Extension of Arithmetic Operations

This section shows that addition and multiplication can be extended continuously to $\mathbb{N}_\infty$ and $\mathbb{N}_k$, which are described in [6]. Some algebraic properties are derived and the remnant rings are constructed. Notation, definitions and results from [4], [5], and [6] will be used as needed.

Products of Uniform Spaces

Let $\{(X_\alpha,\mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty family of uniformities with π_α denoting the standard projection from $\prod_{\alpha\in\Delta} X_\alpha$ onto X_α. As usual each X_α is assumed to be non-empty. The product uniformity $\prod_{\alpha\in\Delta}\mathcal{U}_\alpha$ is the smallest uniformity on $\prod_{\alpha\in\Delta} X_\alpha$ making all the projections uniformly continuous. In the notation of [4] that can be written as:

$$\prod_{\alpha\in\Delta}\mathcal{U}_\alpha = \vee\{\pi_\alpha^{-1}\mathcal{U}_\alpha : \alpha\in\Delta\}$$

The following well-known facts are recorded for the convenience of the reader.

Proposition R12.1.1 Let $\{(X_\alpha,\mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty family of uniformities. Then $\tau(\prod_{\alpha\in\Delta}\mathcal{U}_\alpha) = \prod_{\alpha\in\Delta}\tau(\mathcal{U}_\alpha)$.

Proof: By R7.2.5, for each α, $\tau(\pi_\alpha^{-1}\mathcal{U}_\alpha)$ is $\pi_\alpha^{-1}\tau(\mathcal{U}_\alpha)$, i.e., the smallest topology making π_α continuous. By P2.14 $\tau(\prod_{\alpha\in\Delta}\mathcal{U}_\alpha) = \vee\{\pi_\alpha^{-1}\tau(\mathcal{U}_\alpha) : \alpha\in\Delta\}$, i.e., the smallest topology making all the projections continuous, i.e., the product topology.

Proposition R12.1.2 Let $\{(X_\alpha,\mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty family of totally bounded uniformities. Then $\prod_{\alpha\in\Delta}\mathcal{U}_\alpha$ is also totally bounded.

Proof: By R7.2.6 each $\pi_\alpha^{-1}\mathcal{U}_\alpha$ is totally bounded and so the conclusion is immediate from P2.13.

The next three results will all assume the following: $\{(X_\alpha,\mathcal{U}_\alpha) : \alpha \in \Delta\}$ will denote a non-empty family of totally bounded uniformities, with (Y_α, f_α) a T_2 compactification of X_α such that $\Psi_0(\mathcal{U}_\alpha) = [(Y_\alpha, f_\alpha)]$. (See [3] for the definition of Ψ_0.) The product map from $\prod_{\alpha\in\Delta} X_\alpha$ to $\prod_{\alpha\in\Delta} Y_\alpha$ will be denoted P, i.e., $P(p)(\alpha) = f_\alpha(p(\alpha))$. π_α and ρ_α will denote the projections onto X_α and Y_α respectively.

Lemma R12.1.3 P is a uniform embedding.

Proof: For each α, $\rho_\alpha\circ P = f_\alpha\circ\pi_\alpha$, which is uniformly continuous. Thus P is uniformly continuous. Since each f_α is one-to-one, P is also one-to-one. Finally, given a basic entourage $\cap_{i=1}^n(\pi_{\alpha_i}\times\pi_{\alpha_i})^{-1}[U_{\alpha_i}]$ in the domain of P, let V_{α_i} be in the unique uniformity for Y_{α_i} with $(f_{\alpha_i}\times f_{\alpha_i})[U_{\alpha_i}] = V_{\alpha_i}\cap(f_{\alpha_i}[X_{\alpha_i}]\times f_{\alpha_i}[X_{\alpha_i}])$. It can be shown that

$$(P\times P)[\cap_{i=1}^n(\pi_{\alpha_i}\times\pi_{\alpha_i})^{-1}[U_{\alpha_i}]] = [\cap_{i=1}^n(\rho_{\alpha_i}\times\rho_{\alpha_i})^{-1}[V_{\alpha_i}]]\cap(P[\prod_{\alpha\in\Delta}X_\alpha]\times P[\prod_{\alpha\in\Delta}X_\alpha])$$

The set on the right side of that equation is an entourage in the subspace uniformity on $P[\prod_{\alpha\in\Delta} X_\alpha]$, as required to show P^{-1} uniformly continuous.

Proposition R12.1.4 $(\prod_{\alpha\in\Delta} Y_\alpha, P)$ is a T_2 compactification of $\prod_{\alpha\in\Delta} X_\alpha$.

Proof: As the product of compact T_2 spaces, $\prod_{\alpha\in\Delta} Y_\alpha$ is also compact and T_2. By the previous lemma P is an embedding. It remains to show that the image of P is dense.

A non-empty open subset of $\prod_{\alpha\in\Delta} Y_\alpha$ must contain some $\cap_{i=1}^n(\rho_{\alpha_i}\times\rho_{\alpha_i})^{-1}[V_{\alpha_i}][q]$ where $q \in \prod_{\alpha\in\Delta} Y_\alpha$ and V_{α_i} is in the unique uniformity for Y_{α_i}. Define $p \in \prod_{\alpha\in\Delta} X_\alpha$ by $p(\alpha_i) = x_i$, where $f_{\alpha_i}(x_i) \in V_{\alpha_i}[q(\alpha_i)]$, and $p(\alpha)$ is chosen arbitrarily in X_α if $\alpha \neq \alpha_i$. It can be easily verified that $P(p)$ is in $\cap_{i=1}^n(\rho_{\alpha_i}\times\rho_{\alpha_i})^{-1}[V_{\alpha_i}][q]$.

Proposition R12.1.5 $\Psi_0(\prod_{\alpha\in\Delta}\mathcal{U}_\alpha) = [(\prod_{\alpha\in\Delta} Y_\alpha, P)]$.

Proof:As noted in the proof of R1.4, the uniformity associated with $(\prod_{\alpha\in\Delta} Y_\alpha, P)$ is the one making P a uniform embedding, i.e., by R12.1.3, $\prod_{\alpha\in\Delta}\mathcal{U}_\alpha$.

The final three results of this subsection are stated in the form needed for the sequel.

Lemma R12.1.6 Let $\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ be a non-empty collection of uniformities for the set X, let $\mathcal{U} = \vee\{\mathcal{U}_\alpha : \alpha \in \Delta\}$, and let $(Z, \mathcal{W})$ be a uniform space. Suppose $f : Z \to X$. Then $f : (Z, \mathcal{W}) \to (X, \mathcal{U})$ is uniformly continuous if and only if $f : (Z, \mathcal{W}) \to (X, \mathcal{U}_\alpha)$ is uniformly continuous for every α.

Proof: The necessity of the condition is clear since $\mathcal{U}_\alpha \subseteq \mathcal{U}$ for all α. For the sufficiency, note that basic entourages for $\mathcal{U}$ are of the form $\cap_{i=1}^n U_{\alpha_i}$ where $U_{\alpha_i} \in \mathcal{U}_{\alpha_i}$. By hypothesis $(f\times f)^{-1}[\cap_{i=1}^n U_{\alpha_i}] = \cap_{i=1}^n (f\times f)^{-1}[U_{\alpha_i}] \in \mathcal{W}$.

Lemma R12.1.7 Let $\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ be a non-empty collection of uniformities for the set X. Let $\mathcal{U} = \vee\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. Then $\mathcal{U}\times\mathcal{U} = \vee\{\mathcal{U}_\alpha\times\mathcal{U}_\alpha : \alpha \in \Delta\}$.

Proof: Let π_1 and π_2 denote the projections from $X \times X$ onto X, and let $\mathcal{W} = \vee\{\mathcal{U}_\alpha\times\mathcal{U}_\alpha : \alpha \in \Delta\}$. For every α, $\pi_i : (X \times X, \mathcal{W}) \to (X, \mathcal{U}_\alpha)$ is uniformly continuous and so by the previous lemma $\pi_i : (X \times X, \mathcal{W}) \to (X, \mathcal{U})$ is also uniformly continuous. Thus $\mathcal{U}\times\mathcal{U} \subseteq \mathcal{W}$. Again by R12.1.6, for every α, $\pi_i : (X\times X, \mathcal{U}\times\mathcal{U}) \to (X, \mathcal{U}_\alpha)$ is uniformly continuous so that $\mathcal{U}_\alpha\times\mathcal{U}_\alpha \subseteq \mathcal{U}\times\mathcal{U}$. Thus $\mathcal{W} \subseteq \mathcal{U}\times\mathcal{U}$.

