S-maps

In this section supremum-generated maps from the compactifications of a $T_{3\frac{1}{2}}$ space (X, τ) to the compactifications of another $T_{3\frac{1}{2}}$ space (X, σ), where τ and σ are (not necessarily distinct) topologies on the same set X, are introduced, and some of their basic properties are established. This will be done by focusing on the totally bounded uniformities for X, as in [9].

Because different topologies will be used on the same set X, notational adjustments are needed. Given a $T_{3\frac{1}{2}}$ space (X, τ), $\mathcal{TB}(X, \tau)$ denotes the set $\{\mathcal{U} : \mathcal{U}$ is a totally bounded uniformity for X and $\tau(\mathcal{U}) = \tau\}$. Given a set X, δ denotes the discrete topology on X. The usual notation for the one point compactification of an infinite discrete space will be used: given an infinite set X, $X^+ = X \cup \{\infty\}$, where ∞ denotes some object not in X. X^+ is assumed to have the one-point compactification topology induced by (X, δ).

Basic Facts and Examples

Definition R15.1.1 Let (X, τ) be a $T_{3\frac{1}{2}}$ space, let $\mathcal{V}$ be a totally bounded uniformity on the set X, and let $\sigma = \tau \vee \tau(\mathcal{V})$. The S-map $S_{\mathcal{V}} : \mathcal{TB}(X, \tau) \to \mathcal{TB}(X, \sigma)$ is defined by $S_{\mathcal{V}}(\mathcal{U}) = \mathcal{V} \vee \mathcal{U}$.

In this definition, notice that $\mathcal{V}$ is not assumed to be separated and that the use of $\mathcal{TB}(X, \sigma)$ as the image space is justified by P2.13 and P2.14 in [3]. As shown in [4], $\mathcal{TB}(X, \tau)$ and $\mathcal{TB}(X, \sigma)$ are semi-lattice-theoretic representations of the T_2 compactifications of (X, τ) and (X, σ) respectively, and so any $S_{\mathcal{V}}$ can be regarded as a map of compactifications of the first to compactifications of the second.

Proposition R15.1.2 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{V}$ be a totally bounded uniformity on the set X. Let $\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ be a non-empty family in $\mathcal{TB}(X, \tau)$. Then $S_{\mathcal{V}}(\vee\{\mathcal{U}_\alpha : \alpha \in \Delta\}) = \vee\{S_{\mathcal{V}}(\mathcal{U}_\alpha) : \alpha \in \Delta\}$.

Proof: This follows from the generalized commutivity and associativity of the supremum operation.

The examples in the next definition will be studied in the rest of this section. Given an equivalence relation E on set X, $\mathcal{U}_E$ denotes the uniformity generated by E, as defined in R5.2.1 of [5]. $\mathcal{U}_M$ and $\mathcal{U}_m$ denote the largest, respectively smallest, elements of $\mathcal{TB}(X, \delta)$, as described in [4].

Definition R15.1.3 Let (X, τ) be a $T_{3\frac{1}{2}}$ space. Let E be an equivalence relation on X with finitely many equivalence classes. Then

i) S_E is the S-map with $\mathcal{V} = \mathcal{U}_E$ from $\mathcal{TB}(X, \tau)$ to $\mathcal{TB}(X, \tau \vee \tau(\mathcal{U}_E))$.

ii) S_M is the S-map with $\mathcal{V} = \mathcal{U}_M$ from $\mathcal{TB}(X, \tau)$ to $\mathcal{TB}(X, \delta)$.

iii) S_m is the S-map with $\mathcal{V} = \mathcal{U}_m$ from $\mathcal{TB}(X, \tau)$ to $\mathcal{TB}(X, \delta)$.

Lemma R15.1.4 Let (X, τ) be a $T_{3\frac{1}{2}}$ space, and let $\mathcal{U}$ be in $\mathcal{TB}(X, \tau)$. Then $\mathcal{U} \subseteq \mathcal{U}_M$.

Proof: By R6.3.4 in [6] $\mathcal{U}_M = \{U \subseteq X \times X : \cup_{i=1}^{n} A_i \times A_i \subseteq U$ where $\cup_{i=1}^{n} A_i = X\}$. Since the uniformity $\mathcal{U}$ is totally bounded, every $U \in \mathcal{U}$ must contain some $\cup_{i=1}^{j} O_i \times O_i$, where $O_1, \ldots, O_j$ is a finite τ-open cover of X. The conclusion is immediate.

Example R15.1.5 By the lemma S_M is the constant map, $S_M(\mathcal{U}) = \mathcal{U}_M$. If X is infinite, then S_M is not onto. If $\mathcal{TB}(X, \tau)$ has at least two elements, S_M is not one-to-one.

Example R15.1.6 If $\tau = \delta$, then S_m is the identity map and so one-to-one and onto.

