Mixed Suprema

In this section a generalization of suprema of compactifications is studied. Instead of considering compactifications of a fixed $T_{3\frac{1}{2}}$ space, only the underlying set will be fixed. Given a set X, (Y, f) is a T_2 compactification of X provided there is some $T_{3\frac{1}{2}}$ topology τ for X such that (Y, f) is a T_2 compactification of (X, τ). As in [5], the starting point will be totally bounded uniform spaces.

Basic Facts

Definition R13.1.1 Let X be a set. $TBS\,(X)$ is the set of all totally bounded, separated uniformities on X.

Note that $TBS\,(X)$ is non-empty, since the discrete topology on X is $T_{3\frac{1}{2}}$ and can be generated by a totally bounded separated uniformity. If X is finite, $TBS(X)$ contains exactly one element, the discrete uniformity for X. To avoid considering this trivial case, most results in this section will assume an infinite set X. By P2.13 $TBS(X)$ is closed under arbtrary suprema of uniformities, i.e., it is a complete upper semi-lattice. In addition, if $\mathcal{U}$ is in $TBS(X)$, then $(X, \tau(\mathcal{U}))$ is $T_{3\frac{1}{2}}$ so that a separated completion of $(X,\mathcal{U})$ generates a compactification class of $(X, \tau(\mathcal{U}))$ as shown in R1.2. This compactification class will be denoted $\Psi_0(\mathcal{U})$, as in R1.3.

An element $\mathcal{U}$ of $TBS\,(X)$ can be a complete uniformity. In this case, $(X, \tau(\mathcal{U}))$ is compact and T_2 and the compactitification class will aways be represented by $(X, \tau(\mathcal{U}))$ with the identity map. Note that such a uniformity is a minimal element in $TBS\,(X)$, because a compact Hausdorff space is minimal Hausdorff and has a unique uniformity.

The definition of ordering for compactification classes is identical in form to the standard definition : Let $\mathcal{U}_1$, $\mathcal{U}_2$ be in $TBS\,(X)$ with $\Psi_0(\mathcal{U}_i) = [(Y_i, f_i)]$. $[(Y_1, f_1)] \leq [(Y_2, f_2)]$ if and only if there is a continuous onto map $g : Y_2 \to Y_1$ such that $g \circ f_2 = f_1$.

This ordering, which is reflexive, anti-symmetric, and transitive, might raise a question about which structure ($\mathcal{U}_1$ or $\mathcal{U}_2$? $\tau(\mathcal{U}_1)$ or $\tau(\mathcal{U}_2)$?) is to be used for the set X. The functional equation determines the relationship between these structures, as shown below in R13.1.2 and R13.1.5.

Proposition R13.1.2 Let X be an infinite set and let $\mathcal{U}_1$, $\mathcal{U}_2$ be in $TBS(X)$. Then $\mathcal{U}_1 \subseteq \mathcal{U}_2$ if and only if $\Psi_0(\mathcal{U}_1) \leq \Psi_0(\mathcal{U}_2)$.

Proof: Same as the proof of R1.5.

Corollary R13.1.3 Let X be an infinite set and let $\mathcal{U}_1$, $\mathcal{U}_2$ be in $TBS\,(X)$. Then $\mathcal{U}_1 = \mathcal{U}_2$ if and only if $\Psi_0(\mathcal{U}_1) = \Psi_0(\mathcal{U}_2)$.

Proof: Immediate from R13.1.2.

Proposition R13.1.4 Let (X, τ) be a $T_{3\frac{1}{2}}$ space, where X is infinite. Let (Y, f) be a T_2 compactification of (X, τ). Then there is $\mathcal{U}$ in $TBS\,(X)$ such that $\tau(\mathcal{U}) = \tau$ and $\Psi_0(\mathcal{U}) = [(Y, f)]$.

Proof: Same as the proof of R1.4.

Corollary R13.1.5 Let X be an infinite set and let τ_1 and τ_2 be $T_{3\frac{1}{2}}$ topologies for X. Assume (Y_1, f_1) and (Y_2, f_2) are T_2 compactifications of (X, τ_1) and (X, τ_2) respectively. Then

i) If $[(Y_1, f_1)] \leq [(Y_2, f_2)]$, then $\tau_1 \subseteq \tau_2$.

ii)If (Y_1, f_1) is equivalent to (Y_2, f_2), then $\tau_1 = \tau_2$.

Proof: Let $\mathcal{U}_1, \mathcal{U}_2$ be in $TBS(X)$ with $\Psi_0(\mathcal{U}_i) = [(Y_i, f_i)]$. Under the hypothesis of i), R13.1.2 implies $\mathcal{U}_1 \subseteq \mathcal{U}_2$ and so $\tau_1 = \tau(\mathcal{U}_1) \subseteq \tau(\mathcal{U}_2) = \tau_2$. Part ii) is now immediate.

Of particular interest is the following version of Lubben's theorem.

Theorem R13.1.6 Let X be an infinite set and let Δ be a non-empty set. Assume that, for each $\alpha \in \Delta$, there is a $T_{3\frac{1}{2}}$ topology τ_α for X and (Y_α, f_α), a T_2 compactification of (X, τ_α). Let $\tau = \vee\{\tau_\alpha : \alpha \in \Delta\}$. Then there is (Y, g), a T_2 compactification of (X, τ), that acts as a supremum of $\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$, i.e.

i) $[(Y, g)] \geq [(Y_\alpha, f_\alpha)]$ for all $\alpha \in \Delta$ and

ii) if σ is a $T_{3\frac{1}{2}}$ topology for X and (Z, h) is a T_2 compactification of (X, σ) such that $[(Z, h)] \geq [(Y_\alpha, f_\alpha)]$ for all $\alpha \in \Delta$, then $[(Z, h)] \geq [(Y, g)]$.

Proof: For each α, let $\mathcal{U}_\alpha$ be the element of $TBS(X)$ with $\Psi_0(\mathcal{U}_\alpha) = [(Y_\alpha, f_\alpha)]$ and $\tau(\mathcal{U}_\alpha) = \tau_\alpha$, and let $\mathcal{U} = \vee\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. $\mathcal{U}$ is in $TBS(X)$ as noted above and , by P2.14, $\tau = \tau(\mathcal{U})$. Let (Y, g) be a T_2 compactification of $(X, \tau(\mathcal{U}))$ such that $\Psi_0(\mathcal{U}) = [(Y, g)]$. Properties i) and ii) follow easily from R13.1.2.