Lemma R12.1.8 Let $\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ be a non-empty collection of uniformities for the set X. Let $\mathcal{U} = \vee\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. Assume $F : (X \times X, \mathcal{U}_\alpha\times\mathcal{U}_\alpha) \to (X, \mathcal{U}_\alpha)$ is uniformly continuous for every α. Then $F : (X\times X, \mathcal{U}\times\mathcal{U}) \to (X, \mathcal{U})$ is uniformly continuous.

Proof: R12.1.7 implies that $\mathcal{U}_\alpha\times\mathcal{U}_\alpha \subseteq \mathcal{U}\times\mathcal{U}$ and so $F : (X\times X, \mathcal{U}\times\mathcal{U}) \to (X, \mathcal{U}_\alpha)$ is uniformly continuous for every α. The conclusion is now immediate from R12.1.6.

Extending Addition and Multiplication

Throughout this subsection, $A, M : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ will be the maps defined by $A(m, n) = m+n$ and $M(m, n) = mn$. As in [6], for j, t in $\mathbb{N}$, $\mathcal{U}_j(t)$ will denote $\mathcal{U}_m \vee \mathcal{U}_{E_j(t)}$. The projections from $\mathbb{N}\times\mathbb{N} \to \mathbb{N}$ will be denoted by π_1 and π_2.

Lemma R12.2.1 Both A and M are uniformly continuous from $(\mathbb{N}\times\mathbb{N}, \mathcal{U}_m\times\mathcal{U}_m)$ to $(\mathbb{N}, \mathcal{U}_m)$.

Proof: Let T be co-finite in $\mathbb{N}$. The set $U_T = \{(j, j) : j \in \mathbb{N}-T\} \cup T \times T$ is a basic entourage for $\mathcal{U}_m$. Let l be the largest element in $\mathbb{N}-T$, let $S = \mathbb{N}-\{1, .., l\}$, and let $U_S = \{(j, j) : j \in \mathbb{N}-S\} \cup S \times S$. Since $U_S \in \mathcal{U}_m$, it is sufficient to show $\cap_{i=1}^2(\pi_i\times\pi_i)^{-1}[U_S] \subseteq (A\times A)^{-1}[U_T]$. For $((a, b), (c, d))$ in the intersection, (a, c) and (b, d) are both in U_S. If $a \neq c$ or $b \neq d$, then a, c are both greater than l or b, d are both greater than l. In either case, $a + b > l$ and $c + d > l$. The other possibility, $a = c$ and $b = d$, yields $a + b = c + d$. Thus $(a + b, c + d)$ is in U_T as required. An almost identical argument shows that M is uniformly continuous.

Lemma R12.2.2 Let $n, k \in \mathbb{N}$. Then A and M are both uniformly continuous from $(\mathbb{N}\times\mathbb{N}, \mathcal{U}_{E_n(k)}\times\mathcal{U}_{E_n(k)})$ to $(\mathbb{N}, \mathcal{U}_{E_n(k)})$.

Proof: Here it is sufficient to show that $\cap_{i=1}^{2}(\pi_i \times \pi_i)^{-1}[E_n(k)] \subseteq (M \times M)^{-1}[E_n(k)]$. For $((a,b),(c,d))$ in the intersection, both (a,c) and (b,d) are in $E_n(k)$ so that $a \equiv c \bmod k^n$ and $b \equiv d \bmod k^n$. Then $ab \equiv cd \bmod k^n$, i.e., (ab, cd) is in $E_n(k)$ as required. An almost identical argument shows that A is uniformly continuous.

Lemma R12.2.3 Let $n, k \in \mathbb{N}$. Then A and M are both uniformly continuous from $(\mathbb{N} \times \mathbb{N}, \mathcal{U}_n(k) \times \mathcal{U}_n(k))$ to $(\mathbb{N}, \mathcal{U}_n(k))$.

Proof: This is immediate from the previous two lemmas and R12.1.8.

Theorem R12.2.4 Let $k \in \mathbb{N}$. Then A and M both can be continuously extended to maps from $\mathbb{N}_k \times \mathbb{N}_k \to \mathbb{N}_k$.

Proof: Let $\mathcal{U}_k = \vee\{\mathcal{U}_n(k) : n \in \mathbb{N}\}$. As in the proof of R10.3.8, $\mathcal{U}_k$ is the totally bounded uniformity associated with $\mathbb{N}_k$, i.e., $\Psi_0(\mathcal{U}_k) = [(\mathbb{N}_k, \iota_k)]$. By R12.1.4 and R12.1.5, $(\mathbb{N}_k \times \mathbb{N}_k, \iota_k \times \iota_k)$ is a T_2 compactification of $\mathbb{N}$ with associated uniformity $\mathcal{U}_k \times \mathcal{U}_k$. By R12.2.3 and R12.1.8, both A and M are uniformly continuous from $(\mathbb{N} \times \mathbb{N}, \mathcal{U}_k \times \mathcal{U}_k)$ to $(\mathbb{N}, \mathcal{U}_k)$. Now R7.1.3 implies that both A and M have continuous extensions as required.

Theorem R12.2.5 Both A and M can be continuously extended to maps from $\mathbb{N}_\infty \times \mathbb{N}_\infty \to \mathbb{N}_\infty$.

Proof: Let $\mathcal{U}_\infty$ be the totally bounded uniformity associated with $\mathbb{N}_\infty$, and for each $k \in \mathbb{N}$ let $\mathcal{U}_k$ be as in the previous proof. By R10.1.8, $\mathbb{N}_\infty = \vee\{\mathbb{N}_k : k \in \mathbb{N}\}$, and so it follows easily from R1.5 that $\mathcal{U}_\infty = \vee\{\mathcal{U}_k : k \in \mathbb{N}\}$. The rest of the argument follows the same pattern as the previous proof.

Note that R7.1.3 and R12.2.3 would also imply that A and M extend continuously to the finite-point compactification associated with $\mathcal{U}_n(k)$ for any $n, k \in \mathbb{N}$. All these extensions are inter-related, as illustrated by the following proposition.

In the following subsections, $\mathcal{Z}_k$ and $\mathcal{Z}_\infty$ will denote the normal bases generating $\mathbb{N}_k$ and $\mathbb{N}_\infty$ respectively, as in [6]. By R10.2.8, given a $\mathcal{Z}_\infty$-ultrafilter $\mathcal{F}$, $\mathcal{F} \cap \mathcal{Z}_k$ is a $\mathcal{Z}_k$-ultrafilter and, since $\mathcal{Z}_\infty = \cup\{\mathcal{Z}_k : k \in \mathbb{N}\}$, $\mathcal{F} = \cup\{\mathcal{F} \cap \mathcal{Z}_k : k \subset \mathbb{N}\}$. For the next proposition, the notation from [6] for point ultrafilters will also be used: given $n \in \mathbb{N}$, ${}^k\hat{n}$ and ${}^\infty\hat{n}$ denote the point filter of n in $\mathbb{N}_k$ and $\mathbb{N}_\infty$ respectively.

Proposition R12.2.6 Let $k \in \mathbb{N}$, let A_k and M_k be the continuous extensions of A and M to $\mathbb{N}_k$, and let A_∞ and M_∞ be the continuous extensions of A and M to $\mathbb{N}_\infty$. Let $\mathcal{F}$ and $\mathcal{G}$ be in $\mathbb{N}_\infty$. Then

i) $A_k(\mathcal{F} \cap \mathcal{Z}_k, \mathcal{G} \cap \mathcal{Z}_k) = A_\infty(\mathcal{F}, \mathcal{G}) \cap \mathcal{Z}_k$.

ii) $M_k(\mathcal{F} \cap \mathcal{Z}_k, \mathcal{G} \cap \mathcal{Z}_k) = M_\infty(\mathcal{F}, \mathcal{G}) \cap \mathcal{Z}_k$.

