The Magill-Glasenapp Theorem

Among the results in [3] is the following:

Theorem [Magill and Glasenapp] Let (X, τ) be $T_{3\frac{1}{2}}$ and zero-dimensional. The supremum of any non-empty collection of zero-dimensional T_2 compactifications of X is also zero-dimensional.

The keys to the proof in [3] are two observations. First, the clopen subsets of a 0-dimensional compactification induce a Boolean ring of clopen subsets of the base space. Secondly, the induced Boolean ring can be used to construct a compactification equivalent to the original.

This section will give a uniformity-based proof the theorem stated above. The key, the construction of $\mathcal{M}(\mathcal{U})$, is largely derived from [3]. Notation, definitions, and facts from [4] will be used freely.

Definition R11.1 [2] Let X be a set and let E be an equivalence relation on X. $\mathcal{U}_E = \{U \subseteq X \times X : E \subseteq U\}$.

Since $E \circ E = E$, it is clear that $\mathcal{U}_E$ is a uniformity for X. The uniform space $(X, \mathcal{U}_E)$ will be called the uniform space generated by E.

Definition R11.2 [2] Let $(X, \mathcal{U})$ be a uniform space. $(X, \mathcal{U})$ is an e-uniform space provided $(X, \mathcal{U})$ is the supremum of some non-empty family of uniform spaces generated by equivalence relations.

Given an e-uniform space $(X, \mathcal{U})$, $\mathcal{U}$ will be called an e-uniformity. The following proposition is immediate from the definition.

Proposition R11.3 (Levine [2]) The supremum of a family of e-uniformities for X is also an e-uniformity for X.

Proposition R11.4 (Levine [2]) Let $(X, \mathcal{U})$ be an e-uniform space. Then $(X, \tau(\mathcal{U}))$ is 0-dimensional.

Proof: For an equivalence relation E on X, the equivalence classes are clopen in $(X, \tau(\mathcal{U}_E))$. Such clopen sets form a basis for the topology of a supremum.

Definition R11.5 Let X be a set and let $A \subseteq X$. $E(A) = A \times A \cup (X - A) \times (X - A)$.

Lemma R11.6 Let X be a set and let A_1 and A_2 be subsets of X. Then $E(A_1) \cap E(A_2)$ is a subset of both $E(A_1 \cup A_2)$ and $E(A_1 \cap A_2)$.

Proof: Routine.

Definition R11.7 Let $(X, \mathcal{U})$ be a uniform space.
$\mathcal{R}(\mathcal{U}) = \{A \subseteq X : E(A) \in \mathcal{U}\}$.

Lemma R11.8 Let $(X, \mathcal{U})$ be a uniform space. Then

i) If $A \in \mathcal{R}(\mathcal{U})$, then A is clopen relative to $(X, \tau(\mathcal{U}))$.

ii) $\emptyset$ and X are in $\mathcal{R}(\mathcal{U})$.

iii) If $A \in \mathcal{R}(\mathcal{U})$, then $X - A \in \mathcal{R}(\mathcal{U})$.

iv) $\mathcal{R}(\mathcal{U})$ is closed under finite unions and finite intersections.

Proof: i) holds since $E(A)[x] = A$ if $x \in A$ and $E(A)[x] = X - A$ if $x \in X - A$. ii) is clear and iii) holds since $E(X - A) = E(A)$. iv) follows from R11.6.

Next we recall the definition of algebraic operations on $\mathcal{P}(X)$, the power set of X.

Definition R11.9 Let X be a set. For $A, B \in \mathcal{P}(X)$, $A + B = (A \cup B) - (A \cap B)$ and $A \cdot B = AB = A \cap B$.

As is well-known, $(\mathcal{P}(X), +, \cdot)$ is a commutative ring with multiplicative identity X. The additive identity is $\emptyset$, the negative of A is A since $A + A = \emptyset$, and the ring is Boolean since $A^2 = A$. Because $-A = A$, algebraic subtraction need not be used, although the algebraic difference $X - A = X + (-A) = X + A$ is the complement of A in X so that $X - A$ unambiguously denotes the complement of A.

Proposition R11.10 Let $(X,\mathcal{U})$ be a uniform space. Then $(\mathcal{R}(\mathcal{U}), +, \cdot)$ is a commutative ring with unity.

Proof: By R11.8ii and iv, $\mathcal{R}(\mathcal{U})$ is closed under $\cdot$ and contains the multiplicative identity. Since $A + B = (A \cup B) \cap (X - (A \cap B))$, R11.8iii and iv show $\mathcal{R}(\mathcal{U})$ is also closed under $+$.