Corollary R13.1.7 Let $\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ be a collection of separated, totally bounded uniformities for a set X and let $\mathcal{U} = \vee\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. Then $\Psi_0(\mathcal{U})$ is the compactification class which is the supremum of the classes $\{\Psi_0(\mathcal{U}_\alpha) : \alpha \in \Delta\}$.

Proof: This follows easily from R13.1.2.

Because a T_2 compactification class of an infinite set X is a class and not a set, strictly speaking one cannot treat the compactification classes as a complete upper semi-lattice. However, the correspondence Ψ_0 provides $TBS(X)$ as a set-theoretic representation of this entity. As usual, loose statements such as $(Y, g) = \vee\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$ will be used with the understanding that the equation can be made set-theoretically meaningful by reference to the corresponding members of $TBS(X)$.

Representation of Suprema

The first proposition, a version of R3.1.2 in [6] with uniform spaces used instead of topological spaces, constructs a representation of a finite supremum.

Proposition R13.2.1 Let X be an infinite set, and let $\mathcal{U}_1, \ldots, \mathcal{U}_n$ be in $TBS(X)$ with $\Psi_0(\mathcal{U}_i) = [(Y_i, f_i)]$. Let $\mathcal{U} = \vee_{i=1}^n \mathcal{U}_i$, and let $g : X \to \Pi_{i=1}^n Y_i$ by $g(x) = (f_1(x), \ldots, f_n(x))$. Then

i)$(\overline{g[X]}, g)$ is a T_2 compactification of $(X, \tau(\mathcal{U}))$, where $\overline{g[X]}$ denotes the closure in the product $\Pi_{i=1}^n Y_i$.

ii) $\Psi_0(\mathcal{U}) = [(\overline{g[X]}, g)]$.

Proof: Clearly g is one-to-one and $g[X]$ is dense in the compact T_2 space $\overline{g[X]}$. Let $\mathcal{V}_i$ be the unique uniformity for Y_i and π_i the projection from $\Pi_{i=1}^n Y_i$ to Y_i. By R1.6a, for each i, $\pi_i \circ g = f_i$ is a uniform embedding from $(X, \mathcal{U}_i)$ to $(Y_i, \mathcal{V}_i)$. Since $\mathcal{U}_i \subseteq \mathcal{U}$, $\pi_i \circ g$ is also uniformly continuous from $(X, \mathcal{U})$ to $(Y_i, \mathcal{V}_i)$, and so g is uniformly continuous from from $(X, \mathcal{U})$ to $\Pi_{i=1}^n (Y_i, \mathcal{V}_i)$, because the product uniformity is the weak uniformity generated by the projections. Now consider a basic entourage, $\cap_{i=1}^n U_i$, where each U_i is in $\mathcal{U}_i$, and let V_i be in $\mathcal{V}_i$ such that $(f_i \times f_i)[U_i] = (f_i[X] \times f_i[X]) \cap V_i$. Then $V = \cap_{i=1}^n (\pi_i \times \pi_i)^{-1}[V_i]$ is in the product uniformity on $\Pi_{i=1}^n Y_i$, and it is routine to check the containment $(g[X] \times g[X]) \cap V \subseteq (g \times g)[\cap_{i=1}^n U_i]$. These properties show that g is a uniform embedding, and so i) holds. Property ii) follows from R1.6a.

To represent a supremum of an infinite family of compactifications, the inverse limit of a suitable inverse spectrum will be used, as in [6]. The proofs are very similar, with some minor adjustments needed to incorporate uniformities. Notation consistent with that in Appendix II of Dugundji [2] is used, but the presentation does not refer to Dugundji's general results.

Assumptions for the rest of this subsection: Let X be an infinite set and assume $\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ is a non-empty family from $TBS(X)$. Let $\mathcal{U} = \vee\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. For each $\alpha \in \Delta$, let (Y_α, f_α) be a T_2 compactification of X such that $\Psi_0(\mathcal{U}_\alpha) = [(Y_\alpha, f_\alpha)]$. Let Δ^* denote the set of all non-empty, finite subsets of Δ,and let $\mathcal{C}$ be a cofinal subset of Δ^*. For $\delta \in \mathcal{C}$ of cardinality 1, say $\delta = \{\alpha\}$, let $W_\delta = Y_\alpha$ and $F_\delta = f_\alpha$. If $|\delta| \geq 2$, let $F_\delta : X \to \prod\{Y_\alpha : \alpha \in \delta\}$ be defined by $F_\delta(x)(\alpha) = f_\alpha(x)$ and let $W_\delta = \overline{F_\delta[X]}$, where the closure is in the product. Since δ is finite, R13.2.1 applies: $(W_\delta, F_\delta) = \bigvee\{(Y_\alpha, f_\alpha) : \alpha \in \delta\}$. For $\delta, \gamma \in \mathcal{C}$ with $\delta \subseteq \gamma$, let $\pi_{\gamma\delta}$ denote the restiction to W_γ of the projection described by $\pi_{\gamma\delta}(y)(\alpha) = y(\alpha)$ for $\alpha \in \delta$. Note that $\pi_{\gamma\delta} \circ F_\gamma = F_\delta$ and $\pi_{\gamma\delta}$ is uniformly continuous.

Lemma R13.2.2 For $\delta \subseteq \gamma$, $\pi_{\gamma\delta}$ maps W_γ onto W_δ.

Proof: Same as the proof of R3.2.1.

Lemma R13.2.3 Let δ,γ,ϵ be in $\mathcal{C}$ with $\delta \subseteq \gamma \subseteq \epsilon$. Then $\pi_{\gamma\delta} \circ \pi_{\epsilon\gamma} = \pi_{\epsilon\delta}$.

Proof: Clear from the definition.

Some additional assumptions are needed for this subsection: In $\prod\{W_\delta : \delta \in \mathcal{C}\}$, let $S = \{y : \delta, \gamma \in \mathcal{C} \text{ with } \delta \subseteq \gamma \Rightarrow y(\delta) = \pi_{\gamma\delta}(y(\gamma))\}$. For $x \in X$, y_x in $\prod\{W_\delta : \delta \in \mathcal{C}\}$ is defined by $y_x(\delta) = F_\delta(x)$.