In the next 4 items, which show that S_m is always one-to-one, the following assumptions and notation will be used: Let (X,τ) be a $T_{3\frac{1}{2}}$ space, and let $\mathcal{U}$ and $\mathcal{V}$ be in $\mathcal{TB}(X,\tau)$ with $S_m(\mathcal{U}) = S_m(\mathcal{V})$. Assume $\Psi_0(\mathcal{U}) = [(Y,f)]$ and $\Psi_0(\mathcal{V}) = [(Z,g)]$, where Ψ_0 associates a uniformity with a compactification class as described in [4]. The maps h_1 and h_2 are defined from (X,δ) to $Y \times X^+$, respectively $Z \times X^+$, by $h_1(x) = (f(x),x)$ and $h_2(x) = (g(x),x)$. Since $\Psi_0(\mathcal{U}_m) = [(X^+, Id_X)]$, by R13.2.1 $\Psi_0(\mathcal{U} \vee \mathcal{U}_m) = [(\overline{h_1[X]}, h_1)]$ and $\Psi_0(\mathcal{V} \vee \mathcal{U}_m) = [(\overline{h_2[X]}, h_2)]$. Since $\mathcal{U} \vee \mathcal{U}_m = \mathcal{V} \vee \mathcal{U}_m$, there exists a homeomorphism $\phi : \overline{h_1[X]} \to \overline{h_2[X]}$ such that $\phi \circ h_1 = h_2$. Finally, π_1 denotes the restriction to $\overline{h_2[X]}$ of the projection from $Z \times X^+$ to Z.

Lemma R15.1.7 If $y \in Y$, then there is $t \in X^+$ such that $(y,t) \in \overline{h_1[X]}$.

Proof: Let $\{x_\alpha\}$ be a net in X such that $f(x_\alpha) \to y$ in Y. By compactness there is a subnet $\{x_\beta\}$ converging to some $t \in X^+$ and so $(f(x_\beta), x_\beta)$ converges to (y,t) in $Y \times X^+$, i.e., $(y,t) \in \overline{h_1[X]}$.

Lemma R15.1.8 Let $y \in Y$ and assume $t_1, t_2 \in X^+$ with both (y,t_1) and (y,t_2) in $\overline{h_1[X]}$. Then $\pi_1(\phi(y,t_1)) = \pi_1(\phi(y,t_2))$.

Proof: As a first case, assume both t_1 and t_2 are in X. By R13.3.1i $f(t_1) = y = f(t_2)$. Since f is one-to-one, $t_1 = t_2$ and the conclusion holds. As a second case, assume without loss of generality that $t_1 \in X$ and $t_2 = \infty$. Again by R13.3.1i, $y = f(t_1)$. Let $\{x_\alpha\}$ be a net in X such that $(f(x_\alpha), x_\alpha) \to (y,t_2) = (f(t_1),\infty)$ in $\overline{h_1[X]}$. Then $f(x_\alpha) \to f(t_1)$ in $f[X]$ and, since f is a homeomorphism onto $f[X]$, $x_\alpha \to t_1$ in (X,τ). Thus $(g(x_\alpha), x_\alpha) \to (g(t_1),\infty)$ in $\overline{h_2[X]}$. Since ϕ is continuous, $(g(x_\alpha), x_\alpha) = \phi(f(x_\alpha), x_\alpha) \to \phi(f(t_1),\infty)$. Because $\overline{h_2[X]}$ is T_2, $\phi(f(t_1),\infty) = (g(t_1),\infty)$. Since $\phi(f(t_1),t_1) = (g(t_1),t_1)$, the conclusion holds.

Interchanging the roles of Y and Z and using ϕ^{-1}, one obtains analogous results. These lemmas allow the following definition.

Definition R15.1.9 Define $\overline{\phi} : Y \to Z$ as follows: For $y \in Y$ pick (y,t) in $\overline{h_1[X]}$ and let $\overline{\phi}(y) = \pi_1(\phi(y,t))$. Letting ρ_1 be the restriction to $\overline{h_1[X]}$ of the projection from $Y \times X^+$ to Y, we define $\overline{\psi} : Z \to Y$ as follows: For $z \in Z$ pick (z,t) in $\overline{h_2[X]}$ and let $\overline{\psi}(z) = \rho_1(\phi^{-1}(z,t))$.

Proposition R15.1.10 S_m is one-to-one.

Proof: First $\overline{\phi}$ will be shown to be continuous. Let F be closed in Z. Then $F_1 = \phi^{-1}[(F \times X^+) \cap \overline{h_2[X]}]$ is closed in $\overline{h_1[X]}$. Since ρ_1 (again, the restriction to $\overline{h_1[X]}$ of the projection from $Y \times X^+$ to Y) is closed, $\rho_1[F_1]$ is closed in Y. It is easily checked that $(\overline{\phi})^{-1}[F] = \rho_1[F_1]$ and so $\overline{\phi}$ is continuous. Similarly $\overline{\psi}$ is continuous. Next, for every $x \in X$, $\overline{\phi} \circ f(x) = \pi_1(\phi(f(x),x)) = \pi_1(g(x),x) = g(x)$, i.e., $\overline{\phi} \circ f = g$. This also implies that the dense $g[X]$ is contained in the closed $\overline{\phi}[Y]$ so that $\overline{\phi}$ is onto. Similar calculations show that the continuous $\overline{\psi} \circ \overline{\phi}$ agrees with id_Y (the identity on Y) on the dense $f[X]$ so that $\overline{\psi} \circ \overline{\phi} = id_Y$. Thus $\overline{\phi}$ is one-to-one. The map $\overline{\phi}$ shows that the compactifications (Y,f) and (Z,g) are equivalent. By R1.5 $\mathcal{U} = \mathcal{V}$, i.e., S_m is one-to-one.