Definition R11.11 Let $(X,\mathcal{U})$ be a uniform space. $\mathcal{M}(\mathcal{U})$ denotes the set of maximal ideals in $\mathcal{R}(\mathcal{U})$. For $x \in X$, $M_x = \{A \in \mathcal{R}(\mathcal{U}) : x \notin A\}$.

Lemma R11.12 Let $(X,\mathcal{U})$ be a uniform space. Then

i)If $x \in X$, then $M_x \in \mathcal{M}(\mathcal{U})$.

ii) If $M \in \mathcal{M}(\mathcal{U})$ and $A \in \mathcal{R}(\mathcal{U})$, then either $A \in M$ or $X - A \in M$.

Proof: Clearly $\emptyset \in M_x$ and, since $A + B \subseteq A \cup B$, M_x is closed under addition. For $A \in M_x$ and $C \in \mathcal{R}(\mathcal{U})$, since $CA \subseteq A$, $CA \in M_x$. Thus M_x is an ideal. Now let I be an ideal of $\mathcal{R}(\mathcal{U})$ with $M_x \subseteq I$. Suppose $B \in I$ but $B \notin M_x$. Then $x \in B$ and so $X - B \in M_x$. Thus $B + (X - B) = X \in I$ and so $I = \mathcal{R}(\mathcal{U})$. For the second assertion, let $M \in \mathcal{M}(\mathcal{U})$ and $A \in \mathcal{R}(\mathcal{U})$ be given. Note that $A(X - A) = 0$ in $\mathcal{R}(\mathcal{U})$. In the field $\mathcal{R}(\mathcal{U})/M$, $(A + M)((X - A) + M) = 0 + M$, which implies $A \in M$ or $X - A \in M$.

Definition R11.13 Let $(X,\mathcal{U})$ be a uniform space, and let $A \in \mathcal{R}(\mathcal{U})$. $H_A = \{M \in \mathcal{M}(\mathcal{U}) : A \in M\}$.

Lemma R11.14 Let $(X,\mathcal{U})$ be a uniform space. Then

i) For $A \in \mathcal{R}(\mathcal{U})$, $H_{X-A} = \mathcal{M}(\mathcal{U}) - H_A$.

ii) For $A, B \in \mathcal{R}(\mathcal{U})$, $H_{AB} = H_A \cup H_B$.

iii) There is a topology σ on $\mathcal{M}(\mathcal{U})$ with $\{H_A : A \in \mathcal{R}(\mathcal{U})\}$ as a base for the closed sets.

iv) For each $A \in \mathcal{R}(\mathcal{U})$, H_A is clopen in σ.

v) $\{H_A : A \in \mathcal{R}(\mathcal{U})\}$ is also an open basis for σ.

Proof: For i), note that $H_A \cup H_{X-A} = \mathcal{M}(\mathcal{U})$ by R11.12ii, and $H_A \cap H_{X-A} = \emptyset$ since $A + (X - A)$ equals the multiplicative identity, which can't be in a maximal ideal. The first assertion is immediate from these equations. For ii), let $A, B \in \mathcal{R}(\mathcal{U})$. If $M \in H_A \cup H_B$, then A or B is in M so that AB is in the ideal M, i.e. $M \in H_{AB}$. Now let $M \in H_{AB}$ and suppose $M \notin H_A$. By R11.12.2 the complement of A, i.e. $1 + A$, must be in M, as is $B(1 + A) + AB = B + AB + AB = B$. Thus $M \in H_B$. For iii), first note that $H_X = \emptyset$ and $H_\emptyset = \mathcal{M}(\mathcal{U})$. By ii) $\{H_A : A \in \mathcal{R}(\mathcal{U})\}$ is closed under finite unions. These facts imply that there is a unique topology on $\mathcal{M}(\mathcal{U})$ with $\{H_A : A \in \mathcal{R}(\mathcal{U})\}$ as a base for its closed sets. Part iv) is now immediate from i). By part i) and R11.8iii $\{H_A : A \in \mathcal{R}(\mathcal{U})\} = \{\mathcal{M}(\mathcal{U}) - H_A : A \in \mathcal{R}(\mathcal{U})\}$. Since the complements of a base for the closed sets must form an open basis, v) holds.

For the remainder of this section, $\mathcal{M}(\mathcal{U})$ will always be assumed to have the zero-dimensional topology witn clopen basis $\{H_A : A \in \mathcal{R}(\mathcal{U})\}$. The next proposition, a version of a result which is credited in [3] to Gillman and Jerison [1; p 111,7M], implies by P2.4

that $\mathcal{M}(\mathcal{U})$ has a unique uniformity, the neighborhoods of the diagonal.