Lemma R13.2.4 S is closed in $\prod\{W_\delta : \delta \in \mathcal{C}\}$ and, for all $x \in X$, $y_x \in S$.

Proof: Same as the proof of R3.2.3.

In Dugundji's terminology [2], $\{W_\epsilon; \pi_{\gamma\delta}\}$ is an inverse spectrum of topological spaces over $\mathcal{C}$, with spaces W_ϵ and connecting maps $\pi_{\gamma\delta}$. S is the inverse limit space of the spectrum.

Additional assumptions for this subsection: Let $f : X \to S$ be defined by $f(x) = y_x$. For $\epsilon \in \mathcal{C}$, let ρ_ϵ denote the projection from $\prod\{W_\delta : \delta \in \mathcal{C}\}$ onto W_ϵ.

Lemma R13.2.5 f is a uniform embedding from $(X,\mathcal{U})$.

Proof: First, f is one-to-one as in the proof of R3.2.4. Also, $\rho_\delta \circ f = F_\delta$ is uniformly continuous from $(X, \vee\{\mathcal{U}_\alpha : \alpha \in \delta\})$ for every δ in $\mathcal{C}$ and, since $\mathcal{U}$ is larger, from $(X,\mathcal{U})$. Thus f is uniformly continuous, one-to-one, and onto $f[X]$. Now let $U \in \mathcal{U}$. Since $\mathcal{C}$ is co-final, there is $\epsilon \in \mathcal{C}$ and $\{U_\alpha : \alpha \in \epsilon\}$ such that each $U_\alpha \in \mathcal{U}_\alpha$ (possibly $U_\alpha = X \times X$) and $\cap\{U_\alpha : \alpha \in \epsilon\} \subseteq U$. As in the proof of R13.2.1 F_ϵ is a uniform embedding from $(X, \vee\{U_\alpha : \alpha \in \epsilon\})$, and so there exists V_ϵ in the unique uniformity for W_ϵ such that $F_\epsilon \times F_\epsilon[U] = (F_\epsilon[X] \times F_\epsilon[X]) \cap V_\epsilon$. Then $(\rho_\epsilon \times \rho_\epsilon)^{-1}[V_\epsilon] \cap (S \times S)$ is in the unique uniformity for S and it is easy to check that $(\rho_\epsilon \times \rho_\epsilon)^{-1}[V_\epsilon] \cap (f[X] \times f[X]) \subseteq (f \times f)[U]$. Thus f is also uniformly open and so a uniform embedding from $(X,\mathcal{U})$.

Lemma R13.2.6 Let B be a non-empty open subset of S and let $y \in B$. Then there exist $\gamma \in \mathcal{C}$ and $\{G_\alpha : \alpha \in \gamma\}$ with G_α open in Y_α such that

$$y \in \rho_\gamma^{-1}[(\prod\{G_\alpha : \alpha \in \gamma\}) \cap W_\gamma] \bigcap S \subseteq B.$$

Proof: Same as the proof of R3.2.5.

Proposition R13.2.7 (S, f) is a T_2 compactification of $(X, \tau(\mathcal{U}))$.

Proof: Same as the proof of R3.2.6.

Theorem R13.2.8 $(S, f) = \bigvee\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$.

Proof: By R13.2.5 and R1.6a, $\Psi_0(\mathcal{U}) = [(S, f)]$. The conclusion now follows from R13.1.2.

An Example Related to the Frink Question

Let $X_0 = [0, 1]$, τ_0 the discrete topology on X_0, and $\mathcal{U}_0 = \mathcal{U}_m \vee \mathcal{V}$, where $\mathcal{U}_m$ is the smallest element of $\mathcal{TB}((X_0, \tau_0))$ and $\mathcal{V}$ is the usual uniformity generated by the absolute value metric on X_0. Note that both $\mathcal{V}$ and $\mathcal{U}_m$ are in $TBS\,(X_0)$. Let υ denote $\tau(\mathcal{V})$, the usual topology on X_0. Throughout this subsection X_0, τ_0, υ, $\mathcal{V}$ and $\mathcal{U}_0$ will always represent the structures just described.

In [10] it is shown that every zero-dimensional compactification of a discrete space can be generated by a normal basis and that the compactification associated with $\mathcal{U}_0$ is not zero-dimensional. The question of interest: Is the compactification associated with $\mathcal{U}_0$ generated by a normal basis? A negative answer would settle the Frink question. (See subsection R9.3.) Methods used to obtain a positive answer might suggest a more general approach to it.

Using R13.2.1 a representation of the compactification corresponding to $\mathcal{U}_0$ is constructed and a necessary condition on a normal basis (if one exists) is developed.

Additional notation for this subsection: Given an infinite set X, δ denotes the discrete topology on X and $X^+ = X \cup \{\infty\}$, where ∞ denotes some object not in X. X^+ is assumed to have the one-point compactification topology induced by (X, δ). The closure of A in a topological space is denoted $\overline{A}$.

Lemma R13.3.1 Let (X, τ) be a $T_{3\frac{1}{2}}$ space with X infinite. Let (Y, f) be a T_2 compactification of (X, τ), and let $h : X \to Y \times X^+$ by $h(x) = (f(x), x)$. Then

i)If $(a, b) \in \overline{h[X]}$ and $b \neq \infty$, then $a = f(b)$.

ii) If $x \in X$, then $\{(f(x), x)\}$ is open in $\overline{h[X]}$.

Proof: For i), let $(a, b) \in \overline{h[X]}$ and let $\{x_\alpha\}$ be a net in X such that $\{(f(x_\alpha), x_\alpha)\}$ converges to (a, b). Since $b \neq \infty$, $\{b\}$ is open in X^+ and so $x_\alpha = b$ eventually. Since $\{f(x_\alpha)\} \to a$, $f(x_\alpha) = f(b)$ eventually, and Y is T_2, $a = f(b)$. For ii), note that, for any $x \in X$, part i) implies $\overline{h[X]} \cap (Y \times \{x\}) = \{(f(x), x)\}$.

Definition R13.3.2 $g : X_0 \to X_0 \times X_0^+$ is defined by $g(x) = (x, x)$. X_1 is defined to be $\overline{g[X_0]}$, where the closure is in the product topology determined by using (X_0, υ) as the first factor.