The next example shows that S_E need not be one-to-one.

Example R15.1.11 Let X be an infinite set and let E be an equivalence relation on X with finitely many equivalence classes and at least two distinct infinite equivalence classes. As shown in [5] $\mathcal{U}_m \vee \mathcal{U}_E$ generates an n point compactification with $n \geq 2$ so that $\mathcal{U}_m \vee \mathcal{U}_E \neq \mathcal{U}_m$ in $\mathcal{TB}(X,\delta)$. Clearly $S_E(\mathcal{U}_m \vee \mathcal{U}_E) = S_E(\mathcal{U}_m)$.

The next few results construct a representation of the compactification associated with $S_E(\mathcal{U})$ given the compactification associated with $\mathcal{U}$. R13.2.1 does not usually apply because $\mathcal{U}_E$ is separated only in trivial cases.

We begin with a restatement of some well-known results about disjoint unions in order to establish notation and a point of reference. The coproduct notation follows the usage of Mac Lane [2; p. 62].

Definition R15.1.12 Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ be a family of topological spaces with $\Delta \neq \emptyset$. $\coprod\{X_\alpha : \alpha \in \Delta\}$ is defined to be $\cup\{X_\alpha \times \{\alpha\} : \alpha \in \Delta\}$. Given $\beta \in \Delta$, ϕ_β maps X_β to $\coprod\{X_\alpha : \alpha \in \Delta\}$ by $\phi_\beta(x) = (x, \beta)$ and σ_β is defined to be $\{\phi_\beta[O] : O \in \tau_\beta\} \cup \{\coprod\{X_\alpha : \alpha \in \Delta\}\}$.

Lemma R15.1.13 Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ be a family of topological spaces with $\Delta \neq \emptyset$. Let $\beta \in \Delta$ Then

i) σ_β is a topology on $\coprod\{X_\alpha : \alpha \in \Delta\}$.

ii) ϕ_β is a continuous, open, one-to-one map from (X_β, τ_β) to $(\coprod\{X_\alpha : \alpha \in \Delta\}, \sigma_\beta)$.

Proof: Routine.

Definition R15.1.14 Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ be a family of topological spaces with $\Delta \neq \emptyset$. $\coprod\{\tau_\alpha : \alpha \in \Delta\}$ is defined to be $\vee\{\sigma_\alpha : \alpha \in \Delta\}$.

The space $(\coprod\{X_\alpha : \alpha \in \Delta\}, \coprod\{\tau_\alpha : \alpha \in \Delta\})$ will usually be called the disjoint union of the given family.

Lemma R15.1.15 Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ be a family of topological spaces with $\Delta \neq \emptyset$. $\cup\{\sigma_\alpha : \alpha \in \Delta\}$ is a basis for $\coprod\{\tau_\alpha : \alpha \in \Delta\}$.

Proof: By definition $\cup\{\sigma_\alpha : \alpha \in \Delta\}$ is a subbasis. For $\beta \neq \gamma$ in Δ, $O \in \sigma_\beta$, and $G \in \sigma_\gamma$, either $O \cap G = \emptyset$ or $O \cap G = \coprod\{X_\alpha : \alpha \in \Delta\}$. The conclusion follows easily.

Lemma R15.1.16 Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ be a family of topological spaces with $\Delta \neq \emptyset$. Then

i) $(\coprod\{X_\alpha : \alpha \in \Delta\}, \coprod\{\tau_\alpha : \alpha \in \Delta\})$ is T_2 if and only if (X_α, τ_α) is T_2 $\forall \alpha \in \Delta$.

ii) $(\coprod\{X_\alpha : \alpha \in \Delta\}, \coprod\{\tau_\alpha : \alpha \in \Delta\})$ is compact if and only if Δ is finite and (X_α, τ_α) is compact for every α in Δ.

iii) $X_\alpha \times \{\alpha\}$ is clopen in $(\coprod\{X_\alpha : \alpha \in \Delta\}, \coprod\{\tau_\alpha : \alpha \in \Delta\})$ for every $\alpha \in \Delta$.

iv) If $|\Delta| \geq 2$, then $(\coprod\{X_\alpha : \alpha \in \Delta\}, \coprod\{\tau_\alpha : \alpha \in \Delta\})$ is disconnected.

v) $(\coprod\{X_\alpha : \alpha \in \Delta\}, \coprod\{\tau_\alpha : \alpha \in \Delta\})$ is zero-dimensional if and only if (X_α, τ_α) is zero-dimensional for every $\alpha \in \Delta$.

Proof: Routine

The following notational convenience will be used in the next lemma and proposition: Given an indexed set X_α and $S \subseteq X_\alpha \times X_\alpha$, $S^* = \{((x, \alpha), (y, \alpha)) : (x, y) \in S\}$.