Proposition R11.15 Let $(X,\mathcal{U})$ be a uniform space. Then $\mathcal{M}(\mathcal{U})$ is compact and T_2.

Proof: Let $\{H_A : A \in \mathcal{S}\}$ be a non-empty family with the finite intersection property. For compactness it is sufficient to show that $\cap\{H_A : A \in \mathcal{S}\} \neq \emptyset$. Let $I = \{\Sigma_{i=1}^{n} B_i A_i : B_i \in \mathcal{R}(\mathcal{U})$ and $A_i \in \mathcal{S}\}$. It is easily checked that I is an ideal in $\mathcal{R}(\mathcal{U})$ and $\mathcal{S} \subseteq I$. If X is in I, say $X = \Sigma_{i=1}^{n} B_i A_i$, apply the finite intersection property to select $N \in \cap\{H_{A_i} : i = 1, \ldots, n\}$. Then each A_i is in N and so is $\Sigma_{i=1}^{n} B_i A_i = X$, a contradiction since every maximal ideal is proper. Thus $X \notin I$ and so there is a maximal ideal M with $I \subseteq M$. Since $\mathcal{S} \subseteq I$, $A \in M$ for all A in $\mathcal{S}$, i.e., $M \in \cap\{H_A : A \in \mathcal{S}\}$. For T_2, let $M, N \in \mathcal{M}(\mathcal{U})$ with $M \neq N$. Pick $A \in M$ with $A \notin N$. Then M is in the clopen H_A and N is not.

Definition R11.16 Let $(X,\mathcal{U})$ be a uniform space. $\mathcal{U}(\mathcal{R}(\mathcal{U})) = \vee\{\mathcal{U}_{E(A)} : A \in \mathcal{R}(\mathcal{U})\}$.

Lemma R11.17 Let $(X,\mathcal{U})$ be a uniform space. Then

i) $\mathcal{U}(\mathcal{R}(\mathcal{U}))$ is a totally bounded e-uniformity for X.

ii) $\mathcal{U}(\mathcal{R}(\mathcal{U})) \subseteq \mathcal{U}$.

iii) $\mathcal{R}(\mathcal{U}(\mathcal{R}(\mathcal{U}))) = \mathcal{R}(\mathcal{U})$.

Proof: Since each $\mathcal{U}_{E(A)}$ is totally bounded and the supremum of totally bounded uniformities is also totally bounded by P2.13, i) is clear from the definition of an e-uniformity. Part ii) holds since each $\mathcal{U}_{E(A)}$ is contained in $\mathcal{U}$ by definition of $\mathcal{R}(\mathcal{U})$. Part iii) is easy to check.

Definition R11.18 Let $(X,\mathcal{U})$ be a uniform space. $\mu : X \to \mathcal{M}(\mathcal{U})$ is defined by $\mu(x) = M_x$.

Lemma R11.19 Let $(X,\mathcal{U})$ be a uniform space. Then

i) If $A \in \mathcal{R}(\mathcal{U})$, then $\mu^{-1}[H_A] = X - A$.

ii) $\mu : (X,\mathcal{U}(\mathcal{R}(\mathcal{U}))) \to \mathcal{M}(\mathcal{U})$ is uniformly continuous.

iii) $\mu[X]$ is dense in $\mathcal{M}(\mathcal{U})$.

Proof: Let $A \in \mathcal{R}(\mathcal{U})$. Following the definitions, one easily sees that $\mu^{-1}[H_A] = \{x \in X : A \in M_x\} = X - A$, and so i) holds. For ii), let V be a neighborhood of the diagonal of $\mathcal{M}(\mathcal{U})$. Using compactness and the basis for $\mathcal{M}(\mathcal{U})$, there exist $A_1, \ldots, A_j$ in $\mathcal{R}(\mathcal{U})$ such that $\cup_{i=1}^{j} H_{A_i} \times H_{A_i}$ is an open neighborhood of the diagonal and is contained in V. By i) $\cup_{i=1}^{j}(X - A_i) \times (X - A_i) \subseteq (\mu \times \mu)^{-1}[V]$. Since $H_{A_1}, \ldots, H_{A_j}$ cover $\mathcal{M}(\mathcal{U})$, $\cup_{i=1}^{j}(X - A_i) = X$. By definition $\cap_{i=1}^{j} E(A_i) \in \mathcal{U}(\mathcal{R}(\mathcal{U}))$ and it is easy to check that $\cap_{i=1}^{j} E(A_i) \subseteq \cup_{i=1}^{j}(X - A_i) \times (X - A_i)$ so that $(\mu \times \mu)^{-1}[V]$ is in $\mathcal{U}(\mathcal{R}(\mathcal{U}))$, as required for uniform continuity. For iii) consider a basic open set $H_A \neq \emptyset$ with $A \in \mathcal{R}(\mathcal{U})$. Since $H_X = \emptyset$, $A \neq X$ so that $\mu[X] \cap H_A = \{M_x : x \notin A\}$ is non-empty.