Proposition R13.3.3 (X_1, g) is a T_2 compactification of (X_0, τ_0) and $\Psi_0(\mathcal{U}_0) = [(X_1, g)]$.

Proof: Because (X_0, υ) is compact, $\Psi_0(\mathcal{V}) = [(X_0, id)]$ where id the identity map on X_0. As noted in [7], $\Psi_0(\mathcal{U}_m) = [(X_0^+, id)]$. The result is now immediate from R13.2.1.

The next lemma makes a connection that allows the use of X_1 in the investigation of the Frink question for the example. Notation from [4] is used.

Lemma R13.3.4 Let (X, τ) be a $T_{3\frac{1}{2}}$ space with normal basis $\mathcal{Z}$. Let (Y, f) be a T_2 compactification of (X, τ) with (Y, f) equivalent to $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$. Let $h : \omega(\mathcal{Z}) \to Y$ be the homeomorphism such that $h \circ \iota_{\mathcal{Z}} = f$. Then

i) For every $Z \in \mathcal{Z}$, $h[Z^\omega] = \overline{f[Z]}$.

ii)$\{\overline{f[Z]} : Z \in \mathcal{Z}\}$ is a normal basis for Y.

iii) For every $Z_1, Z_2 \in \mathcal{Z}$, $\overline{f[Z_1] \cap f[Z_2]} = \overline{f[Z_1]} \cap \overline{f[Z_2]}$.

Proof: For i), let Z be in $\mathcal{Z}$. It is not hard to check that $c(\iota_{\mathcal{Z}}[Z]) = Z^\omega$, where c denotes the closure in $\omega(\mathcal{Z})$. Since h preserves closure and $h \circ \iota_{\mathcal{Z}} = f$, $h[Z^\omega] = \overline{f[Z]}$. For ii), as noted in P3.6, $\{Z^\omega : Z \in \mathcal{Z}\}$ is a normal basis for $\omega(\mathcal{Z})$. The homeomorphism transfers the normal basis the domain to a normal basis in the range, and so the conclusion follows. For iii), let Z_1 and Z_2 be in $\mathcal{Z}$. Since $(Z_1 \cap Z_2)^\omega = Z_1^\omega \cap Z_2^\omega$, f is one-to-one, and i) holds, we have

$$\overline{f[Z_1] \cap f[Z_2]} = \overline{f[Z_1 \cap Z_2]} = h((Z_1 \cap Z_2)^\omega) = h[Z_1^\omega] \cap h[Z_2^\omega] = \overline{f[Z_1]} \cap \overline{f[Z_2]}.$$

The previous lemma takes a small step in the direction of the following theorem, which can be found in Chandler [1; pp. 93-94]. Only the lemma will be used in this subsection.

Theorem [Steiner-Šapiro] Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let (Y, f) be a T_2 compactification of (X, τ). Then the following are equivalent

i) There is a normal basis $\mathcal{Z}$ for (X, τ) such that (Y, f) is equivalent to $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$.

ii) There is a base ring of closed sets $\mathcal{A}$ in Y such that $\overline{A \cap f[X]} = A$ for every $A \in \mathcal{A}$.

iii) There is a base ring $\mathcal{B}$ of closed sets in Y such that, if $B \in \mathcal{B}$ and $B \neq \emptyset$, then $B \cap f[X] \neq \emptyset$.

The next definition and lemma are standard in introductory topology texts. They are recorded here to establish terminology and a point of reference.

Definition R13.3.5 Let (X, τ) be a topological space and let $A \subseteq X$. A point $x \in X$ is a limit point of A provided every open set containing x contains a point of A other than x. The derived set of A, denoted A', is the set of all limit points of A.

Lemma R13.3.6 Let (X, τ) be a topological space and let A, B be subsets of X. Then

i) $\overline{A} = A \cup A'$.

ii) A is closed if and only if $A' \subseteq A$.

iii) If $A \subseteq B$, then $A' \subseteq B'$.

iv) $(A \cup B)' = A' \cup B'$ and $(A \cap B)' \subseteq A' \cap B'$.

v) If X is T_1, then A' is closed.

In discussions of the example (X_0 etc.) in this subsection, given $A \subseteq X_0$, the derived set of A will always be taken relative to (X_0, υ), i.e., X_0 with the usual topology. For $x \in X_0$, $B_\epsilon(x)$ denotes the ϵ-ball centered at x, i.e., simply $(x - \epsilon, x + \epsilon) \cap X_0$.

Lemma R13.3.7 Let $A \subseteq X_0$. Then $\overline{g[A]} = \{(x, x) : x \in A\} \cup \{(x, \infty) : x \in A'\}$.

Proof: Note first that $\{(x, x) : x \in A\} = g[A] \subseteq \overline{g[A]}$. Now let $x \in A'$ and let G be open in X_1 with $(x, \infty) \in G$. There exist $\epsilon > 0$ and S co-finite in X_0 s.t. $(x, \infty) \in [B_\epsilon(x) \times (S \cup \{\infty\})] \cap X_1 \subseteq G$. Since S is co-finite, there is $\gamma > 0$ with $\gamma < \epsilon$ such that $B_\gamma(x) \cap (X_0 - S) \subseteq \{x\}$. For $a \neq x$ in $B_\gamma(x) \cap A$, $a \in S$ and so $g(a) = (a, a) \in [B_\gamma(x) \times (S \cup \{\infty\})] \cap X_1 \subseteq G$, i.e., $G \cap g[A] \neq \emptyset$. Thus $(x, \infty) \in \overline{g[A]}$.

For the opposite containment, if $b \in X_0 - A$, by R13.3.1ii $\{(b, b)\}$ is an open set in X_1. It has an empty intersection with $g[A]$, i.e., $(b, b) \notin \overline{g[A]}$. Thus $\overline{g[A]} \cap g[X_0] = g[A]$. Now let $(x, \infty) \in \overline{g[A]}$ and let $\epsilon > 0$. The X_1-open set $[B_\epsilon(x) \times ((X_0 - \{x\}) \cup \{\infty\})] \cap X_1$ contains (x, ∞) and so must contain $g(a) = (a, a)$ for some $a \in A$. Thus $a \neq x$ and $a \in B_\epsilon(x)$. Thus $x \in A'$.