Lemma R15.1.17 Let $Y_1, \ldots, Y_n$ be compact, T_2 spaces with unique uniformities $\mathcal{V}_1, \ldots, \mathcal{V}_n$. Then V is in the unique uniformity for $Y = \coprod_{i=1}^n Y_i$ if and only if there exist $V_1, \ldots, V_n$ with $V_i \in \mathcal{V}_i$ such that $\cup_{i=1}^n V_i^* \subseteq V$.

Proof: First, assume V is in the unique uniformity for Y, i.e., V is a neighborhood of the diagonal. There exist $G_1, \ldots, G_m$ open in Y with $\cup_{j=1}^m G_j = Y$ and $\cup_{j=1}^m (G_j \times G_j) \subseteq V$. For each i, j let $O_j^i = \phi_i^{-1}[G_j]$, which is open in Y_i. Since $\cup_{j=1}^m O_j^i = Y_i$, $\mathcal{V}_i$ must contain $V_i = \cup_{j=1}^m O_j^i \times O_j^i$, and it is easy to check that $\cup_{i=1}^n V_i^* \subseteq V$. For the converse, assume $\cup_{i=1}^n V_i^* \subseteq V$, where $V_i \in \mathcal{V}_i$. For each i there exist Y_i-open sets $O_1^i, \ldots, O_{k(i)}^i$ such that $Y_i = \cup_{j=1}^{k(i)} O_j^i$ and $\cup_{j=1}^{k(i)} (O_j^i \times O_j^i) \subseteq V_i$. For each i, j, $O_j^i \times \{i\}$ is open in Y and

$\cup_{i=1}^{n} \cup_{j=1}^{k(i)} (O_j^i \times \{i\}) = Y$. Since $(O_j^i \times \{i\}) \times (O_j^i \times \{i\}) = (O_j^i \times O_j^i)^*$, it follows that $\cup_{i=1}^{n} \cup_{j=1}^{k(i)} (O_j^i \times O_j^i)^*$ must be in the unique uniformity for Y. It is easy to check that this entourage must be a subset of V, and so the conclusion holds.

Lemma R15.1.18 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space, and let $\Psi_0(\mathcal{U}) = [(Y,f)]$. Let $A \subseteq X$ and let $\mathcal{V}$ be the subspace uniformity induced on A by $\mathcal{U}$. Then $\Psi_0(\mathcal{V}) = [(\overline{f[A]}, f|_A)]$, where $\overline{f[A]}$ denotes the Y-closure of $f[A]$.

Proof: Clearly $\overline{f[A]}$ is compact and T_2, $f[A]$ is dense in $\overline{f[A]}$, and $f|_A$ is one-to-one and uniformly continuous. Let $V \in \mathcal{V}$. For some $U \in \mathcal{U}$, $V = (A \times A) \cap U$. By hypothesis there is W in the unique uniformity for Y such that $(f \times f)[U] = (f[X] \times f[X]) \cap W$. It is easy to check that $(f|_A \times f|_A)[V] = (f[A] \times f[A]) \cap W$, and so $f|_A$ is a uniform embedding into $\overline{f[A]}$. The conclusion now follows from R1.6a.

Proposition R15.1.19 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space, and let $\Psi_0(\mathcal{U}) = [(Y,f)]$. Let E be an equivalence relation on X with finitely many equivalence classes $C_1,\ldots,C_n$. For $1 \le i \le n$ let $Y_i = \overline{f[C_i]}$ and $f_i = f|_{C_i}$, where $\overline{f[C_i]}$ denotes the closure of $f[C_i]$ in Y. Let $g : X \to \coprod_{i=1}^{n} Y_i$ by $g(x) = (f_i(x), i)$ for $x \in C_i$. Then $(\coprod_{i=1}^{n} Y_i, g)$ is a T_2 compactification of $(X, \tau(\mathcal{U} \vee \mathcal{U}_E))$ and $\Psi_0(S_E(\mathcal{U})) = [(\coprod_{i=1}^{n} Y_i, g)]$.

Proof: $\coprod_{i=1}^{n} Y_i$ is compact and T_2 by R15.1.16. To see that $g[X]$ is dense, let $p \in O$, where O is open in $\coprod_{i=1}^{n} Y_i$. By R15.1.15 for some i between 1 and n there is O_i open in Y_i such that $p \in O_i \times \{i\} \subseteq O$. Since $f_i[C_i]$ is dense in Y_i, there is $x \in C_i$ such that $f_i(x) \in O_i$. It follows that $(f(x), i) \in O_i \times \{i\}$, i.e., $g[X] \cap O \neq \emptyset$. Next note that g is one-to-one, since $g(a) = g(b)$ implies a, b are in the same C_i and $f_i(a) = f_i(b)$ implies $a = b$. For the uniform continuity of g, by R15.1.17 it is sufficient to show that $\cup_{i=1}^{n} (g \times g)^{-1}[V_i^*]$ is in $\mathcal{U} \vee \mathcal{U}_E$, where each V_i is in the unique uniformity for Y_i. For each i, since $(g \times g)^{-1}[V_i^*] = (f_i \times f_i)^{-1}[V_i]$ and f_i is uniformly continuous by R15.1.18, there is $U_i \in \mathcal{U}$ such that $(g \times g)^{-1}[V_i^*] = (C_i \times C_i) \cap U_i$. Let $U = \cap_{i=1}^{n} U_i$. It is easy to check that $U \cap E \subseteq \cup_{i=1}^{n} (g \times g)^{-1}[V_i^*]$ and so the required conclusion holds. Next let $U \in \mathcal{U}$, and let V_i be in the unique uniformity for Y_i such that $(f_i \times f_i)[(C_i \times C_i) \cap U] = (f_i[X] \times f_i[X]) \cap V_i$. Then $(g \times g)[U \cap E] = (g[X] \times g[X]) \cap (\cup_{i=1}^{n} V_i^*)$, which is in the subspace uniformity on $g[X]$ by R15.1.17. This shows that g is a uniform embedding from $(X, \mathcal{U} \vee \mathcal{U}_E)$. It is now immediate that $(\coprod_{i=1}^{n} Y_i, g)$ is a T_2 compactification of $(X, \tau(\mathcal{U} \vee \mathcal{U}_E))$. The second conclusion follows from R1.6a.

