F} \cap \mathcal{Z}(E_m(k))$. For the converse, let $Z \in \mathcal{Z}_k$ with $\mathcal{F} \notin Z^\omega$. Since $\mathcal{Z}_k^\omega$ is a closed base for $\mathbb{N}_k$, it is sufficient to show that $\hat{x}_n \notin Z^\omega$ eventually. Pick m with $Z \in \mathcal{Z}(E_m(k))$. By convergence at level m, $x_n \notin Z$ eventually, which is equivalent to $\hat{x}_n \notin Z^\omega$ eventually.

Proposition R10.2.11 Let $k \in \mathbb{N}$, let $\{x_n\}$ be a sequence in $\mathbb{N}$, and let $\mathcal{F} \in \mathbb{N}_k$ be a non-point ultrafilter associated with $\{y_n\}$. Then $\hat{x}_n \to \mathcal{F}$ in $\mathbb{N}_k$ if and only if

i) $\{x_n\}$ is unbounded and

ii)for every $m \in \mathbb{N}$, x_n is eventually in $C_m^{y_m}(k)$

Proof: First assume $\hat{x}_n \to \mathcal{F}$ in $\mathbb{N}_k$. If $\{x_n\}$ were bounded, then $\{\hat{x}_n : n \in \mathbb{N}\}$ would be finite and so closed in $\mathbb{N}_k$. Its open complement would contain the non-point $\mathcal{F}$, which would contradict the given convergence. Next let $m \in \mathbb{N}$ and let $\mathcal{F}^m = \mathcal{F} \cap \mathcal{Z}(E_m(k))$. By definition $\mathcal{F}^m = \{Z \in \mathcal{Z}(E_m(k)) : Z$ is associated with $\Delta \subseteq \{1, \ldots, k^m\} - \{y_m\}\}$ and so $C_m^{y_m}(k) \in \mathcal{F}^m$. Let $Z = \mathbb{N} - C_m^{y_m}(k)$. $Z \in \mathcal{Z}(E_m(k))$ since $\mathcal{Z}(E_m(k))$ is closed under complementation and, by the definition of a filter, $Z \notin \mathcal{F}^m$. In other words, $\mathcal{F}^m$ is not in the $\omega(\mathcal{Z}(E_m(k)))$-closed set Z^ω. By R10.2.10 $\hat{x}_n^m \to \mathcal{F}^m$ in $\omega(\mathcal{Z}(E_m(k)))$ and so eventually $\hat{x}_n^m$ is in the complement of Z^ω. From that it easily follows that x_n is eventually in $C_m^{y_m}(k)$.

For the converse, assume $\{x_n\}$ has the two properties. Since $\mathcal{Z}_k^\omega$ is a closed base for $\mathbb{N}_k$, it is sufficient to consider $\mathcal{F}$ in an open set which is the complement of Z^ω for some $Z \in \mathcal{Z}_k$. Pick m with $Z \in \mathcal{Z}(E_m(k))$ and let Z be associated with $\Delta \subseteq \{1, \ldots, k^m\}$. Since $Z \notin \mathcal{F}$, $Z \notin \mathcal{F} \cap \mathcal{Z}(E_m(k))$. By definition of y_m, $y_m \in \Delta$ so that $Z \cap C_m^{y_m}(k)$ is finite. Since $\{x_n\}$ is unbounded and eventually in $C_m^{y_m}(k)$, eventually $x_n \notin Z$. In other words, eventually $\hat{x}_n$ is in the complement of Z^ω, as required for convergence.

Proposition R10.2.12 Let $k \in \mathbb{N}$ and let $\{x_n\}$ be an unbounded sequence in $\mathbb{N}$. Assume for every $m \in \mathbb{N}$ there is $y_m \in \{1, \ldots, k^m\}$ such that x_n is eventually in $C_m^{y_m}(k)$. Then there is a non-point $\mathcal{Z}_k$-ultrafilter $\mathcal{F}$ such that $\mathcal{F}$ is associated with $\{y_n\}$ and $\hat{x}_n \to \mathcal{F}$ in $\mathbb{N}_k$.