Proof: As in the proof of R9.1.1iii, the unique continuous surjection from $\mathbb{N}_\infty$ to $\mathbb{N}_k$ that preserves the embeddings is given by $h(\mathcal{F}) = \mathcal{F} \cap \mathcal{Z}_k$. Let P be the continuous product map from $\mathbb{N}_\infty \times \mathbb{N}_\infty \to \mathbb{N}_k \times \mathbb{N}_k$ described by $P(\mathcal{F}, \mathcal{G}) = (h(\mathcal{F}), h(\mathcal{G}))$. Since A_∞ and A_k extend addition, for $m, n \in \mathbb{N}$, $A_\infty({}^\infty\hat{m}, {}^\infty\hat{n}) = {}^\infty\widehat{m+n}$ and $A_k({}^k\hat{m}, {}^k\hat{n}) = {}^k\widehat{m+n}$. Since $h({}^\infty\hat{x}) = {}^k\hat{x}$, $h \circ A_\infty = A_k \circ P$ on the dense subset $\{({}^\infty\hat{x}, {}^\infty\hat{y}) : x, y \in \mathbb{N}\}$ and so $h \circ A_\infty = A_k \circ P$ everywhere on $\mathbb{N}_\infty \times \mathbb{N}_\infty$. This is the functional version of the equation stated as i). The proof for ii) is analogous.

Note that the functional equations $h \circ A_\infty = A_k \circ P$ and $h \circ M_\infty = M_k \circ P$ mean algebraically that h preserves the extended operations, i.e., h is an additive and multiplicative homomorphism.

The following corollary essentially restates the last proposition from a 'bottom up' point-of-view.

Corollary R12.2.7 Let $k \in \mathbb{N}$, let A_k and M_k be the continuous extensions of A and M to $\mathbb{N}_k$, and let A_∞ and M_∞ be the continuous extensions of A and M to $\mathbb{N}_\infty$. Let $\mathcal{F}$ and $\mathcal{G}$ be in $\mathbb{N}_\infty$. Then

i) $A_\infty(\mathcal{F},\mathcal{G}) = \cup\{A_k(\mathcal{F}\cap\mathcal{Z}_k,\mathcal{G}\cap\mathcal{Z}_k) : k \in \mathbb{N}\}$.

ii) $M_\infty(\mathcal{F},\mathcal{G}) = \cup\{M_k(\mathcal{F}\cap\mathcal{Z}_k,\mathcal{G}\cap\mathcal{Z}_k) : k \in \mathbb{N}\}$.

Proof:As noted just before R12.2.6, $A_\infty(\mathcal{F},\mathcal{G}) = \cup\{A_\infty(\mathcal{F},\mathcal{G})\cap\mathcal{Z}_k : k \in \mathbb{N}\}$ and $M_\infty(\mathcal{F},\mathcal{G}) = \cup\{M_\infty(\mathcal{F},\mathcal{G})\cap\mathcal{Z}_k : k \in \mathbb{N}\}$. The corollary follows from R12.2.6 by substitution.

Algebraic Properties

In the remaining subsections, some of the more cumbersome notation from the previous section will be simplified at the risk of introducing some ambiguity. In particular, ordinary algebraic notation will be used for the continuous extensions of addition and multiplication, and $\hat{x}$ will denote a point filter of $x \in \mathbb{N}$, without indicating the underlying compactification in either case. For example, the extension properties at any level will now be written $\widehat{m+n} = \hat{m}+\hat{n}$ and $\widehat{mn} = \hat{m}\hat{n}$, where $m, n \in \mathbb{N}$. As another example, the following propositions apply to $\mathbb{N}_k$ for all $k \in \mathbb{N}$ as well as to $\mathbb{N}_\infty$.

Proposition R12.3.1 Extended addition is commutative and associative.

Proof: These properties follow by straight-forward transference proofs from the corresponding properties in $\mathbb{N}$. One will be done in detail as an illustration: Let $\mathcal{F}$ and $\mathcal{G}$ be ultrafilters, and pick sequences of natural numbers, $\{x_n\}$ and $\{y_n\}$, such that $\hat{x}_n \to \mathcal{F}$ and $\hat{y}_n \to \mathcal{G}$. By continuity of the extended operation, $\hat{x}_n + \hat{y}_n \to \mathcal{F}+\mathcal{G}$ and $\hat{y}_n + \hat{x}_n \to \mathcal{G}+\mathcal{F}$. Since $\hat{x}_n + \hat{y}_n = \widehat{x_n + y_n} = \widehat{y_n + x_n} = \hat{y}_n + \hat{x}_n$ and limits are unique in a Hausdorf space, $\mathcal{F}+\mathcal{G} = \mathcal{G}+\mathcal{F}$.

Proposition R12.3.2 Extended multiplication is commutative, associative, and has an identity. It is also distributive over extended addition.

Proof: These propertities also transfer from $\mathbb{N}$ by arguments simliar to the one illustrated in the previous proof. The multiplicative identity is $\hat{1}$.

Cancellation will be considered in another subsection.

Describing the Operations

This subsection calculates extended addition and multiplication in two cases of interest: first, when a point filter is combined with a non-point ultrafilter and, secondly, when two non-point ultrafilters are combined. Extensive use will be made of the sequence associated with a non-point ultrafilter in $\mathbb{N}_k$, as defined in R10.2.3.

Lemma R12.4.1 Let $\mathcal{F}$ be a non-point ultrafilter in $\mathbb{N}_\infty$, and suppose $\{s_n\}$ ia a sequence in $\mathbb{N}$ such that $\hat{s}_n \to \mathcal{F}$. Then $\{s_n\}$ is unbounded.

Proof: Let $k \in \mathbb{N}$. By R10.2.8 and the proof of R9.1.1iii, the unique continuous surjection from $\mathbb{N}_\infty$ to $\mathbb{N}_k$ that preserves the embeddings is given by $h(\mathcal{G}) = \mathcal{G}\cap\mathcal{Z}_k$. Then $h(\hat{s}_n) = \hat{s}_n \to \mathcal{F}\cap\mathcal{Z}_k$, a non-point ultrafilter. By R10.2.11, $\{s_n\}$ is unbounded.

The next proposition, like the results in the preceding subsection, is stated generically. It should be interpreted as applying to $\mathbb{N}_k$ for any $k \in \mathbb{N}$ and to $\mathbb{N}_\infty$.

Proposition R12.4.2 Let $\mathcal{F}$ and $\mathcal{G}$ be non-point ultrafilters and let $j \in \mathbb{N}$. Then $\hat{j}+\mathcal{F}$, $\hat{j}\mathcal{F}$, $\mathcal{F}+\mathcal{G}$, and $\mathcal{F}\mathcal{G}$ are all non-point ultrafilters.

Proof: Since $\mathbb{N}$ is locally compact and discrete, for any $l \in \mathbb{N}$, $\{\hat{l}\}$ is open in any compactification. Thus a sequence of point filters converging to a point filter must be eventually constant, and so bounded. Let $\{s_n\}$ and $\{t_n\}$ be sequences in $\mathbb{N}$ such that $\hat{s}_n \to \mathcal{F}$ and $\hat{t}_n \to \mathcal{G}$. Since the extended operations are continuous, $\widehat{j+s_n} \to \hat{j}+\mathcal{F}$, $\widehat{js_n} \to \hat{j}\mathcal{F}$,$\widehat{s_n+t_n} \to \mathcal{F}+\mathcal{G}$, and $\widehat{s_n \cdot t_n} \to \mathcal{F}\mathcal{G}$. By R10.2.11 (or R12.4.1) both $\{s_n\}$ and $\{t_n\}$ are unbounded and consequently $\{j+s_n\}$, $\{js_n\}$, $\{s_n+t_n\}$, and $\{s_n \cdot t_n\}$ are also unbounded. The conclusion follows immediately.

Proposition R12.4.3 Let $k, j \in \mathbb{N}$. Let $\mathcal{F}$ be a non-point ultrafilter in $\mathbb{N}_k$ associated with the sequence $\{w_n\}$. Assume $\hat{j}+\mathcal{F}$ and $\hat{j}\mathcal{F}$ are associated with $\{x_n\}$ and $\{y_n\}$ respectively. Then, for every $m \in \mathbb{N}$, $x_m \equiv j + w_m \mod k^m$ and $y_m \equiv jw_m \mod k^m$.

Proof: Let $\{s_n\}$ be a sequence in $\mathbb{N}$ such that $\hat{s}_n \to \mathcal{F}$ in $\mathbb{N}_k$. By continuity of the extensions, $\widehat{j+s_n} \to \hat{j}+\mathcal{F}$ and $\widehat{js_n} \to \hat{j}\mathcal{F}$ in $\mathbb{N}_k$. Let m be in $\mathbb{N}$. By R10.2.11, s_n is eventually in $C_m^{w_m}(k)$, $j+s_n$ is eventually in $C_m^{x_m}(k)$, and js_n is eventually in $C_m^{y_m}(k)$. For sufficiently large n, $s_n \equiv w_m \mod k^m$, $j+s_n \equiv x_m \mod k^m$, and $js_n \equiv y_m \mod k^m$. The conclusions follow immediately.