The previous proposition also provides a representation of certain compactifications of simple extensions of a $T_{3\frac{1}{2}}$ space by a closed set. The following definition is equivalent to the definition given by Levine in [1].

Definition R15.1.20 Let (X, τ) be a topological space and let $A \subseteq X$. The simple extension of τ by A is $\tau_A = \tau \vee \{\emptyset, A, X\}$.

Lemma R15.1.21 Let $(X,\mathcal{U})$ be a uniform space and let F be closed relative to $\tau(\mathcal{U})$. Let $E(F) = (F \times F) \cup ((X - F) \times (X - F))$. Then $(\tau(\mathcal{U}))_F = \tau(\mathcal{U} \vee \mathcal{U}_{E(F)})$.

Proof: By P2.14 $\tau(\mathcal{U} \vee \mathcal{U}_{E(F)}) = \tau(\mathcal{U}) \vee \tau(\mathcal{U}_{E(F)})$. Since $\tau(\mathcal{U}_{E(F)}) = \{\emptyset, F, X - F, X\}$ and $X - F \in \tau(\mathcal{U})$, the conclusion follows easily from the definition of simple extension.

Corollary R15.1.22 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space, and let $\Psi_0(\mathcal{U}) = [(Y,f)]$. Let F be closed relative to $\tau(\mathcal{U})$. Let $Y_1 = \overline{f[F]}$ and $Y_2 = \overline{f[X - F]}$ be the closures of $f[F]$ and $f[X - F]$ in Y. Let $g : X \to Y_1 \coprod Y_2$ by $g(x) = (f(x), 1)$ if $x \in F$ and $g(x) = (f(x), 2)$ if $x \in X - F$. Then $(Y_1 \coprod Y_2, g)$ is a T_2 compactification of X

with the simple extension of $\tau(\mathcal{U})$ by F.

Proof: Since $E(F)$ has two equivalence classes and the previous lemma holds, this is a direct application of R15.1.19.

Zero-Dimensionality and Images of S-maps

The notion of S-maps naturally leads to questions of the following form: Given a property P, for which uniformities does $S(\mathcal{U})$ correspond to a compactification with property P? In what follows the question is answered for S_E and S_m when P is the property of being a finite point compactification or of being zero-dimensional.

Proposition R15.2.1 Let $(X,\mathcal{U})$ be a totally bounded, separated uniform space with $\Psi_0(\mathcal{U}) = [(Y,f)]$. Let E be an equivalence relation on X with finitely many equivalence classes $C_1, C_2, \ldots, C_n$. Let $\Psi_0(S_E(\mathcal{U})) = [(Z,g)]$. Then (Z,g) is a finite point compactification of (X,δ) if and only if $\overline{f[C_i]} - f[C_i]$ is finite for every i, where $\overline{f[C_i]}$ denotes the closure of $f[C_i]$ in Y.

Proof: Without loss of generality assume (Z,g) is the disjoint union of $\overline{f[C_1]}, \ldots, \overline{f[C_n]}$ with embedding g as described in R15.1.19. Because the union is disjoint, $Z - g[X] = \cup_{i=1}^{n}(\overline{f[C_i]} - f[C_i])$. Clearly $Z - g[X]$ is finite if and only if $\overline{f[C_i]} - f[C_i]$ is finite for all i.

Corollary R15.2.2 Let $(X,\mathcal{U})$ be a totally bounded, separated uniform space with $\Psi_0(\mathcal{U}) = [(Y,f)]$. Let E be an equivalence relation on X with finitely many equivalence classes. Let $\Psi_0(S_E(\mathcal{U})) = [(Z,g)]$. If (Z,g) is a finite point compactification of (X,δ), then (Y,f) is a finite point compactification of $(X,\tau(\mathcal{U}))$

Proof: Let $C_1, C_2, \ldots, C_n$ be the distinct equivalence classes of E. Since $f[X] = \cup_{i=1}^{n} f[C_i]$ and $f[X]$ is dense, $Y = \cup_{i=1}^{n}\overline{f[C_i]}$ and so $Y - f[X] \subseteq \cup_{i=1}^{n}(\overline{f[C_i]} - f[C_i])$. The conclusion is now immediate from R15.2.1.

The converse of R15.2.2 is false, as the following example shows.