Proposition R13.3.8 Assume $\mathcal{Z}$ is a normal basis for (X_0, τ_0) and $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$ is equivalent to (X_1, g). Let Z_1, Z_2 be in $\mathcal{Z}$. Then $(Z_1 \cap Z_2)' = Z_1' \cap Z_2'$.

Proof: By R13.3.6iv it is sufficient to show $(Z_1 \cap Z_2)' \supseteq Z_1' \cap Z_2'$. Let $x \in Z_1' \cap Z_2'$. By R13.3.7 (x, ∞) is in $\overline{g[Z_1]} \cap \overline{g[Z_2]}$, which equals $\overline{g[Z_1] \cap g[Z_2]}$ by R13.3.4iii. Since g is one-to-one, $(x, \infty) \in \overline{g[Z_1 \cap Z_2]}$. Another application of R13.3.7 shows $x \in (Z_1 \cap Z_2)'$.

Note that $\{Z \subseteq X_0 : Z$ is finite or co-finite$\}$, the normal basis for the one-point compactification of (X_0, τ_0), has the property in the conclusion of R13.3.8. The next example does not.

In the example the set $\mathcal{A}$ of all finite unions of closed intervals in X_0 will be needed. The usual interval notation, e.g., $[a, b] = \{x : a \leq x \leq b\}$, is used, and the cases $a > b$ (the empty interval) and $a = b$ are included. With these assumptions $\mathcal{A}$ is closed under finite unions and finite intersections.

Example R13.3.9 Let $\mathcal{Z} = \{A \cap S : A \in \mathcal{A}$ and S is co-finite in $X_0\}$. Then $\mathcal{Z}$ is a normal basis for (X_0, τ_0) but $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$ is not equivalent to (X_1, g).

Proof: Since $\mathcal{Z}$ contains all finite and co-finite subsets of X_0, it is a closed base for the discrete space (X_0, τ_0) and it satisfies P3.1iii, the disjunctive property of a normal base. Clearly it is closed under finite intersections. For finite unions, $(A_1 \cap S_1) \cup (A_2 \cap S_2) = (A_1 \cup A_2) \cap [(S_1 \cup S_2) \cap (A_1 \cup S_2) \cap (A_2 \cup S_1)]$, which is in $\mathcal{Z}$ since the co-finite sets are closed under finite intersections and supersets. Before verifying the normality property (P3.1iv), we observe that $\mathcal{Z}$ is closed under complementation: First, $X_0 - [a, b] = [0, a) \cup (b, 1] = ([0, a] \cup [b, 1]) \cap (X_0 - \{a, b\})$, which is in $\mathcal{Z}$. This combined with the closure of $\mathcal{Z}$ under finite intersections and DeMorgan's law shows that, if $A \in \mathcal{A}$, then $X_0 - A$ is in $\mathcal{Z}$. Since $\mathcal{Z}$ contains all complements of co-finite sets and is closed under finite unions, another application of DeMorgan's law shows that $\mathcal{Z}$ is closed under complementation. Now let $Z_1, Z_2 \in \mathcal{Z}$ with $Z_1 \cap Z_2 = \emptyset$. Let $C_i = X_0 - Z_i$. Then $C_i \in \mathcal{Z}$, $C_1 \cup C_2 = X_0$, and $Z_i \cap C_i = \emptyset$ so that P3.1iv holds and $\mathcal{Z}$ is a normal basis. Next note that $Z_1 = [0, \frac{1}{2})$, $Z_2 = (\frac{1}{2}, 1]$ are both in $\mathcal{Z}$, $(Z_1 \cap Z_2)' = \emptyset$, and $Z_1' \cap Z_2' = \{\frac{1}{2}\}$. By R13.3.8 $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$ is not equivalent to (X_1, g).

Lifting Property

This subsection shows that the proposition about lifting extensions to a supremum (R3.3.5 or R7.1.5) also generaizes to mixed suprema. The term 'extension' will be used as in definition R3.3.1.

Proposition R13.4.1 Let Δ be a non-empty set, let X and Z be infinite sets, and let $h : X \to Z$. Assume, for every $\alpha \in \Delta$, there exist $\mathcal{U}_\alpha$ in $TBS(X)$ and $\mathcal{V}_\alpha$ in $TBS(Z)$ such that $h : (X, \mathcal{U}_\alpha) \to (Z, \mathcal{V}_\alpha)$ is uniformly continuous.
Then $h : (X, \vee\{\mathcal{U}_\alpha : \alpha \in \Delta\}) \to (Z, \vee\{\mathcal{V}_\alpha : \alpha \in \Delta\})$ is also uniformly continuous.

Proof: A basic set in $\vee\{\mathcal{V}_\alpha : \alpha \in \Delta\}$ has the form $\cap\{V_\alpha : \alpha \in \delta\}$, where δ is a finite subset of Δ and each V_α is in $\mathcal{V}_\alpha$. $(h \times h)^{-1}[\cap\{V_\alpha : \alpha \in \delta\}] = \cap\{(h \times h)^{-1}[V_\alpha] : \alpha \in \delta\}$. By the hypotheses and the definition of the supremum of uniformities, this is clearly in $\vee\{\mathcal{U}_\alpha : \alpha \in \Delta\}$.

Proposition R13.4.2 Let Δ be a non-empty set, let X and Z be infinite sets, and let $h : X \to Z$. Assume, for every $\alpha \in \Delta$, there exist T_2 compactifications (Y_α, f_α) and (W_α, g_α) of X and Z respectively and $H_\alpha : Y_\alpha \to W_\alpha$, a continuous extension of h. Then there exists $H : \vee\{Y_\alpha : \alpha \in \Delta\} \to \vee\{W_\alpha : \alpha \in \Delta\}$, a continuous extension of h.

Proof: For each $\alpha \in \Delta$ there exist $\mathcal{U}_\alpha$ in $TBS(X)$ and $\mathcal{V}_\alpha$ in $TBS(Z)$ such that $\Psi_0(\mathcal{U}_\alpha) = [(Y_\alpha, f_\alpha)]$ and $\Psi_0(\mathcal{V}_\alpha) = [(W_\alpha, g_\alpha)]$. By hypothesis H_α extends h and is continuous from the compact Y_α to W_α. Thus H_α is uniformly continuous. By R1.6a $h = g_\alpha^{-1} \circ H_\alpha \circ f_\alpha$ is uniformly continuous from $(X, \mathcal{U}_\alpha)$ to $(Z, \mathcal{Z}_\alpha)$. By R13.4.1 $h : (X, \vee\{\mathcal{U}_\alpha : \alpha \in \Delta\}) \to (Z, \vee\{\mathcal{V}_\alpha : \alpha \in \Delta\})$ is also uniformly continuous. Since the two uniformity suprema correspond via Ψ_0 to the suprema of the compactifications, by R7.1.3 there exists $H : \vee\{Y_\alpha : \alpha \in \Delta\} \to \vee\{W_\alpha : \alpha \in \Delta\}$, a continuous extension of h.