Proposition R12.4.4 Let $k \in \mathbb{N}$. Let $\mathcal{F}$ and $\mathcal{G}$ be a non-point ultrafilters in $\mathbb{N}_k$ associated with the sequences $\{w_n\}$ and $\{x_n\}$ respectively. Assume $\mathcal{F}+\mathcal{G}$ and $\mathcal{F}\mathcal{G}$ are associated with $\{y_n\}$ and $\{z_n\}$ respectively. Then, for every $m \in \mathbb{N}$, $y_m \equiv w_m + x_m \mod k^m$ and $z_m \equiv w_m x_m \mod k^m$.

Proof: Let $\{s_n\}$ and $\{t_n\}$ be sequences in $\mathbb{N}$ such that $\hat{s}_n \to \mathcal{F}$ and $\hat{t}_n \to \mathcal{G}$ in $\mathbb{N}_k$. By continuity of the extended operations, $\widehat{s_n+t_n} \to \mathcal{F}+\mathcal{G}$ and $\widehat{s_nt_n} \to \mathcal{F}\mathcal{G}$. Let $m \in \mathbb{N}$. By R10.2.11, s_n is eventually in $C_m^{w_m}(k)$, t_n is eventually in $C_m^{x_m}(k)$, s_n+t_n is eventually in $C_m^{y_m}(k)$, and s_nt_n is eventually in $C_m^{z_m}(k)$. For sufficiently large n, $s_n \equiv w_m \mod k^m$, $t_n \equiv x_m \mod k^m$, $s_n+t_n \equiv y_m \mod k^m$, and $s_nt_n \equiv z_m \mod k^m$. The conclusions follow immediately.

Note that by the definition and results in [6] the equivalences in the previous two propositions uniquely determine the results of the extended operations in $\mathbb{N}_k$.

As a result of R12.2.7 and results in [6], the extended operations in $\mathbb{N}_\infty$ are also determined by these propositions. This is illustrated more clearly in the following subsections. In principle, one proceeds as follows: As noted in the proof of R10.2.9, every non-point ultrafilter $\mathcal{F}$ in $\mathbb{N}_\infty$ is associated with a unique sequence of sequences, where, for a given $k \in \mathbb{N}$, the kth sequence (or k-level sequence) is the unique sequence associated with the non-point $\mathcal{Z}_k$-ultrafilter $\mathcal{F} \cap \mathbb{N}_k$. The results of extended addition and multiplication in $\mathbb{N}_\infty$ can be determined by applying the congruences of the last two propositions at every level k.

Cancellation Properties

Because addition and multiplication in $\mathbb{N}_k$ and $\mathbb{N}_\infty$ extend the ordinary operations, the focus here is on situations involving at least one non-point ultrafilter. Cancellation works for addition with the exception of one case in $\mathbb{N}_1$: If $\mathcal{F}$ is the unique non-point ultrafilter in $\mathbb{N}_1$, then $\mathcal{F}+\hat{j} = \mathcal{F}+\hat{l}$ for any $j, l \in \mathbb{N}$. As one might expect, cancellation is more complicated for multiplication.

Proposition R12.5.1 Let $k, j, l \in \mathbb{N}$. Let $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ be non-point ultrafilters in $\mathbb{N}_k$. Then

i) If $\hat{j} + \mathcal{G} = \hat{j} + \mathcal{H}$, then $\mathcal{G} = \mathcal{H}$.

ii) If $k \geq 2$ and $\mathcal{F} + \hat{j} = \mathcal{F} + \hat{l}$, then $j = l$.

iii) If $\mathcal{F} + \mathcal{G} = \mathcal{F} + \mathcal{H}$, then $\mathcal{G} = \mathcal{H}$.

Proof: Let $\{x_n\}$,$\{y_n\}$, and $\{z_n\}$ be the sequences associated with $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ respectively. If $\hat{j} + \mathcal{G} = \hat{j} + \mathcal{H}$, by R12.4.3 $j + y_m \equiv j + z_n \mod k^m$ and so $y_m \equiv z_m \mod k^m$ for all m. By definition y_m and z_m are both in $\{1, 2, \ldots, k^m\}$ so that $y_m = z_m$. By R10.2.4 $\mathcal{G} = \mathcal{H}$. The argument for iii) is similar: The hypothesis and R12.4.4 imply $x_m + y_m \equiv x_m + z_m \mod k^m$, which leads to $y_m = z_m$ for every m and so $\mathcal{G} = \mathcal{H}$. For ii) pick m large enough that both j and l are smaller than k^m. By R12.4.3 $x_m + j \equiv x_m + l \mod k^m$ and so $j = l$.

Corollary R12.5.2 Let $k \in \mathbb{N}$ with $k \geq 2$. Let $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ be in $\mathbb{N}_k$. If $\mathcal{F} + \mathcal{G} = \mathcal{F} + \mathcal{H}$, then $\mathcal{G} = \mathcal{H}$.

Proof: This summarizes the cases discussed above.

Proposition R12.5.3 Let $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ be in $\mathbb{N}_\infty$. If $\mathcal{F} + \mathcal{G} = \mathcal{F} + \mathcal{H}$, then $\mathcal{G} = \mathcal{H}$.

Proof: For every $k \in \mathbb{N}$, by R12.2.6i, $(\mathcal{F} + \mathcal{G}) \cap \mathcal{Z}_k = (\mathcal{F} \cap \mathcal{Z}_k) + (\mathcal{G} \cap \mathcal{Z}_k)$ and $(\mathcal{F} + \mathcal{H}) \cap \mathcal{Z}_k = (\mathcal{F} \cap \mathcal{Z}_k) + (\mathcal{H} \cap \mathcal{Z}_k)$. Using the hypothesis and R12.5.3, $\mathcal{G} \cap \mathcal{Z}_k = \mathcal{H} \cap \mathcal{Z}_k$ for every k and so $\mathcal{G} = \mathcal{H}$.

In what follows $a|b$ will mean a divides b.

Lemma R12.5.4 let $k \in \mathbb{N}$. Let $\mathcal{F}$ and $\mathcal{G}$ be non-point ultrafilters in $\mathbb{N}_k$ associated with $\{w_n\}$ and $\{x_n\}$ respectively. If, for some $m, t \in \mathbb{N}$, $k^m|(w_{m+t} - x_{m+t})$, then $k^m|(w_m - x_m)$.

Proof: By R10.2.5ii, k^m divides both $w_{m+t} - w_m$ and $x_{m+t} - x_m$. Thus $w_m - x_m = w_{m+t} - x_{m+t} + lk^m$ for some integer l. The conclusion is now clear.

Proposition R12.5.5 Let $k, j \in \mathbb{N}$. Let $\mathcal{F}$ and $\mathcal{G}$ be non-point ultrafilters in $\mathbb{N}_k$. If $\hat{j}\mathcal{F} = \hat{j}\mathcal{G}$, then $\mathcal{F} = \mathcal{G}$.

Proof: Assume $\mathcal{F}$ and $\mathcal{G}$ are associated with $\{w_n\}$ and $\{x_n\}$ respectively. By the hypothesis and R12.4.3 $jw_n \equiv jx_n \mod k^n$ for every $n \in \mathbb{N}$. If p is a prime which divides j but not k, then p is invertible mod k^n so that $(j/p)w_n \equiv (j/p)x_n \mod k^n$. Thus assume without loss of generality that every prime which divides j also divides k. Under that condition, there is $t \in \mathbb{N}$ such that $j|k^t$. Now fix $m \in \mathbb{N}$. $jw_{m+t} \equiv jx_{m+t} \mod k^{m+t}$ so that $j(w_{m+t} - x_{m+t}) = lk^{m+t}$ for some integer l. Since $j|k^t$, $k^m|(w_{m+t} - x_{m+t})$. By R12.5.4 $k^m|(w_m - x_m)$. By the definition of the associated sequences, w_m and x_m are in $\{1, 2, \ldots, k^m\}$. Thus $w_m = x_m$. By R10.2.4 $\mathcal{F} = \mathcal{G}$.

Proposition R12.5.6 Let $j \in \mathbb{N}$. Let $\mathcal{F}$ and $\mathcal{G}$ be non-point ultrafilters in $\mathbb{N}_\infty$. If $\hat{j}\mathcal{F} = \hat{j}\mathcal{G}$, then $\mathcal{F} = \mathcal{G}$.