Proof: Let $m \in \mathbb{N}$. By R10.1.2ii, $C_m^{y_m}(k) = \bigcup\{C_{m+1}^{y_m + jk^m}(k) : j = 0, \ldots k-1\}$. Since the equivalence classes form a partition and x_n is eventually in both $C_m^{y_m}(k)$ and $C_{m+1}^{y_{m+1}}(k)$, y_{m+1} must be in $\{y_m + jk^m : j \in \{0, 1, \ldots, k-1\}\}$. By R10.2.6 there is a unique non-point $\mathcal{Z}_k$-ultrafilter $\mathcal{F}$ associated with $\{y_n\}$. By R10.2.11 $\hat{x}_n \to \mathcal{F}$ in $\mathbb{N}_k$.

Lattice Structure and Divisibility

Well-known facts about divisibility will be used in this subsection. For $a, b \in \mathbb{N}$, $a|b$ means a divides b, $[a,b]$ denotes the least common multiple (LCM) of a and b, and (a,b) the greatest commom divisor (GCD).

Lemma R10.3.1 Let $a, b \in \mathbb{N}$. Then

i) If $a|b$, then $\mathbb{N}_a \leq \mathbb{N}_b$.

ii) For any $j \in \mathbb{N}$, $\mathbb{N}_{a^j}$ is equivalent to $\mathbb{N}_a$.

iii) $\mathbb{N}_a \vee \mathbb{N}_b \leq \mathbb{N}_{[a,b]}$.

iv) $\mathbb{N}_{(a,b)} \leq \mathbb{N}_a \wedge \mathbb{N}_b$.

Proof: If $a|b$, it is clear from the definitions that $E_n(b) \subseteq E_n(a)$ and so, by R10.1.1 and R9.1.8, $[(\omega(\mathcal{Z}(E_n(a))), \iota_{\mathcal{Z}(E_n(a))})] \leq [(\omega(\mathcal{Z}(E_n(b))), \iota_{\mathcal{Z}(E_n(b))})]$. This immediately yields $\mathbb{N}_a \leq \mathbb{N}_b$. Similarly, $E_n(a^j) = E_{nj}(a)$ so that $\mathbb{N}_{a^j} \leq \mathbb{N}_a$. Part i) gives the reverse inequality so that ii) follows. Since both a and b divide $[a,b]$, iii) is immediate from i). Part iv) follows in the same way because (a,b) divides both a and b.

Lemma R10.3.2 Let $a, b \in \mathbb{N}$ and let $m = [a,b]$. Let $j \in \mathbb{N}$. If $C_j^r(a) \cap C_j^s(b) \neq \emptyset$ for some $r \in \{1, \ldots, a^j\}$,$s \in \{1, \ldots, b^j\}$, then there is a unique $t \in \{1, \ldots, m^j\}$ such that $C_j^r(a) \cap C_j^s(b) \subseteq C_j^t(m)$.

Proof: Let x be in the intersection. There is a unique $t \in \{1, \ldots, m^j\}$ such that $x \in C_j^t(m)$. If y is also in the intersection, then $a^j|(x-y)$ and $b^j|(x-y)$ so that $m^j|(x-y)$, i.e., x and y must be in the same $E_j(m)$ equivalence class.

The rest of this subsection will use the following notation: for $t, x \in \mathbb{N}$, ${}^t\hat{x}$ will denote the point-ultrafilter determined by x in $\mathbb{N}_t$. In the next proposition the compactification $\mathbb{N}_a \vee \mathbb{N}_b$ will be represented as a subspace of $\mathbb{N}_a \times \mathbb{N}_b$ as in R3.1.2 of [3].

Proposition R10.3.3 Let $a, b \in \mathbb{N}$. Then $\mathbb{N}_a \vee \mathbb{N}_b$ is equivalent to $\mathbb{N}_{[a,b]}$.

Proof: Let $m = [a,b]$. By R10.3.1iii there is $h : \mathbb{N}_m \to \mathbb{N}_a \vee \mathbb{N}_b$ continuous, closed and onto such that $h({}^m\hat{x}) = ({}^a\hat{x}, {}^b\hat{x})$ for all x in $\mathbb{N}$. It is sufficient to show that h is one-to-one.