Lattice Questions

It is natural to ask whether the complete upper semi-lattice of T_2 compactifications based on a particular set can be made into a lattice. The answer is easier and less interesting than that of the similar question in the case of the compactifications of a fixed $T_{3\frac{1}{2}}$ space.

Lemma R13.5.1 Let X be a set, let $\mathcal{U}_1$ and $\mathcal{U}_2$ be in $TBS(X)$, and let $\Psi(\mathcal{U}_i) = [(Y_i, f_i)]$ for $i = 1, 2$. Assume (Z, g) is a T_2 compactification of X which is an infimum, i.e., $[(Z, g)] \le [(Y_i, g_i)]$ for $i = 1, 2$ and, if (W, h) is a T_2 compactification of X with $[(W, h)] \le [(Y_i, g_i)]$ for $i = 1, 2$, then $[(W, h)] \le [(Z, g)]$. Then $\mathcal{U}_1 \wedge \mathcal{U}_2$ is in $TBS(X)$ and $\Psi_0(\mathcal{U}_1 \wedge \mathcal{U}_2) = [(Z, g)]$.

Proof: Let $\mathcal{V}$ be the element of $TBS(X)$ such that $\Psi_0(\mathcal{V}) = [(Z, g)]$. By R13.1.2 and the first hypothesis, $\mathcal{V} \subseteq \mathcal{U}_1 \wedge \mathcal{U}_2$, and so the latter must be separated. As a subset of a totally bounded uniformity, it is also totally bounded, i.e., $\mathcal{U}_1 \wedge \mathcal{U}_2$ is in $TBS(X)$. Let $[(W, h)] = \Psi_0(\mathcal{U}_1 \wedge \mathcal{U}_2)$. By R13.1.2 and the second hypothesis, $[(W, h)] \le [(Z, g)]$. Using R13.1.2 again, $\mathcal{U}_1 \wedge \mathcal{U}_2 \subseteq \mathcal{V}$. Thus they are equal and the second conclusion follows.

Proposition R13.5.2 Let X be a set, let $\mathcal{U}_1$ and $\mathcal{U}_2$ be in $TBS(X)$, and let $\Psi(\mathcal{U}_i) = [(Y_i, f_i)]$ for $i = 1, 2$. There is a T_2 compactification of X which is an infimum of $[(Y_1, f_1)]$ and $[(Y_2, f_2)]$ if and only if $\mathcal{U}_1 \wedge \mathcal{U}_2$ is separated.

Proof: If an infimum exists, $\mathcal{U}_1 \wedge \mathcal{U}_2$ is in $TBS(X)$ by R13.5.1 and so is separted. Conversely, if $\mathcal{U}_1 \wedge \mathcal{U}_2$ is separated, as a subset of a totally bounded uniformity, it is also totally bounded, i.e., $\mathcal{U}_1 \wedge \mathcal{U}_2$ is in $TBS(X)$. Let $[(W, h)] = \Psi_0(\mathcal{U}_1 \wedge \mathcal{U}_2)$. That $[(W, h)]$ is the required infimum follows easily from R13.1.2.

Proposition R13.5.3 Let X be a countably infinite set. Then there exist $\mathcal{U}_1$ and $\mathcal{U}_2$ in $TBS(X)$ such that $\mathcal{U}_1 \wedge \mathcal{U}_2$ is not separated.

Proof: Let $S = \{0\} \cup \{\frac{1}{n} : n$ is a positive integer$\}$ have the usual absolute value topology. Let $a, b \in X$ with $a \ne b$. Let $f : S \to X$ be bijective with $f(0) = a$. Define $g : S \to X$ by $g(0) = b$, $g(f^{-1}(b)) = a$, and $g(s) = f(s)$ for all other $s \in S$. Note that g is also bijective. Let τ_1 and τ_2 be the topologies for X which make f, respectively g, homeomorphisms. Since (X, τ_1) is compact and T_2, there is a unique uniformity $\mathcal{U}_1$ in $TBS(X)$ such that $\tau(\mathcal{U}_1) = \tau_1$. Similarly let $\mathcal{U}_2$ be the unique uniformity in $TBS(X)$ such that $\tau(\mathcal{U}_2) = \tau_2$. To see that $\mathcal{U}_1 \wedge \mathcal{U}_2$ is not separated, suppose otherwise, and pick U in $\mathcal{U}_1 \wedge \mathcal{U}_2$ such that $(a, b) \notin U$. Pick $V \in \mathcal{U}_1 \wedge \mathcal{U}_2$ with $V = V^{-1}$ and $V \circ V \subseteq U$. Now V is also in $\mathcal{U}_1$ and so, by the construction of τ_1, $V[a]$, a τ_1-neighborhood of a, must contain all but finitely many elements of X. Using τ_2 similarly, we have that $V[b]$ also contains all but finitely many elements of X. These facts guarantee there is $t \in V[a] \cap V[b]$. But then (a, t) and (t, b) are both in V so that (a, b) is in $V \circ V$, a contradiction.

Let S be a set. For rest of this subsection $\mathcal{U}_m(S)$ will denote the smallest element of $\mathcal{TB}((S, \delta))$, where δ is the discrete topology on S.

Definition R13.5.4 Let X be a set and let $D \subseteq X$. For $\mathcal{U}$ in $TBS(D)$, $\mathcal{U}^* = \{U \subseteq X \times X : V \cup W \subseteq U$ for some $V \in \mathcal{U}$ and some $W \in \mathcal{U}_m(X - D)\}$.

Lemma R13.5.5 Let X be a set and let $D \subseteq X$. Let $\mathcal{U}$ be in $TBS(D)$. Then $\mathcal{U}^*$ is in $TBS(X)$.