Proof: For every $k \in \mathbb{N}$, by R12.2.6ii, $(\hat{j}\mathcal{F}) \cap \mathcal{Z}_k = (\hat{j} \cap \mathcal{Z}_k)(\mathcal{F} \cap \mathcal{Z}_k)$ and $(\hat{j}\mathcal{G}) \cap \mathcal{Z}_k = (\hat{j} \cap \mathcal{Z}_k)(\mathcal{G} \cap \mathcal{Z}_k)$. Using the hypothesis, the fact (R9.1.1ii) that $\hat{j} \cap \mathcal{Z}_k$ is the point-filter of j at level k, and R12.5.5, $\mathcal{F} \cap \mathcal{Z}_k = \mathcal{G} \cap \mathcal{Z}_k$ for every k and so $\mathcal{F} = \mathcal{G}$.

Lemma R12.5.7 Let $k \in \mathbb{N}$. Then there is a unique non-point ultrafilter in $\mathbb{N}_k$ associated with the sequence $\{k^n\}$.

Proof: This is immediate from R10.2.6.

Definition R12.5.8 Let $k \in \mathbb{N}$. The non-point ultrafilter in $\mathbb{N}_k$ associated with the sequence $\{k^n\}$ will be denoted $\mathcal{O}_k$.

It turns out that $\mathcal{O}_k$ is 'nearly' an additive identity, in the sense described in the following proposition.

Proposition R12.5.9 Let $k \in \mathbb{N}$. Then

i) If $\mathcal{F}$ is a non-point ultrafilter in $\mathbb{N}_k$, then $\mathcal{O}_k + \mathcal{F} = \mathcal{F}$.

ii) If $j \in \mathbb{N}$, then $\mathcal{O}_k + \hat{j}$ is the non-point ultrafilter associated with $\{x_n\}$, where $x_n \equiv j \bmod k^n$ for every n.

Proof: For $t \in \mathbb{N}$ and all n, $t + k^n \equiv t \bmod k^n$. The two assertions follow easily from this combined with R12.4.3, R12.4.4, and R10.2.4.

The next result shows that $\mathcal{O}_k$ cannot be canceled.

Proposition R12.5.10 Let $k \in \mathbb{N}$ and let $\mathcal{F} \in \mathbb{N}_k$. Then $\mathcal{O}_k\mathcal{F} = \mathcal{O}_k$.

Proof: If $\mathcal{F}$ is a non-point ultrafilter associated with $\{x_n\}$, then $\mathcal{O}_k\mathcal{F}$ is associated with $\{y_n\}$, where $y_n \equiv k^n x_n \bmod k^n$ for all n. If $\mathcal{F}$ is the point ultrafilter of j in $\mathbb{N}$, $\mathcal{O}_k\mathcal{F}$ is associated with $\{y_n\}$, where $y_n \equiv k^n j \bmod k^n$ for all n. In either case, since y_n must be in $\{1, 2, \ldots, k^n\}$, $y_n = k^n$ for all n and the conclusion follows from R10.2.4.

Here is another example in which multiplicative cancellation fails.

Example R12.5.11 Let $k \in \mathbb{N}$ with $k \geq 2$. Let $\{x_n\}$ and $\{y_n\}$ be constant sequences with $x_n = 1$ and $y_n = 2$ for every n. By R10.2.6 $\mathbb{N}_k$ contains non-point ultrafilters $\mathcal{F}$ associated with $\{x_n\}$ and $\mathcal{G}$ associated with $\{y_n\}$. By R12.4.3 and R12.4.4, $\hat{2}\mathcal{F} = \mathcal{F}\mathcal{G}$. But $\mathcal{G} \neq \hat{2}$ since $\mathcal{G}$ is not a point filter.

R12.5.5 describes one case in which multiplicative cancellation is possible for $\mathbb{N}_k$. The following results identify some others.

Lemma R12.5.12 Let $k \in \mathbb{N}$ with $k \geq 2$. Let $\mathcal{F}$ be a non-point ultrafilter in $\mathbb{N}_k$ associated with $\{x_n\}$. Let $a \in \mathbb{N}$ be such that $a|k$. Assume a^j does not divide x_m for some m and some integer $1 \leq j \leq m$. Then a^j does not divide x_{m+t} for all $t \in \mathbb{N}$.

Proof: Use induction. For $t = 1$, by R10.2.5, $x_{m+1} = x_m + sk^m$ for some integer s between 0 and $k-1$. Since $a^j|k^m$, if $a^j|x_{m+1}$, then a^j would also divide x_m, contradicting the hypothesis. Thus a^j does not divide x_{m+1}. The argument for the induction step begins with the equation $x_{m+t+1} = x_{m+t} + sk^{m+t}$ and is almost identical.

Corollary R12.5.13 Let $k \in \mathbb{N}$ with $k \geq 2$. Let $\mathcal{F}$ be a non-point ultrafilter in $\mathbb{N}_k$ associated with $\{x_n\}$ such that $\mathcal{F}$ is not $\mathcal{O}_k$. Let u, v be in $\mathbb{N}$ and assume $\hat{u}\mathcal{F} = \hat{v}\mathcal{F}$. Then $u = v$.

Proof: Pick m such that $x_m \neq k^m$. Since $x_m \in \{1, 2, \ldots, k^m - 1\}$, there must be a prime p and positive integer r such that $p^r|k$ and an integer j, $1 \leq j \leq m$, such that p^{rj} does not divide x_m. Using R12.4.3 and the hypothesis, we conclude that k^{m+t}, and so $p^{r(m+t)}$, divides $x_{m+t}(u - v)$ for every t in $\mathbb{N}$. By the preceding lemma with $a = p^r$, p^{rj} does not divide x_{m+t} and so $p^{r(m+t-j)+1}$ must divide $u - v$. Since t can be arbitrarily large, $u = v$.

Lemma R12.5.14 Let $k \in \mathbb{N}$ with $k \geq 2$. Let $\mathcal{G}$ and $\mathcal{H}$ be non-point ultrafilters in $\mathbb{N}_k$ associated with $\{y_n\}$ and $\{z_n\}$ respectively. Let $a|k$ and assume a^j does not divide $(y_m - z_m)$ for some m and some integer $1 \leq j \leq m$. Then a^j does not divide $y_{m+t} - z_{m+t}$ for all $t \in \mathbb{N}$.

Proof: This is an induction argument very similar to that of R12.5.12. By R10.2.5, $y_{m+t+1} - z_{m+t+1} = y_{m+t} - z_{m+t} + lk^{m+t}$ for some integer l.

Lemma R12.5.15 Let $k \in \mathbb{N}$ with $k \geq 2$. Let $\mathcal{G}$ and $\mathcal{H}$ be non-point ultrafilters in $\mathbb{N}_k$ associated with $\{y_n\}$ and $\{z_n\}$ respectively. If $y_r = z_r$, then $y_n = z_n$ for all $n \leq r$.

Proof: Use induction on r. For $r = 1$ the conclusion obviously holds. Now suppose the conclusion holds for r and $y_{r+1} = z_{r+1}$. By R10.2.5 $y_r + ik^r = z_r + jk^r$ where $0 \leq i, j \leq k-1$. Since y_r and z_r are both in $\{1, 2, \ldots, k^r\}$, $k^r|(y_r - z_r)$ implies $y_r = z_r$. By the induction hypothesis $y_n = z_n$ for all $n \leq r$, and the conclusion follows.

Proposition R12.5.16 Let $p \in \mathbb{N}$ be a prime. Let $\mathcal{F}$,$\mathcal{G}$ and $\mathcal{H}$ be non-point ultrafilters in $\mathbb{N}_p$ associated with $\{x_n\}$,$\{y_n\}$ and $\{z_n\}$ respectively. Assume $\mathcal{F}$ is not $\mathcal{O}_p$.
If $\mathcal{F}\mathcal{G} = \mathcal{F}\mathcal{H}$, then $\mathcal{G} = \mathcal{H}$.

Proof: Pick m such that $x_m \neq p^m$. Since $x_m \in \{1, 2, \ldots, p^m - 1\}$, p^m does not divide x_m. Suppose for some $r \geq m$, $y_r \neq z_r$. Since $0 < |y_r - z_r| \leq p^r - 1$, p^r does not divide $y_r - z_r$. By R12.5.12 and R12.5.14 p^{m+r} cannot divide $x_t(y_t - z_t)$ for $t \geq r$. But by the hypothesis, for any $t \in \mathbb{N}$, $x_t(y_t - z_t) \equiv 0 \bmod p^t$. When $t \geq m + r$, there is a contradiction. Thus $y_r = z_r$ for all $r \geq m$. By R12.5.15 $y_n = z_n$ for all n and so $\mathcal{G} = \mathcal{H}$ by R10.2.4.