Suppose $h(\mathcal{F}) = h(\mathcal{G}) = (\mathcal{H}_1, \mathcal{H}_2)$ where $\mathcal{F}, \mathcal{G}$ are in $\mathbb{N}_m$, $\mathcal{H}_1$ is in $\mathbb{N}_a$, and $\mathcal{H}_2$ is in $\mathbb{N}_b$. By R10.1.10iii and density, there exist sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{N}$ such that ${}^m\hat{x}_n \to \mathcal{F}$ and ${}^m\hat{y}_n \to \mathcal{G}$ in $\mathbb{N}_m$. By continuity of h and the fact that h 'respects' the embeddings, ${}^a\hat{x}_n \to \mathcal{H}_1$ and ${}^a\hat{y}_n \to \mathcal{H}_1$ in $\mathbb{N}_a$. Similarly, ${}^b\hat{x}_n \to \mathcal{H}_2$ and ${}^b\hat{y}_n \to \mathcal{H}_2$ in $\mathbb{N}_b$.

If $\mathcal{H}_1$ or $\mathcal{H}_2$ is a point-ultrafilter, then both $\{x_n\}$ and $\{y_n\}$ must be eventually constant with the same constant, which would imply that $\mathcal{F} = \mathcal{G}$. If $\mathcal{F}$ or $\mathcal{G}$ is a point-ultrafilter, both $\mathcal{H}_1$ and $\mathcal{H}_2$ are point-ultrafilters, which again yields $\mathcal{F} = \mathcal{G}$. Thus we can assume that $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}_1$, and $\mathcal{H}_2$ are non-point ultrafilters associated with $\{p_j\}$, $\{q_j\}$, $\{r_j\}$, and $\{s_j\}$ respectively.

Now fix $j \in \mathbb{N}$. Applying R10.2.11 repeatedly to the convergence facts described above, x_n is eventually in $C_j^{p_j}(m)$, y_n is eventually in $C_j^{q_j}(m)$, both x_n and y_n are eventually in $C_j^{r_j}(a)$, and both x_n and y_n are eventually in $C_j^{s_j}(b)$. By R10.3.2 there must be a unique t such that both x_n and y_n are eventually in $C_j^t(m)$ and so $p_j = t = q_j$. By R10.2.4 $\mathcal{F} = \mathcal{G}$.

Proposition R10.3.4 $\mathbb{N}_\infty$ is equivalent to $\vee\{\mathbb{N}_p : p \text{ is prime}\}$.

Proof: From R10.1.8 it is clear that $\vee\{\mathbb{N}_p : p \text{ is prime}\} \leq \mathbb{N}_\infty$. Now let $k \in \mathbb{N}$. If $k = 1$, then $\mathbb{N}_1$ is the one-point compactification and so $\mathbb{N}_1 \leq \mathbb{N}_p$ for any prime. If

$k > 1$, let the prime factorization of k be $\prod_{i=1}^{n} p_i^{j_i}$. By R10.3.1ii each $\mathbb{N}_{p^j}$ is equivalent to $\mathbb{N}_p$ and, since k is the LCM of $\{p_1^{j_1}, \ldots, p_n^{j_n}\}$, a routine inductive extension of R10.3.3 shows that $\mathbb{N}_k$ is equivalent to $\vee\{\mathbb{N}_t : t \in \{p_1, \ldots, p_n\}\}$. Thus $\mathbb{N}_k \leq \vee\{\mathbb{N}_p : p \text{ is prime}\}$ for all k. It follows that $\mathbb{N}_\infty \leq \vee\{\mathbb{N}_p : p \text{ is prime}\}$.

Proposition R10.3.5 Let p, q be distinct primes in $\mathbb{N}$. Then $\mathbb{N}_p \not\leq \mathbb{N}_q$.