Example R15.2.3 Let $(X,\mathcal{U})$ be $[0,1)$ with the usual metric uniformity. $\mathcal{U}$ determines the one point compactification of X, which can be represented as $([0,1], id_X)$. Let E be the equivalence relation with two classes: A, the set of rationals in X, and B, the set of irrationals in X. Since $\overline{A} = [0,1] = \overline{B}$, the compactification corresponding to $S_E(\mathcal{U})$ is not a finite point compactification by R15.2.1.

Proposition R15.2.4 Let $(X,\mathcal{U})$ be a totally bounded, separated uniform space with $\Psi_0(\mathcal{U}) = [(Y,f)]$. Let E be an equivalence relation on X with finitely many equivalence classes $C_1, C_2, \ldots, C_n$. Let $\Psi_0(S_E(\mathcal{U})) = [(Z,g)]$. Then Z is zero-dimensional if and only if $\overline{f[C_i]}$ is zero-dimensional for every i, where $\overline{f[C_i]}$ denotes the closure of $f[C_i]$ in Y.

Proof: This is immediate from R15.1.19 and R15.1.16v.

To deal with similar questions for $S_m(\mathcal{U})$, a representation of the corresponding compactification is needed. This is obtained below as an application of R13.2.1.

Lemma R15.2.5 Assume X is infinite and let $(X,\mathcal{U})$ be a totally bounded, separated uniform space with $\Psi_0(\mathcal{U}) = [(Y,f)]$. Let $g : X \to Y \times X^+$ by $g(x) = (f(x), x)$. Let $\overline{g[X]}$ be the closure of $g[X]$ in $Y \times X^+$. Then $\Psi_0(S_m(\mathcal{U})) = [(\overline{g[X]}, g)]$.

Proof: As noted in [5] $\Psi_0(\mathcal{U}_m) = [(X^+, id_X)]$. Since $S_m(\mathcal{U}) = \mathcal{U} \vee \mathcal{U}_m$, the conclusion is immediate from R13.2.1

In what follows limit points, in the meaning of R13.3.5, will be used. The context, given a uniform space $(X,\mathcal{U})$, will involve two topologies, $\tau(\mathcal{U})$ and δ. For $A \subseteq X$ the derived set A' will denote the set of limit points of A relative to $(X,\tau(\mathcal{U}))$.

The next lemma is a generalization of R13.3.7.

Lemma R15.2.6 Assume X is infinite and let $(X,\mathcal{U})$ be a totally bounded, separated uniform space with $\Psi_0(\mathcal{U}) = [(Y,f)]$. Let $g : X \to Y \times X^+$ by $g(x) = (f(x), x)$. For $A \subseteq X$ let $\overline{g[A]}$ denote the closure of $g[A]$ in $Y \times X^+$ and $c(f[A])$ the closure of $f[A]$ in Y. Then $\overline{g[A]} = g[A] \cup \{(f(x), \infty) : x \in A'\} \cup \{(y, \infty) : y \in c(f[A]) - f[A]\}$.

Proof: Let $(s,t) \in \overline{g[A]}$. If $t \in X$, then by R13.3.1i $s = f(t)$ and it follows easily from R13.3.1ii that the given pair is in $g[A]$. Thus assume that $t = \infty$ and let $\{a_\alpha\}$ be a net in A such that $(f(a_\alpha), a_\alpha) \to (s, \infty)$ in $Y \times X^+$. Clearly $s \in c(f[A])$ and, if $s \notin f[X]$, (s,t) is in $\{(y,\infty) : y \in c(f[A]) - f[X]\}$. Thus assume $s = f(x)$ and let x be in O for some $O \in \tau(\mathcal{U})$. If $O \cap A = \{x\}$, then $a_\alpha = x$ eventually. But $a_\alpha \to \infty$ in X^+ implies a_α is eventually in $X^+ - \{x\}$, a contradiction. Thus $x \in A'$ as required.

The opposite containment will be verified by cases. Always $g[A] \subseteq \overline{g[A]}$. For $x \in A'$ let $(f(x), \infty)$ be in $O \times G$, where O is open in Y and G is open in X^+. Since $\infty \in G$, $F = X - G$ is compact in (X, δ), i.e., F is finite. Then $f^{-1}[O] \cap (X - (F - \{x\}))$, which is open in $\tau(\mathcal{U})$, contains x and so some $a \in A$ with $a \neq x$. $(f(a), a)$, i.e., $g(a)$ is in $(O \times G) \cap g[A]$ and so $(f(x), \infty) \in \overline{g[A]}$. Finally let $y \in c(f[A]) - f[A]$ and let (y, ∞) be in $O \times G$, where O is open in Y and G is open in X^+. As before $F = X - G$ is finite. $O \cap (Y - (f[F] - \{y\}))$, which is open in Y, contains y and so some $f(a)$ where $a \in A$. By hypothesis $y \neq f(a)$ and so $a \notin F$ since, otherwise, $f(a)$ would be in $f[F] - \{y\}$. Thus $(f(a), a)$ is in $g[A] \cap (O \times G)$ and so $(y, \infty) \in \overline{g[A]}$.