The next two lemmas establish a way to construct elements of $\mathbb{N}_\infty$. As in [5], $E_n(k)$ denotes equivalence mod k^n in $\mathbb{N}$.

Lemma R12.5.17 Let $k, l \in \mathbb{N}$. If $k|l$, then

i) $\mathcal{Z}_k \subseteq \mathcal{Z}_l$.

ii) If $\mathcal{F}$ is in $\mathbb{N}_l$, then $\mathcal{F} \cap \mathcal{Z}_k$ is in $\mathbb{N}_k$.

Proof: Since $k^n|l^n$, $E_n(l) \subseteq E_n(k)$ and so $\mathcal{Z}(E_n(k)) \subseteq \mathcal{Z}(E_n(l))$ by R10.1.1. By definition R10.1.3, part i) follows. For part ii) it is sufficient to show that R9.1.5 applies. Let $A \in \mathcal{Z}_k$ and $B \in \mathcal{Z}_l$ with $A \cap B = \emptyset$ so that $B \subseteq \mathbb{N} - A$. By definition R10.1.3 and R9.1.7, $C = \mathbb{N} - A$ is in $\mathcal{Z}_k$. C is the set needed to verify the hypothesis of R9.1.5.

Lemma R12.5.18 For each $i \in \mathbb{N}$, let $\mathcal{F}_i$ be a $\mathcal{Z}_i$-ultrafilter. Assume that if $k|l$, then $\mathcal{F}_k \subseteq \mathcal{F}_l$. Then $\mathcal{F} = \cup\{\mathcal{F}_i : i \in \mathbb{N}\}$ is a $\mathcal{Z}_\infty$-ultrafilter and $\mathcal{F} \cap \mathcal{Z}_i = \mathcal{F}_i$ for all $i \in \mathbb{N}$.

Proof: Clearly $\mathcal{F}$ is a non-empty collection of non-empty $\mathcal{Z}_\infty$-sets. Let $A \subseteq B$ with $A \in \mathcal{F}$ and $B \in \mathcal{Z}_\infty$. Then A is in $\mathcal{F}_i$ and B is in $\mathcal{Z}_j$ for some $i, j \in \mathbb{N}$. By R12.5.17i A, B are both in $\mathcal{Z}_{ij}$ and by hypothesis A is in $\mathcal{F}_{ij}$ so that $B \in \mathcal{F}_{ij} \subseteq \mathcal{F}$. Next let C, D be in $\mathcal{F}$. If $C \in \mathcal{F}_i$ and $D \in \mathcal{F}_j$, by hypothesis both are in $\mathcal{F}_{ij}$ as is $C \cap D$. Thus $C \cap D$ is in $\mathcal{F}$. Finally, suppose $\mathcal{G}$ is a $\mathcal{Z}_\infty$-ultrafilter with $\mathcal{F} \subseteq \mathcal{G}$. Then, for the $\mathcal{Z}_i$-ultrafilters $\mathcal{F}_i$ and (by R10.2.8) $\mathcal{G} \cap \mathcal{Z}_i$, $\mathcal{F}_i$ is a subset of $\mathcal{G} \cap \mathcal{Z}_i$, which implies their equality. Clearly, $\mathcal{G} = \cup\{\mathcal{G} \cap \mathcal{Z}_i : i \in \mathbb{N}\}$ so that $\mathcal{G} = \mathcal{F}$. We now have that $\mathcal{F}$ is a $\mathcal{Z}_\infty$-ultrafilter and so $\mathcal{F} \cap \mathcal{Z}_i$, which contains $\mathcal{F}_i$, is a $\mathcal{Z}_i$-ultrafilter by R10.2.8. By hypothesis, $\mathcal{F}_i = \mathcal{F} \cap \mathcal{Z}_i$.

Lemma R12.5.19 Let $k, l \in \mathbb{N}$ with $k|l$. Let $\mathcal{F}$ be a non-point ultrafilter in $\mathbb{N}_k$ associated with $\{x_n\}$, and let $\mathcal{G}$ be a non-point ultrafilter in $\mathbb{N}_l$ associated with $\{y_n\}$. Then $\mathcal{F} \subseteq \mathcal{G}$ if and only if $x_n \equiv y_n \bmod k^n$ for all n.

Proof: Assume $\mathcal{F} \subseteq \mathcal{G}$, and let n be in $\mathbb{N}$. Since $C_n^{x_n}(k)$ is associated with $\{1, \ldots, k^n\} - \{x_n\}$, $C_n^{x_n}(k)$ is in $\mathcal{F}$. As an element of $\mathcal{G}$, it must be associated with $\Delta \subseteq \{1, \ldots, l^n\} - \{y_n\}$. Therefore $(\mathbb{N} - C_n^{x_n}(k)) \cap C_n^{y_n}(l)$ is finite, and so $C_n^{x_n}(k)) \cap C_n^{y_n}(l) \neq \emptyset$. Since $k|l$, it follows that $x_n \equiv y_n$ mod k^n. For the converse, note that $\mathcal{G} \cap \mathcal{Z}_k$ is a $\mathcal{Z}_k$-

ultrafilter by R12.5.17ii. It is clearly non-point and so is associated with a sequence $\{z_n\}$. By the first half of this lemma $z_n \equiv y_n \mod k^n$ and so by hypothesis $z_n \equiv x_n \mod k^n$ for all n. Since x_n and z_n are in $\{1, \ldots, k^n\}$, we have $x_n = z_n$ for all n. By R10.2.4 $\mathcal{F} = \mathcal{G} \cap \mathcal{Z}_k$.

Lemma R12.5.20 Let $k, l \in \mathbb{N}$ with $k|l$. Then $\mathcal{O}_k \subseteq \mathcal{O}_l$.

Proof: $\mathcal{O}_k$ is associated with $\{k^n\}$ and $\mathcal{O}_l$ is associated with $\{l^n\}$. Since $k|l$, $k^n \equiv l^n \mod k^n$ for all n so that $\mathcal{O}_k \subseteq \mathcal{O}_l$ by R12.5.19.

Definition R12.5.21 $\mathcal{O}_\infty = \cup\{\mathcal{O}_i : i \in \mathbb{N}\}$.

Note that by R12.5.20 and R12.5.18 $\mathcal{O}_\infty \in \mathbb{N}_\infty$ and $\mathcal{O}_\infty \cap \mathcal{Z}_i$ is the non-point $\mathcal{O}_i$ for all i in $\mathbb{N}$, and so $\mathcal{O}_\infty$ is non-point. The next proposition uses the following fact, which is easy but has not been explicitly stated.

Lemma R12.5.22 If $\mathcal{F}$ is a non-point ultrafilter in $\mathbb{N}_\infty$, then $\mathcal{F} \cap \mathcal{Z}_k$ is a non-point ultrafilter in $\mathbb{N}_k$ for all k.

Proof: $\mathcal{F} \cap \mathcal{Z}_k$ is a $\mathcal{Z}_k$-ultrafilter by R10.2.8. If $\mathcal{F} \cap \mathcal{Z}_k$ is the point filter of j, then $\{j\}$ is in $\mathcal{F} \cap \mathcal{Z}_k$ and so also in $\mathcal{F}$.

Proposition R12.5.23 $\mathcal{O}_\infty$ has the following propertities.

i) If $\mathcal{F}$ is a non-point ultrafilter in $\mathbb{N}_\infty$, then $\mathcal{F} + \mathcal{O}_\infty = \mathcal{F}$.

ii) Let j be in $\mathbb{N}$ and $\hat{j} + \mathcal{O}_\infty = \mathcal{G}$. Then $\mathcal{G} \cap \mathcal{Z}_k$ is associated with $\{{}^k x_n\}$, where ${}^k x_n \equiv j \mod k^n$ for all $k, n \in \mathbb{N}$.

iii) If $\mathcal{F} \in \mathbb{N}_\infty$, then $\mathcal{F}\mathcal{O}_\infty = \mathcal{O}_\infty$.