Proof: Deny. Then there is a continuous onto map $f : \mathbb{N}_q \to \mathbb{N}_p$ such that $f({}^q\hat{x}) = {}^p\hat{x}$ for all x in $\mathbb{N}$. Use R10.2.12 with $k = q$ and $y_m = q^m$ to see that the sequence ${}^q\widehat{q^n}$ converges to a non-point ultrafilter in $\mathbb{N}_q$. By the continuity of f the sequence ${}^p\widehat{q^n}$ converges to a non-point ultrafilter in $\mathbb{N}_p$. Applying R10.2.11 to the second sequence with $m = 1$, we can find $t \in \{1, \ldots, p\}$ such that q^n is eventually in $C_1^t(p)$. Thus, for sufficiently large n, $q^n \equiv q^{n+1} \bmod p$. Since q is invertible mod p, this yields $q \equiv 1 \bmod p$. Similarly by R10.2.12 the sequence ${}^q\widehat{nq^n}$ converges to a non-point ultrafilter in $\mathbb{N}_q$ and so ${}^p\widehat{nq^n}$ converges to a non-point ultrafilter in $\mathbb{N}_p$. Applying R10.2.11 to the last sequence with $m = 1$, we find $s \in \{1, \ldots, p\}$ such that nq^n is eventually in $C_1^s(p)$, i.e., for sufficiently large n, $nq^n \equiv (n+1)q^{n+1} \bmod p$. Since $q \equiv 1 \bmod p$, this implies $n \equiv n + 1 \bmod p$ for sufficiently large n, a contradiction.

Since the roles of p and q can be reversed, R10.3.5 says that $\mathbb{N}_p$ and $\mathbb{N}_q$ are not related in any way. Although the question of whether the topological space $\mathbb{N}_p$ is homeomorphic to $\mathbb{N}_q$ remains open, the following corollary is immediate.

Corollary R10.3.6 Let p, q be distinct primes in $\mathbb{N}$. Then $\mathbb{N}_p$ and $\mathbb{N}_q$ are not equivalent compactifications.

This subsection concludes with a sharpened version of R10.3.1iv. In [2] each totally bounded uniformity, $\mathcal{U}$, for a $T_{3\frac{1}{2}}$ space X determines a compactification class $\Psi_0(\mathcal{U})$. R5.2.4 and R5.3.8 show that , if X is an infinite discrete space and E is an n-compatible equivalence relation on X, then $\Psi_0(\mathcal{U}_m \vee \mathcal{U}_E)$ is compactification class of $(\omega(\mathcal{Z}(E)), \iota_{\mathcal{Z}(E)})$, where $\mathcal{U}_m$ is the totally bounded uniformity associated with the one-point compactification and $\mathcal{U}_E$ is the uniformity is the uniformity based on E as defined in R5.2.1. The next two lemmas use these ideas to describe the uniformity associated with $\mathbb{N}_k$.

Lemma R10.3.7 Let X be an infinite discrete space. Suppose Δ is non-empty and for each $\alpha \in \Delta$ E_α is an n_α-compatible equivalence relation on X such that $\{\mathcal{Z}(E_\alpha) : \alpha \in \Delta\}$ is a directed set under containment. Let $\mathcal{Z} = \cup\{\mathcal{Z}(E_\alpha) : \alpha \in \Delta\}$ and $\mathcal{U}_\alpha$ be the totally bounded uniformity for X such that $\Psi_0(\mathcal{U}_\alpha) = [(\omega(\mathcal{Z}(E_\alpha)), \iota_{\mathcal{Z}(E_\alpha)})]$.
Then $\Psi_0(\vee\{\mathcal{U}_\alpha : \alpha \in \Delta\}) = [(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})]$.

Proof: Let $\mathcal{U}$ be the totally bounded uniformity for X such that $\Psi_0(\mathcal{U}) = [(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})]$. R1.5 and R9.2.4 clearly imply that $\mathcal{U} = \vee\{\mathcal{U}_\alpha : \alpha \in \Delta\}$.

In the rest of this subsection, for j, t in $\mathbb{N}$, $\mathcal{U}_j(t)$ will denote $\mathcal{U}_m \vee \mathcal{U}_{E_j(t)}$.