Proposition R15.2.7 Let $(X,\mathcal{U})$ be a totally bounded, separated uniform space with $\Psi_0(\mathcal{U}) = [(Y,f)]$. Let $\Psi_0(S_m(\mathcal{U})) = [(Z,g)]$. Then (Z,g) is a finite point compactification of (X,δ) if and only if (Y,f) is a finite point compactification of $(X,\tau(\mathcal{U}))$ and X' is finite.

Proof: If X is finite, $\mathcal{U} = S_m(\mathcal{U})$ and we can assume that $(Y,f) = (Z,g) = (X, id_X)$. Thus the result is true and trivial. In the infinite case, by R15.2.5 we can assume that $g : X \to Y \times X^+$ by $g(x) = (f(x), x)$ and $Z = \overline{g[X]}$, where the closure is taken in $Y \times X^+$. By applying R15.2.6 with $A = X$, $Z - g[X]$ is finite if and only if both X' and $Y - f[X]$ are finite.

Corollary R15.2.8 Let $(X,\mathcal{U})$ be a totally bounded, separated uniform space. Let $\Psi_0(S_m(\mathcal{U})) = [(Z,g)]$. If (Z,g) is a finite point compactification of (X,δ), then $(X,\tau(\mathcal{U}))$ is zero-dimensional.

Proof: For any p not in X', $\{p\}$ is $\tau(\mathcal{U})$-open. It is also closed by T_1. Now let $x \in X'$. For each $t \in X'$ with $t \neq x$, apply the Hausdorff property to obtain O_t, G_t in $\tau(\mathcal{U})$ such that $x \in O_t, t \in G_t$ and $O_t \cap G_t = \emptyset$. Let $O_x = \cap\{O_t : t \in X' \text{ and } t \neq x\}$. Since X' is finite, O_x is open. It is routine to check that every open subset of O_x containing x is clopen. From these facts it follows easily that $(X,\tau(\mathcal{U}))$ is zero-dimensional.

Example R15.1.6 shows that the converse of R15.2.8 is false. The property of $S_m(\mathcal{U})$ determining a finite point compactification is uniform, not topological.

For the rest of this subsection some ideas from [8] will be needed: For a set X and $A \subseteq X$, $E(A) = (A \times A) \cup ((X - A) \times (X - A))$. For a uniform space $(X,\mathcal{U})$, applying definitions R11.7 and R11.16 to the uniformity of interest, $\mathcal{R}(\mathcal{U} \vee \mathcal{U}_m) = \{A : E(A) \in \mathcal{U} \vee \mathcal{U}_m\}$ and $\mathcal{U}(\mathcal{R}(\mathcal{U} \vee \mathcal{U}_m)) = \vee\{\mathcal{U}_{E(A)} : A \in \mathcal{R}(\mathcal{U} \vee \mathcal{U}_m)\}$.

Proposition R15.2.9 Let $(X,\mathcal{U})$ be a totally bounded, separated uniform space. Let $\Psi_0(S_m(\mathcal{U})) = [(Z,g)]$. Then Z is zero-dimensional if and only if $\mathcal{U} \vee \mathcal{U}_m = \mathcal{U}(\mathcal{R}(\mathcal{U} \vee \mathcal{U}_m))$.

Proof: If Z is zero-dimensional, the equality $\mathcal{U} \vee \mathcal{U}_m = \mathcal{U}(\mathcal{R}(\mathcal{U} \vee \mathcal{U}_m))$ is part of the conclusion of R11.24. Now assume $\mathcal{U} \vee \mathcal{U}_m = \mathcal{U}(\mathcal{R}(\mathcal{U} \vee \mathcal{U}_m))$. It follows easily that $\mathcal{U} \vee \mathcal{U}_m = \vee\{\mathcal{U}_m \vee \mathcal{U}_{E(A)} : A \in \mathcal{R}(\mathcal{U} \vee \mathcal{U}_m)\}$. By R5.2.4 each $\mathcal{U}_m \vee \mathcal{U}_{E(A)}$ corresponds to a finite point (in fact, a two point) compactification of (X, δ). Applying R13.1.2 in the usual way, we see that (Z, g) is a supremum of two point compactifications and so by R9.3.3 Z is zero-dimensional.

Another characterization, which will be now be developed, requires some non-standard terminology.

Definition R15.2.10 Let (X, τ) be a topological space and let $A \subseteq X$. A is near-clopen relative to τ provided there is T, a co-finite subset of X, such that $A \cap T$ is clopen in T with the subspace topology from τ.

Note that in an arbitrary space all finite and co-finite subsets are near-clopen. The phrase 'relative to τ' will sometimes be omitted when there is no possible ambiguity.

Lemma R15.2.11 Let (X, τ) be a topological space and let A and B be near-clopen subsets of X. Then

i)$X - A$ is also near-clopen relative to τ.

ii) $A \cap B$ is near-clopen relative to τ.

Proof: Let S and T be co-finite subsets of X such that $A \cap S = O \cap S = F \cap S$ and $B \cap T = G \cap T = H \cap T$, where $O, G \in \tau$ and F, H are closed relative to τ. Then $S \cap T$ is co-finite in X, $(X - A) \cap S = (X - O) \cap S = (X - F) \cap S$, and $(A \cap B) \cap (S \cap T) = (O \cap G) \cap (S \cap T) = (F \cap H) \cap (S \cap T)$.