Proof: In algebraic notation R12.2.7i states $\mathcal{F} + \mathcal{O}_\infty = \cup\{(\mathcal{F} \cap \mathcal{Z}_k) + \mathcal{O}_k : k \in \mathbb{N}\}$ for any $\mathcal{F}$ in $\mathbb{N}_\infty$. The first assertion follows from this and R12.5.9i. Simlarly, $\hat{j} + \mathcal{O}_\infty = \cup\{\hat{j} + \mathcal{O}_k : k \in \mathbb{N}\}$ so that $\mathcal{G} \cap \mathcal{Z}_k = \hat{j} + \mathcal{O}_k$ and ii) holds by R12.5.9ii. Finally, by R12.2.7ii, $\mathcal{F}\mathcal{O}_\infty = \cup\{(\mathcal{F} \cap \mathcal{Z}_k)\mathcal{O}_k : k \in \mathbb{N}\}$ for any $\mathcal{F}$ in $\mathbb{N}_\infty$ so that iii) follows from R12.5.10.

It is also possible to contruct an analog of example R12.5.11.

Example R12.5.24 For each $k \in \mathbb{N}$ with $k \geq 2$, let $\mathcal{F}_k$ and $\mathcal{G}_k$ be associated with the constant sequences $\{1\}$ and $\{2\}$ respectively. Let $\mathcal{F}_1 = \mathcal{G}_1$ be the unique non-point ultrafilter in $\mathbb{N}_1$. By R12.5.19 and R12.5.18, $\mathcal{F} = \cup\{\mathcal{F}_i : i \in \mathbb{N}\}$ and $\mathcal{G} = \cup\{\mathcal{G}_i : i \in \mathbb{N}\}$ are non-point ultrafilters in $\mathbb{N}_\infty$. Using R12.2.7ii, R12.4.3, and R12.4.4, we have $\hat{2}\mathcal{F} = \mathcal{F}\mathcal{G}$. But $\mathcal{G} \neq \hat{2}$ since $\mathcal{G}$ is not a point filter.

The Remnant Rings

This subsection will show that, if all point-filters are removed from $\mathbb{N}_\infty$ or $\mathbb{N}_k$ for $k \geq 2$, what remains is a compact topological ring. These rings contain dense, non-discrete, sub-semi-rings which are algebraically isomorphic to $\mathbb{N}$. Ursal's monograph [1] contains an extensive bibliography on topological rings as well as elementary and advanced results.

Lemma R12.6.1 Let $k \in \mathbb{N}$. Let $\mathcal{F}$ be a non-point $\mathcal{Z}_k$-ultrafilter associated with $\{x_n\}$. For each n let y_n be the element of $\{1, \ldots, k^n\}$ with $y_n \equiv k^n - x_n \mod k^n$. Then there is a non-point $\mathcal{Z}_k$-ultrafilter associated with $\{y_n\}$.

Proof: If $k = 1$, then $y_n = 1 = x_n$ for all n and the conclusion holds. Now assume $k \geq 2$ and fix $n \in \mathbb{N}$. By R10.2.5i $x_{n+1} = x_n + rk^n$ where r is an integer and $0 \leq r \leq k-1$. By R10.2.6 it is sufficient to show $y_{n+1} = y_n + sk^n$ for some integer s with $0 \leq s \leq k-1$. As a first case, assume $1 \leq x_n \leq k^n - 1$. It follows that $1 \leq x_{n+1} \leq k^{n+1} - 1$, $y_n = k^n - x_n$, and $y_{n+1} = k^{n+1} - x_{n+1}$. Routine algebra shows that $y_{n+1} = y_n + (k - 1 - r)k^n$ and

$0 \le k-1-r \le k-1$. In the remaining case, $x_n = k^n$ and so $y_n = k^n$. If $r = k-1$, then $x_{n+1} = k^{n+1} = y_{n+1}$ and the required equation holds with $s = k-1$. If $r \le k-2$, then $y_{n+1} = y_n + (k-2-r)k^n$ and $0 \le k-2-r \le k-2$.

Definition R12.6.2 Let $k \in \mathbb{N}$. Let $\mathcal{F}$ be a non-point $\mathcal{Z}_k$-ultrafilter associated with $\{x_n\}$. ${}^-\mathcal{F}$ is the non-point $\mathcal{Z}_k$-ultrafilter associated with $\{y_n\}$, where $y_n \in \{1, \ldots, k^n\}$ and $y_n \equiv k^n - x_n \bmod k^n$.

Proposition R12.6.3 Let $k \in \mathbb{N}$. Let $\mathcal{F}$ be a non-point $\mathcal{Z}_k$-ultrafilter. Then $\mathcal{F} +^- \mathcal{F} = \mathcal{O}_k$.

Proof: Let $\mathcal{F}$ be associated with $\{x_n\}$. $\mathcal{F} +^- \mathcal{F}$ is a non-point ultrafilter associated with $\{z_n\}$. By definition ${}^-\mathcal{F}$ is associated with $\{y_n\}$, where $y_n \equiv k^n - x_n \bmod k^n$. By R12.4.4 $z_n \equiv x_n + (k^n - x_n) \equiv k^n \bmod k^n$. Since $z_n \in \{1, \ldots, k^n\}$, $z_n = k^n$ for all n and so $\mathcal{F} +^- \mathcal{F} = \mathcal{O}_k$.

Definition R12.6.4 Let $k \in \mathbb{N}$ with $k \ge 2$. $\mathbf{R}_k = \mathbb{N}_k - \{\hat{j} : j \in \mathbb{N}\}$.

Theorem R12.6.5 For $k \in \mathbb{N}$ with $k \ge 2$, $\mathbf{R}_k$ is a compact, metrizable, zero-dimensional, commutative topological ring with unity.

Proof: By R12.4.2 $+$ and $\cdot$ are binary operations on $\mathbf{R}_k$. In addition to the needed algebraic properties derived in R12.3.1 and R12.3.2, $\mathcal{O}_k$ is an additive identity by R12.5.9i, R12.6.3 shows the existence of additive inverses, and $\hat{1} + \mathcal{O}_k$ is the multiplicative identity. Note that $\hat{1} + \mathcal{O}_k \neq \mathcal{O}_k$ since $k \ge 2$. By R12.2.4 both operations are continuous and, since $\mathbf{R}_k$ is a ring with unity, the continuity of the map $\mathcal{F} \mapsto {}^-\mathcal{F}$ follows automatically. Since $\mathbb{N}$ is locally compact, it is embedded as an open set in the compactification $\mathbb{N}_k$. Thus $\mathbf{R}_k$ is a closed subspace of the compact, T_2 space $\mathbb{N}_k$. Metrizability and zero-dimensionality follow from R10.1.10, since these properties are inherited by subspaces.

Definition R12.6.6 Let $k \in \mathbb{N}$ with $k \ge 2$. For each $j \in \mathbb{N}$, $\mathcal{F}_k(j)$ is defined to be the non-point ultrafilter $\mathcal{O}_k + \hat{j}$.

Note that by R12.5.9ii $\mathcal{F}_k(j)$ is associated with $\{x_n\}$, where $x_n \in \{1, \ldots, k^n\}$ and $x_n \equiv j \bmod k^n$ for all n. Clearly x_n is eventually constant with value j. If $j \neq l$, then R12.5.1ii implies $\mathcal{F}_k(j) \neq \mathcal{F}_k(l)$.

Definition R12.6.7 Let $k \in \mathbb{N}$ with $k \ge 2$. $P_k = \{\mathcal{F}_k(j) : j \in \mathbb{N}\}$.

The next two propositions show that P_k is an isomorphic image of $\mathbb{N}$, but that $\mathbf{R}_k$ is not a compactification of $\mathbb{N}$ via the isomorphism of R12.6.8.

Proposition R12.6.8 Let $k \in \mathbb{N}$ with $k \ge 2$. P_k is isomorphic to $\mathbb{N}$ via the map $j \mapsto \mathcal{F}_k(j)$.

Proof: The observation just before R12.6.7 shows that the map is one-to-one. Since $(\widehat{j+l}) + \mathcal{O}_k = \hat{j} + \hat{l} + \mathcal{O}_k = (\hat{j} + \mathcal{O}_k) + (\hat{l} + \mathcal{O}_k)$, addition is preserved. Using algebraic propertities derived above, one easily checks that $(\hat{j} + \mathcal{O}_k)(\hat{l} + \mathcal{O}_k) = \hat{j}\hat{l} + \mathcal{O}_k = \widehat{jl} + \mathcal{O}_k$ and so multiplication is also preserved. Lastly, the image of 1 is is the multiplicative identity, as noted in the proof of R12.6.5.

Proposition R12.6.9 Let $k \in \mathbb{N}$ with $k \ge 2$. P_k is a dense, non-discrete subset of the topological space $\mathbf{R}_k$.