Lemma R10.3.8 Let k be in $\mathbb{N}$. Let $\mathcal{U}$ be the totally bounded uniformity for $\mathbb{N}$ such that $\Psi_0(\mathcal{U})$ is the compactification class of $\mathbb{N}_k$. Then $\mathcal{U} = \cup\{\mathcal{U}_j(k) : j \in \mathbb{N}\}$.

Proof: The hypotheses of R10.3.7 are satisfied and so $\mathcal{U} = \vee\{\mathcal{U}_j(k) : j \in \mathbb{N}\}$. By R10.1.2iii $\mathcal{Z}(E_j(k)) \subseteq \mathcal{Z}(E_{j+1}(k))$ for all j in $\mathbb{N}$. By R9.1.8 and R1.5 $\mathcal{U}_j(k) \subseteq \mathcal{U}_{j+1}(k)$ for all j. Since $\{\mathcal{U}_j(k) : j \in \mathbb{N}\}$ is a chain, $\vee\{\mathcal{U}_j(k) : j \in \mathbb{N}\} = \cup\{\mathcal{U}_j(k) : j \in \mathbb{N}\}$.

Proposition R10.3.9 Let $a, b \in \mathbb{N}$. Then $\mathbb{N}_{(a,b)} = \mathbb{N}_a \wedge \mathbb{N}_b$.

Proof: Let $c = (a, b)$ so that $c^j = (a^j, b^j)$ for any $j \in \mathbb{N}$. Let (Y, f) be a T_2 compactification of $\mathbb{N}$ with $(Y, f) \leq \mathbb{N}_a$ and $(Y, f) \leq \mathbb{N}_b$. Let $\mathcal{V}$ be the totally bounded uniformity for $\mathbb{N}$ with $\Psi_0(\mathcal{V}) = [(Y, f)]$. By R10.3.8, R1.5, and R10.3.1iv it is sufficient to show that $\mathcal{V} \subseteq \cup\{\mathcal{U}_j(c) : j \in \mathbb{N}\}$. Let $V \in \mathcal{V}$ and pick $W \in \mathcal{V}$ with $W = W^{-1}$ and $W \circ W \subseteq V$. The assumptions about Y along with R1.5 and R10.3.8 imply that W is in both $\cup\{\mathcal{U}_j(a) : j \in \mathbb{N}\}$ and $\cup\{\mathcal{U}_j(b) : j \in \mathbb{N}\}$. Since both $\{\mathcal{U}_j(a) : j \in \mathbb{N}\}$ and $\{\mathcal{U}_j(b) : j \in \mathbb{N}\}$ are chains, there is $l \in \mathbb{N}$ such that $W \in \mathcal{U}_l(a) \cap \mathcal{U}_l(b)$. Then there is $U_1 \in \mathcal{U}_m$ with $U_1 \cap E_l(a) \subseteq W$ and $U_1 \cap E_l(b) \subseteq W$. Because $\mathcal{U}_m$ is the uniformity associated with the one-point compactification of $\mathbb{N}$, there is B such that $B \times B \subseteq U_1$ and $\mathbb{N} - B$ is finite. Let $U = B \times B \cup \{(x,x) : x \in \mathbb{N} - B\}$. $U \in \mathcal{U}_m$ and so the proof can be completed by showing $U \cap E_l(c) \subseteq V$. Let $(x,y) \in U \cap E_l(c)$. Without loss of generality, assume $x > y$ so that $x, y \in B$ and $x - y = tc^l$ for some t in $\mathbb{N}$. Since c^l is the GCD of a^l and b^l, there are infinitely many integer solutions of $c^l = \alpha a^l + \beta b^l$ with $\alpha < 0$ and $\beta > 0$. Thus, since $\mathbb{N} - B$ is finite, we can find one such solution with $z = x - \alpha t a^l = y + \beta t b^l$ in B. Then both (x,z) and (z,y) are in W so that $(x,y) \in W \circ W \subseteq V$.

Corollary R10.3.10 Let $k \in \mathbb{N}$. Then $\mathbb{N}_k$ is not equivalent to $\mathbb{N}_\infty$.