Lemma R11.20 Let $(X,\mathcal{U})$ be a uniform space. Then

i) μ is one-to-one if and only if $\mathcal{U}(\mathcal{R}(\mathcal{U}))$ is separated.

ii) If $\mathcal{U}(\mathcal{R}(\mathcal{U}))$ is separated, then $\mu : (X,\mathcal{U}(\mathcal{R}(\mathcal{U}))) \to \mathcal{M}(\mathcal{U})$ is a uniform embedding onto $\mu[X]$.

Proof: First assume μ is one-to-one and let a, b be in X with $a \neq b$. Since $M_a \neq M_b$, there is $A \in \mathcal{R}(\mathcal{U})$ with $a \in A$ and $b \notin A$. Then $(a, b) \notin E(A)$ so that $(a, b) \notin \cap\{U : U \in \mathcal{U}(\mathcal{R}(\mathcal{U}))\}$, i.e, $\mathcal{U}(\mathcal{R}(\mathcal{U}))$ is separated. For the converse, let a, b be in X

with $a \neq b$. Since $(a,b) \notin \cap\{U : U \in \mathcal{U}(\mathcal{R}(\mathcal{U}))\}$, there exist $A_1, \ldots, A_j$ in $\mathcal{R}(\mathcal{U})$ such that $(a,b) \notin \cap_{i=1}^{j} E(A_i)$. Pick i with $(a,b) \notin E(A_i)$. Then A_i is in one of M_a, M_b and not the other, i.e., $M_a \neq M_b$. Thus μ is one-to-one. For ii), because of the first part and R11.19ii, it is only necessary to show that $\mu \times \mu[U]$ is an entourage in the subspace uniformity on $\mu[X]$ for any $U \in \mathcal{U}(\mathcal{R}(\mathcal{U}))$. Since $\mu \times \mu[E(A)] = (\mu[X] \times \mu[X]) \cap ((H_{X-A} \times H_{X-A}) \cup (H_A \times H_A))$ and $\mu \times \mu$ is one-to-one, this follows easily from the definition of $\mathcal{U}(\mathcal{R}(\mathcal{U}))$.

Definition R11.21 Let $(X,\mathcal{U})$ be a uniform space. $\mathcal{R}(\mathcal{U})$ generates $\mathcal{U}$ if and only if $\mathcal{U} = \mathcal{U}(\mathcal{R}(\mathcal{U}))$.

Lemma R11.22 Let $(X,\mathcal{U})$ be a uniform space. Then $\mathcal{R}(\mathcal{U})$ generates $\mathcal{U}$ if and only if $\mathcal{U}$ is a totally bounded e-uniformity.

Proof: The necessity is immediate from R11.17i. Now assume $\mathcal{U} = \vee\{\mathcal{U}_E : E \in \mathcal{E}\}$ is totally bounded, where $\mathcal{E}$ is a non-empty family of equivalence relations on X. For $E \in \mathcal{E}$, $\mathcal{U}_E$ is also totally bounded and so E has finitely many distinct equivalence classes $C_1, \ldots, C_j$. Since $E \subseteq E(C_i)$, each C_i is in $\mathcal{R}(\mathcal{U})$. It is easy to check that $E = \cap_{i=1}^{j} E(C_i)$ and so $E \in \mathcal{U}(\mathcal{R}(\mathcal{U}))$.The conclusion follows.

In the next few results some notation from [5] will be used. Given a $T_{3\frac{1}{2}}$ space (X,τ), $\mathcal{TB}(X)$ will denote the set of totally bounded uniformities that generate τ and, for $\mathcal{U} \in \mathcal{TB}(X)$, $\Psi_0(\mathcal{U})$ is the class of T_2 compactifications determined by $\mathcal{U}$.