Proof: For $S, S_1 \subseteq D \times D$ and $T, T_1 \subseteq (X - D) \times (X - D)$, $(S \cup T)^{-1} = S^{-1} \cup T^{-1}$ and, since D and $X - D$ are disjoint, $(S \cup T) \circ (S \cup T) \subseteq (S \circ S) \cup (T \circ T)$. Also, $(S \cap S_1) \cup (T \cap T_1) \subseteq (S \cup T) \cap (S_1 \cup T_1)$. Those facts make it routine to check that $\mathcal{U}^*$ is a uniformity for X. Since both $\mathcal{U}$ and $\mathcal{U}_m(X - D)$ are separated and totally bounded, $\mathcal{U}^*$ also has those properties, i.e., $\mathcal{U}^*$ is in $TBS(X)$.

Proposition R13.5.6 Let X be a set. Then $TBS(X)$ is a lattice if and only if X is finite.

Proof: If X is finite, then $TBS(X)$ is the trivial lattice containing exactly one element. Now assume X is infinite, and let D be a countably infinite subset of X. By R13.5.3 there exist $\mathcal{U}_1$ and $\mathcal{U}_2$ in $TBS(D)$ such that $\mathcal{U}_1 \wedge \mathcal{U}_2$ is not separated. Let $(a, b) \in D \times D$ with $a \neq b$ and $(a, b) \in \cap\{U : U \in \mathcal{U}_1 \wedge \mathcal{U}_2\}$. It follows easily that $(a, b) \in \cap\{U : U \in \mathcal{U}_1^* \wedge \mathcal{U}_2^*\}$ so that $\mathcal{U}_1^* \wedge \mathcal{U}_2^*$ is not separated. Thus $TBS(X)$ is not a lattice.

Of course, given any $\mathcal{U}_0$ in $TBS(X)$, $\{\mathcal{U} : \mathcal{U} \in TBS(X)$ and $\mathcal{U}_0 \subseteq \mathcal{U}\}$ is a complete lattice. R8.3 in [9] shows that such a lattice need not be distributive.

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Added Comment 2007

Chandler and Faulkner [11] state that the Frink question was resolved by Ul'yanov [12], who proved that "the assertion that every Hausdorff bicompact extension of an arbitrary separable completely regular space is an extension of Wallman type is equivalent to the continuum hypothesis."

Additional References

An asterisk indicates a reference not seen by me.

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Added Subsection 2012

The representation of an infinite supremum as an inverse limit in R13.2 used a specific representation of finite suprema. This subsection verifies the unsurprising fact that arbitrary representations of the finite suprema can be substituted in the argument.

Assumptions for the rest of this subsection: Let X be an infinite set and assume $\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ is a non-empty family from $TBS\,(X)$. Let $\mathcal{U} = \vee\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. Let Δ^* denote the set of all non-empty, finite subsets of Δ, and let $\mathcal{C}$ be a cofinal subset of Δ^*. For each $\delta \in \mathcal{C}$ assume (Y_δ, F_δ) is a compactification corresponding to $\vee\{\mathcal{U}_\alpha : \alpha \in \delta\}$. For $\delta, \gamma \in \mathcal{C}$ with $\delta \subseteq \gamma$, let $\pi_{\gamma\delta}$ denote the unique continuous surjection from Y_γ to Y_δ such that $\pi_{\gamma\delta} \circ F_\gamma = F_\delta$. Note that $\pi_{\gamma\delta}$ is uniformly continuous.

Lemma R13.Add.1 Let δ,γ,ϵ be in $\mathcal{C}$ with $\delta \subseteq \gamma \subseteq \epsilon$. Then $\pi_{\gamma\delta} \circ \pi_{\epsilon\gamma} = \pi_{\epsilon\delta}$.

Proof: $(\pi_{\gamma\delta} \circ \pi_{\epsilon\gamma}) \circ F_\epsilon = \pi_{\gamma\delta} \circ (\pi_{\epsilon\gamma} \circ F_\epsilon) = \pi_{\gamma\delta} \circ F_\gamma = F_\delta$. Since $\pi_{\epsilon\delta}$ is the unique such map, the equality follows.

Some additional definitions are needed for this subsection: In $\prod\{Y_\delta : \delta \in \mathcal{C}\}$, let $S = \{y : \delta, \gamma \in \mathcal{C}$ with $\delta \subseteq \gamma \Rightarrow y(\delta) = \pi_{\gamma\delta}(y(\gamma))\}$. For $x \in X$, y_x in $\prod\{Y_\delta : \delta \in \mathcal{C}\}$ is defined by $y_x(\delta) = F_\delta(x)$.

Lemma R13.Add.2 S is closed in $\prod\{Y_\delta : \delta \in \mathcal{C}\}$ and, for all $x \in X$, $y_x \in S$.

Proof: Let $\{x_t\}$ be a net in S converging to p and $\delta \subseteq \gamma$ for some $\gamma, \delta \in \mathcal{C}$. By definition of S, for every t, $\pi_{\gamma\delta}(x_t(\gamma)) = x_t(\delta)$. Since convergence in the product is pointwise, $\{x_t(\gamma)\}$ converges to $p(\gamma)$ and $\{x_t(\delta)\}$ to $p(\delta)$. Since $\pi_{\gamma\delta}$ is continuous, $\pi_{\gamma\delta}(x_t(\gamma))$ converges to $\pi_{\gamma\delta}(p(\gamma))$. Since limits are unique in a T_2 space, $\pi_{\gamma\delta}(p(\gamma)) = p(\delta)$. Thus $p \in S$. Now let $x \in X$ and $\delta \subseteq \gamma$ for some $\gamma, \delta \in \mathcal{C}$. By the choice of $\pi_{\gamma\delta}$, $\pi_{\gamma\delta}(y_x(\gamma)) = \pi_{\gamma\delta}(F_\gamma(x)) = F_\delta(x) = y_x(\delta)$. Thus $y_x \in S$.

In Dugundji's terminology [2], $\{Y_\epsilon; \pi_{\gamma\delta}\}$ is an inverse spectrum of topological spaces over $\mathcal{C}$, with spaces Y_ϵ and connecting maps $\pi_{\gamma\delta}$. S is the inverse limit space of the spectrum.