Proof: If $k = 1$, the conclusion follows because $\mathbb{N}_1$ is the one-point compactification and $\mathbb{N}_\infty$ is not a finite-point compactification. If $k > 1$, let p be a prime with $(k,p) = 1$ and suppose $\mathbb{N}_k$ is equivalent to $\mathbb{N}_\infty$. Then $\mathbb{N}_k \geq \mathbb{N}_p$ so that $\mathbb{N}_k \wedge \mathbb{N}_p = \mathbb{N}_p$, which contradicts R10.3.9.

Albert J. Klein 2005
http://www.susanjkleinart.com/compactification/

References

1. This Website, P3: Normal Bases
2. This Website, R1: Existence of the Supremum via Uniform Space Theory
3. This Website, R3: Representation of Suprema
4. This Website, R5: Finite-Point Compactifications
5. This Website, R6: Suprema of Two-point Compactifications
6. This Website, R9: Directed Sets of Normal Bases

Added Comment 2019

In R26.Add.19 it is shown that, for $k, l \in \mathbb{N}$ with $k, l \geq 2$, $\mathbb{N}_k$ and $\mathbb{N}_l$ are homeomorphic topological spaces.

Added Reference

7. This Website, R26: The Remnant Rings Are Homeomorphic

Added 2021

$\mathbb{N}_\infty$ is the compactification generated by $\mathcal{Z}_\infty$, which is $\cup\{\mathcal{Z}_k : k \in \mathbb{N}\}$. By R10.3.4 $\mathbb{N}_\infty$ is equivalent to $\vee\{\mathbb{N}_p : p \text{ is prime}\}$, a fact that might suggest the question of whether $\cup\{\mathcal{Z}_p : p \text{ is prime}\}$ is equal to $\mathcal{Z}_\infty$. The next example shows that it is only a proper subset.

Example R10.Add.1 $C_1^1(6) \in \mathcal{Z}(E_1(6))$ since the set is associated with $\{1,2,3,4,5\}$. Thus $C_1^1(6)$ is in $\mathcal{Z}_6 \subseteq \mathcal{Z}_\infty$. If $C_1^1(6)$ is in $\mathcal{Z}_2$, then for some n it is in $\mathcal{Z}(E_n(2))$ and as such is associated with some $\Delta \subseteq \{1,2,\ldots,2^n\}$. By definition $i \in \Delta$ implies $C_1^1(6) \cap C_n^i(2)$ is finite and $i \notin \Delta$ implies $(\mathbb{N} - C_1^1(6)) \cap C_n^i(2)$ is finite. For every $k \in \mathbb{N}$, $1+3k \cdot 2^n \in C_1^1(6) \cap C_n^1(2)$ and so $1 \notin \Delta$. For every $j \in \mathbb{N}$ with $(j,3)=1$, $1+j\cdot 2^n \in (\mathbb{N} - C_1^1(6)) \cap C_n^1(2)$, which contradicts $1 \notin \Delta$. Now suppose $C_1^1(6)$ is in $\mathcal{Z}_p$ for some odd prime p. Then for some n it is in $\mathcal{Z}(E_n(p))$ and as such is associated with some $\Delta \subseteq \{1,2,\ldots,p^n\}$. By definition $i \in \Delta$ implies $C_1^1(6) \cap C_n^i(p)$ is finite and $i \notin \Delta$ implies $(\mathbb{N} - C_1^1(6)) \cap C_n^i(p)$ is finite. For every $k \in \mathbb{N}$ $1+6k\cdot p^n$ is in $C_1^1(6) \cap C_n^1(p)$ and so $1 \notin \Delta$. For every k odd $1+kp^n$ is in $(\mathbb{N} - C_1^1(6)) \cap C_n^1(p)$, which contradicts $1 \notin \Delta$. Thus $C_1^1(6) \notin \cup\{\mathcal{Z}_p : p \text{ is prime}\}$.

It is easy to check that $C_1^1(6) = C_1^1(2) \cap C_1^1(3)$ and so the example also shows that $\cup\{\mathcal{Z}_p : p \text{ is prime}\}$ is not a normal basis for $\mathbb{N}$ with the discrete topology.