Proof: Let $\mathcal{F} \in \mathbf{R}_k$ and let $\{j_n\}$ be a sequence in $\mathbb{N}$ such that $\widehat{j_n} \to \mathcal{F}$ in $\mathbb{N}_k$. By continuity of addition and R12.5.9i $\mathcal{F}_k(j_n) = \mathcal{O}_k + \widehat{j_n} \to \mathcal{O}_k + \mathcal{F} = \mathcal{F}$ in $\mathbb{N}_k$ and so in $\mathbf{R}_k$. Thus $\mathcal{F}$ is in the $\mathbf{R}_k$ -closure of P_k, i.e., P_k is dense. Now suppose $\{\mathcal{F}_k(1)\} = G \cap P_k$ for some G open in $\mathbb{N}_k$. Let $\{j_n\}$ be a sequence in $\mathbb{N}$ such that $\widehat{j_n} \to \mathcal{F}_k(1)$ in $\mathbb{N}_k$.

As in the density argument, $\mathcal{F}_k(j_n)$ converges to $\mathcal{F}_k(1)$ in $\mathbb{N}_k$ and so must eventually be in G. Since $\{j_n\}$ is unbounded by R10.2.11, this contradicts the assumption that the singleton is open in P_k.

The rest of this subsection develops results analogous results for $\mathbb{N}_\infty$.

Lemma R12.6.10 Let $\mathcal{F}$ be a non-point ultrafilter in $\mathbb{N}_\infty$. Let $k, l \in \mathbb{N}$ with $k|l$. Then $^-(\mathcal{F} \cap \mathcal{Z}_k) \subseteq^- (\mathcal{F} \cap \mathcal{Z}_l)$.

Proof: Since $k|l$, $\mathcal{Z}_k \subseteq \mathcal{Z}_l$ by R12.5.17i. Let the non-point ultrafilters $\mathcal{F} \cap \mathcal{Z}_k$ and $\mathcal{F} \cap \mathcal{Z}_l$ be associated with $\{a_n\}$ and $\{b_n\}$ respectively. Let $^-(\mathcal{F} \cap \mathcal{Z}_k)$ and $^-(\mathcal{F} \cap \mathcal{Z}_l)$ be associated with $\{c_n\}$ and $\{d_n\}$ respectively. Fix n. By R12.5.19 $a_n \equiv b_n \bmod k^n$, and so $c_n \equiv k^n - a_n \equiv l^n - b_n \equiv d_n \bmod k^n$. The conclusion now follows from R12.5.19.

Definition R12.6.11 Let $\mathcal{F}$ be a non-point ultrafilter in $\mathbb{N}_\infty$. The negative of $\mathcal{F}$ is $^-\mathcal{F} = \cup\{^-(\mathcal{F} \cap \mathcal{Z}_k) : k \in \mathbb{N}\}$.

Proposition R12.6.12 Let $\mathcal{F}$ be a non-point ultrafilter in $\mathbb{N}_\infty$. Then $^-\mathcal{F}$ is a non-point ultrafilter in $\mathbb{N}_\infty$ and $\mathcal{F} +^- \mathcal{F} = \mathcal{O}_\infty$.

Proof: $^-\mathcal{F}$ is in $\mathbb{N}_\infty$ by R12.6.10 and R12.5.18. From R12.5.18 we also conclude $^-\mathcal{F} \cap \mathcal{Z}_k =^-(\mathcal{F} \cap \mathcal{Z}_k)$, which is non-point for each k. Thus $^-\mathcal{F}$ is also non-point. Also, by R12.2.7i and R12.6.3, we have $\mathcal{F} +^- \mathcal{F} = \cup\{(\mathcal{F} \cap \mathcal{Z}_k) +^- (\mathcal{F} \cap \mathcal{Z}_k) : k \in \mathbb{N}\} = \cup\{\mathcal{O}_k : k \in \mathbb{N}\} = \mathcal{O}_\infty$.

Definition R12.6.13 $\mathbf{R}_\infty = \mathbb{N}_\infty - \{\hat{j} : j \in \mathbb{N}\}$.

Theorem R12.6.14 $\mathbf{R}_\infty$ is a compact, metrizable, zero-dimensional, commutative topological ring with unity.

Proof: By R12.4.2 $+$ and $\cdot$ are binary operations on $\mathbf{R}_\infty$. In addition to the needed algebraic properties derived in R12.3.1 and R12.3.2, $\mathcal{O}_\infty$ is an additive identity by R12.5.23i, R12.6.12 shows the existence of additive inverses, and $\hat{1} + \mathcal{O}_\infty$ is the multiplicative identity. By R12.2.5 both operations are continuous and, since $\mathbf{R}_\infty$ is a ring with unity, the continuity of the map $\mathcal{F} \mapsto^- \mathcal{F}$ follows automatically. Since $\mathbb{N}$ is locally compact, it is embedded as an open set in the compactification $\mathbb{N}_\infty$. Thus $\mathbf{R}_\infty$ is a closed subspace of the compact, T_2 space $\mathbb{N}_\infty$. Metrizability and zero-dimensionality follow from R10.1.10, since these properties are inherited by subspaces.

Definition R12.6.15 Let $j \in \mathbb{N}$. $\mathcal{F}_\infty(j) = \hat{j} + \mathcal{O}_\infty$.

Note that, since $\mathcal{O}_\infty \cap \mathcal{Z}_k = \mathcal{O}_k$, by R12.2.6i $\mathcal{F}_\infty(j) \cap \mathcal{Z}_k = \mathcal{F}_k(j)$ for all $k \in \mathbb{N}$ with $k \geq 2$. If $j \neq l$, since $\mathcal{F}_k(j) \neq \mathcal{F}_k(l)$, this obsevation shows that $\mathcal{F}_\infty(j) \neq \mathcal{F}_\infty(l)$.

Definition R12.6.16 $P_\infty = \{\mathcal{F}_\infty(j) : j \in \mathbb{N}\}$.

The next two propositions show that P_∞ is an isomorphic image of $\mathbb{N}$, but that $\mathbf{R}_\infty$ is not a compactification of $\mathbb{N}$ via the isomorphism of R12.6.17.

Proposition R12.6.17 P_∞ is isomorphic to $\mathbb{N}$ via the map $j \mapsto \mathcal{F}_\infty(j)$.

Proof: The comment just before R12.6.16 shows that the map is one-to-one. Since $(\widehat{j+l}) + \mathcal{O}_\infty = \hat{j} + \hat{l} + \mathcal{O}_\infty = (\hat{j} + \mathcal{O}_\infty) + (\hat{l} + \mathcal{O}_\infty)$, addition is preserved. Using algebraic propertities derived above, one easily checks that $(\hat{j} + \mathcal{O}_\infty)(\hat{l} + \mathcal{O}_\infty) = \hat{j}\hat{l} + \mathcal{O}_\infty = \widehat{jl} + \mathcal{O}_\infty$ and so multiplication is also preserved. Lastly, the image of 1 is the multiplicative identity, as noted in the proof of R12.6.14.

Proposition R12.6.18 P_∞ is a dense, non-discrete subset of the topological space $\mathbf{R}_\infty$.

Proof: Let $\mathcal{F} \in \mathbf{R}_\infty$ and let $\{j_n\}$ be a sequence in $\mathbb{N}$ such that $\widehat{j_n}$ converges to $\mathcal{F}$ in

$\mathbb{N}_\infty$. By continuity of addition and R12.5.23i $\mathcal{F}_\infty(j_n) = \mathcal{O}_\infty + \widehat{j_n} \to \mathcal{O}_\infty + \mathcal{F} = \mathcal{F}$ in $\mathbb{N}_\infty$ and so in $\mathbf{R}_\infty$. Thus $\mathcal{F}$ is in the $\mathbf{R}_\infty$-closure of P_∞, i.e., P_∞ is dense. Now suppose $\{\mathcal{F}_\infty(1)\} = G \cap P_\infty$ for some G open in $\mathbb{N}_\infty$. Let $\{j_n\}$ be a sequence in $\mathbb{N}$ such that $\widehat{j_n} \to \mathcal{F}_\infty(1)$ in $\mathbb{N}_\infty$. As in the density argument, $\mathcal{F}_\infty(j_n) \to \mathcal{F}_\infty(1)$ in $\mathbb{N}_\infty$ and so $\mathcal{F}_\infty(j_n)$ must eventually be in G. Since $\{j_n\}$ is unbounded by R12.4.1, this contradicts the assumption that the singleton is open in P_∞.

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