Additional definitions for this subsection: Let $f : X \to S$ be defined by $f(x) = y_x$. For $\epsilon \in \mathcal{C}$, let ρ_ϵ denote the projection from $\prod\{Y_\delta : \delta \in \mathcal{C}\}$ onto Y_ϵ.

Lemma R13.Add.3 Let $\gamma \in \mathcal{C}$ and let $U \in \vee\{\mathcal{U}_\alpha : \alpha \in \gamma\}$. Assume V is in the unique uniformity for Y_γ with $(F_\gamma \times F_\gamma)[U] = V \cap (F_\gamma[X] \times F_\gamma[X])$.
Then $(\rho_\gamma \times \rho_\gamma)^{-1}[V] \cap (f[X] \times f[X]) \subseteq (f \times f)[U]$.

Proof: Let $(y_a, y_b) \in (\rho_\gamma \times \rho_\gamma)^{-1}[V] \cap (f[X] \times f[X])$ which means $(y_a(\gamma), y_b(\gamma)) = (F_\gamma(a), F_\gamma(b))$ is in $V \cap (F_\gamma[X] \times F_\gamma[X])$. Thus $(a, b) \in U$, i.e., $(y_a, y_b) \in (f \times f)[U]$.

Lemma R13.Add.4 f is a uniform embedding from $(X,\mathcal{U})$.

Proof: First, f is one-to-one since each F_δ is one-to-one. Also, $\rho_\delta \circ f = F_\delta$, which is uniformly continuous from $(X, \vee\{\mathcal{U}_\alpha : \alpha \in \delta\})$ for every δ in $\mathcal{C}$ by R1.6a and, since $\mathcal{U}$ is larger, from $(X,\mathcal{U})$. Thus f is uniformly continuous, one-to-one, and onto $f[X]$.

Now let $U \in \mathcal{U}$. Since $\mathcal{C}$ is co-final, there is $\epsilon \in \mathcal{C}$ and $\{U_\alpha : \alpha \in \epsilon\}$ such that each $U_\alpha \in \mathcal{U}_\alpha$ (possibly $U_\alpha = X \times X$) and $\cap\{U_\alpha : \alpha \in \epsilon\} \subseteq U$. Since F_ϵ is a uniform embedding from $(X, \vee\{\mathcal{U}_\alpha : \alpha \in \epsilon\})$, there exists V in the unique uniformity for Y_ϵ such that $F_\epsilon \times F_\epsilon[U] = (F_\epsilon[X] \times F_\epsilon[X]) \cap V$. Then $(\rho_\epsilon \times \rho_\epsilon)^{-1}[V] \cap (S \times S)$ is in the unique uniformity for S and by the previous lemma $(\rho_\epsilon \times \rho_\epsilon)^{-1}[V] \cap (f[X] \times f[X]) \subseteq (f \times f)[U]$. Thus f is also uniformly open and so a uniform embedding from $(X,\mathcal{U})$.

Lemma R13.Add.5 Let D be a finite, non-empty subset of $\mathcal{C}$ and, for each $\gamma \in D$, suppose O_γ is open in Y_γ. Let $\epsilon \in \mathcal{C}$ be such that $\cup\{\gamma : \gamma \in D\} \subseteq \epsilon$ and, for each $\gamma \in D$, let $G_\gamma = \pi_{\epsilon\gamma}^{-1}[O_\gamma]$. Then $\cap\{\rho_\gamma^{-1}[O_\gamma] : \gamma \in D\} \cap S = \cap\{\rho_\epsilon^{-1}[G_\gamma] : \gamma \in D\} \cap S$.

Proof: Let $p \in \cap\{\rho_\gamma^{-1}[O_\gamma] : \gamma \in D\} \cap S$. For each $\gamma \in D$, $\rho_\gamma(p) = p(\gamma) = \pi_{\epsilon\gamma}(p(\epsilon))$ is in O_γ so that $p(\epsilon) \in G_\gamma$. Thus $p \in \cap\{\rho_\epsilon^{-1}[G_\gamma] : \gamma \in D\} \cap S$. Each step is reversible and so the converse also holds.

Lemma R13.Add.6 $f[X]$ is dense in S.

Proof: A typical basic open set B in S has the form $B = \cap\{\rho_\gamma^{-1}[O_\gamma] : \gamma \in D\} \cap S$, where D is a finite subset of $\mathcal{C}$ and, for each $\gamma \in D$, O_γ is open in Y_γ. Since $\mathcal{C}$ is co-final, there is $\epsilon \in \mathcal{C}$ such that $\cup\{\gamma : \gamma \in D\} \subseteq \epsilon$. For each $\gamma \in D$, let $G_\gamma = \pi_{\epsilon\gamma}^{-1}[O_\gamma]$. By the previous lemma $B = \cap\{\rho_\epsilon^{-1}[G_\gamma] : \gamma \in D\} \cap S$. If $B \neq \emptyset$, $\cap\{G_\gamma : \gamma \in D\}$ is non-empty, and so, by the density of $F_\epsilon[X]$ in Y_ϵ, there is $x \in X$ with $F_\epsilon(x) \in \cap\{G_\gamma : \gamma \in D\}$. Since $y_x(\epsilon) = F_\epsilon(x)$, $f(x) = y_x \in \cap\{\rho_\epsilon^{-1}[G_\gamma] : \gamma \in D\} \cap S = B$.

Proposition R13.Add.7 (S, f) is a T_2 compactification of $(X, \tau(\mathcal{U}))$.

Proof: As a closed subset of the compact, T_2 space $\prod\{Y_\delta : \delta \in \mathcal{C}\}$, S is compact and T_2. This, R13.Add.4, and R13.Add.6 yield the conclusion.

Proposition R13.Add.8 $[(S, F)]$ corresponds to the uniformity $\mathcal{U} = \vee\{\mathcal{U}_\alpha : \alpha \in \Delta\}$.

Proof: This is immediate from R13.Add.4, R13.Add.6, and R1.6a.

Proposition R13.Add.9 The compactification class of (S, F) is the supremum of the classes of (Y_γ, F_γ) over all $\gamma \in \mathcal{C}$.

Proof: Since $\mathcal{U} = \vee\{\mathcal{U}_\alpha : \alpha \in \Delta\} = \vee\{\vee\{\mathcal{U}_\alpha : \alpha \in \gamma\} : \gamma \in \mathcal{C}\}$, this follows easily from R13.1.2.