Proposition R11.23 Let $(X,\mathcal{U})$ be a uniform space. If $\mathcal{U}$ is separated and $\mathcal{R}(\mathcal{U})$ generates $\mathcal{U}$, then

i) $(\mathcal{M}(\mathcal{U}), \mu)$ is a T_2 compactification of $(X, \tau(\mathcal{U}))$.

ii) $\Psi_0(\mathcal{U}) = [(\mathcal{M}(\mathcal{U}), \mu)]$.

Proof: With the given assumptions, part i) is immediate from R11.15, R11.19iii, and R11.20ii. Let $\mathcal{V}$ in $\mathcal{TB}(X)$ be such that $\Psi_0(\mathcal{V}) = [(\mathcal{M}(\mathcal{U}), \mu)]$. By R1.6a, $\mathcal{V}$ makes μ a uniform embedding, where $\mu[X]$ has the subspace uniformity from the unique uniformity of $\mathcal{M}(\mathcal{U})$. $\mathcal{U}$ also makes μ a uniform embedding so that $\mathcal{U} = \mathcal{V}$.

The final two results could be stated without assuming that X is zero-dimensional, since that is implied by the existence of a zero-dimensional T_2 compactification of X. As usual, a slightly loose set-like expression is used to indicate a supremum of compactifications.

Proposition R11.24 Let (X,τ) be a zero-dimensional $T_{3\frac{1}{2}}$ space, let (Y,f) be a zero-dimensional T_2 compactification of X, and let $\mathcal{U} \in \mathcal{TB}(X)$ be the uniformity with $\Psi_0(\mathcal{U}) = [(Y,f)]$. Then $\mathcal{R}(\mathcal{U})$ generates $\mathcal{U}$ and $(\mathcal{M}(\mathcal{U}), \mu)$ is equivalent to (Y,f).

Proof: First note, for any C clopen in Y, $A = f^{-1}[C]$ is clopen in X. In addition, $(C \times C) \cup ((Y-C) \times (Y-C))$ is in the unique uniformity for Y and $E(A) = (f \times f)^{-1}[(C \times C) \cup ((Y-C) \times (Y-C))]$. Since $f : (X,\mathcal{U}) \to Y$ is uniformly continuous, $E(A) \in \mathcal{U}$ and $A \in \mathcal{R}(\mathcal{U})$. Now let $U \in \mathcal{U}$. There is V, a neighborhood of the diagonal in Y, such that $(f \times f)[U] = (f[X] \times f[X]) \cap V$. By the compactness and zero-dimensionality of Y, there exist clopen sets $C_1, \ldots, C_n$ of Y such that $Y = \cup_{i=1}^{n} C_i$ and $\cup_{i=1}^{n} C_i \times C_i \subseteq V$. Let $A_i = f^{-1}[C_i]$. It is easy to check that $\cap_{i=1}^{n} E(A_i) \subseteq U$, which implies $U \in \mathcal{U}(\mathcal{R}(\mathcal{U}))$. Thus $\mathcal{R}(\mathcal{U})$ generates $\mathcal{U}$. By R11.23ii $\Psi_0(\mathcal{U}) = [(\mathcal{M}(\mathcal{U}), \mu)]$ so that $(\mathcal{M}(\mathcal{U}), \mu)$ is equivalent to (Y,f).

Theorem R11.25 [Magill-Glasenapp] Let (X,τ) be a zero-dimensional $T_{3\frac{1}{2}}$ space. Let $\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$ be a non-empty family of zero-dimensional T_2 compactifications of

X and let $(Y, f) = \vee\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$. Then (Y, f) is a zero-dimensional T_2 compactification of X.

Proof: For each α, let $\mathcal{U}_\alpha$ be the uniformity in $\mathcal{TB}(X)$ with $\Psi_0(\mathcal{U}_\alpha) = [(Y_\alpha, f_\alpha)]$. Let $\mathcal{U} = \vee\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. By R1.5 $\Psi_0(\mathcal{U}) = [(Y, f)]$. By R11.24 and R11.22 each $\mathcal{U}_\alpha$ is a totally bounded e-uniformity. By P2.13 and R11.3 $\mathcal{U}$ is also a totally bounded e-uniformity so that, by R11.22 again, $\mathcal{R}(\mathcal{U})$ generates $\mathcal{U}$. By R11.23 $[(Y, f)] = [(\mathcal{M}(\mathcal{U}), \mu)]$. Since $\mathcal{M}(\mathcal{U})$ is zero-dimensional, so is Y.

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Added Comments 2009

Herrlich and Strecker [6, p. 315] state that MacGill and Glasenapp rediscovered a result known earlier. Apparently it was originally due to Banaschewski [7] in 1955.

Additional References

An asterisk indicates a reference not seen by me.

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