Definition R15.2.12 Let $(X, \mathcal{U})$ be a uniform space and let $A \subset X$. A is uniformly near-clopen relative to $\mathcal{U}$ provided there exist T, a co-finite subset of X, and $U \in \mathcal{U}$ such that, for every $t \in A \cap T$, $(U[t] \cap T) \subseteq A$.

Note that if $X = [0, 1]$ has the usual metric uniformity and $A = [0, \frac{1}{2}]$, A is near-clopen but not uniformly near-clopen.

Lemma R15.2.13 Let $(X, \mathcal{U})$ be a uniform space and let A and B be subsets of X. Assume A and B are uniformly near-clopen relative to $\mathcal{U}$. Then

i) $X - A$ is uniformly near-clopen relative to $\mathcal{U}$.

ii) $A \cap B$ is uniformly near-clopen relative to $\mathcal{U}$.

iii) A is near-clopen relative to $\tau(\mathcal{U})$.

Proof: Let T be co-finite in X and $U \in \mathcal{U}$ such that $(U[t] \cap T) \subseteq A$ for all t in $A \cap T$. Since every uniformity has a symmetric base, also assume $U = U^{-1}$. For i), let $x \in (X - A) \cap T$ and suppose $t \in U[x] \cap T$. If $t \notin (X - A)$, then $t \in A \cap T$ and so $x \in (U[t] \cap T) \subseteq A$, a contradiction. Thus i) holds by definition. For iii) recall that $U[t]$ is always a $\tau(\mathcal{U})$-neighborhood of t. Since $A \cap T = \cup\{U[t] \cap T : t \in A \cap T\}$, $A \cap T$ is open in T. Similarly, $(X - A) \cap T$, the complement of $A \cap T$ in T, is open in T. Thus $A \cap T$ is clopen in T. For ii) let S be co-finite in X and $V \in \mathcal{U}$ such that, for every $x \in B \cap S$, $V[x] \cap S \subseteq B$. $S \cap T$ is co-finite in X, $U \cap V \in \mathcal{U}$, and it is easy to check that, for every $x \in (A \cap B) \cap (S \cap T)$, $(U \cap V)[x] \cap (S \cap T) \subseteq A \cap B$.

Lemma R15.2.14 Let $(X, \mathcal{U})$ be a totally bounded, separated uniform space. Let E be an equivalence relation on X with $E \in \mathcal{U} \vee \mathcal{U}_m$. Then E has finitely many equivalence classes and each equivalence class is uniformly near-clopen relative to $\mathcal{U}$.

Proof: Since $\mathcal{U} \vee \mathcal{U}_m$ is totally bounded, E clearly must have finitely many equivalence classes. By R14.1.5 there exist $U \in \mathcal{U}$ and T co-finite in X such that $U \cap U_T \subseteq E$, where $U_T = T \times T \cup \{(x,x) : x \notin T\}$. Let C be an equivalence class of E and $t \in C \cap T$. Then $U[t] \cap T = (U \cap U_T)[t] \subseteq E[t] = C$, as required by the definition.

Definition R15.2.15 Let $(X,\mathcal{U})$ be a uniform space and let E be an equivalence relation on X. E is zd-suitable relative to $\mathcal{U}$ provided E has finitely many equivalence classes, each of which is uniformly near-clopen relative to $\mathcal{U}$.

Lemma R15.2.16 Let $(X,\mathcal{U})$ be a uniform space and let E_1 and E_2 be zd-suitable equivalence relations relative to $\mathcal{U}$. Then $E_1 \cap E_2$ is zd-suitable relative to $\mathcal{U}$

Proof: The equivalence classes of $E_1 \cap E_2$ are of the form $C \cap D$, where C is an E_1-equivalence class and D is an E_2-equivalence class. Thus there are only finitely many $E_1 \cap E_2$-classes, each of which is uniformly near-clopen relative to $\mathcal{U}$ by R15.2.13ii.

Recall that the diagonal of a set X is $\Delta_X = \{(x,x) : x \in X\}$.

Definition R15.2.17 Let $(X,\mathcal{U})$ be a uniform space and let $\mathcal{P}$ be a collection of subsets of $X \times X$ such that $\Delta_X \subseteq P$ for every $P \in \mathcal{P}$. $\mathcal{P}$ is a pseudo-basis for $\mathcal{U}$ if and only if for every $U \in \mathcal{U}$ there is $P \in \mathcal{P}$ with $P \subseteq U$.

By itself the existence of a pseudo-basis is not informative, since $\{\Delta_X\}$ is a pseudo-basis for any uniformity on a set X. However the notion can play a role in combination with other properties, as in the following proposition.

Proposition R15.2.18 Let $(X,\mathcal{U})$ be a totally bounded, separated uniform space with $\Psi_0(\mathcal{U} \vee \mathcal{U}_m) = [(Y,f)]$. Then Y is zero-dimensional if and only if $\mathcal{U}$ has a pseudo-basis of zd-suitable equivalence relations.

Proof: First assume Y is zero-dimensional, so that $\mathcal{U}(\mathcal{R}(\mathcal{U} \vee \mathcal{U}_m)) = \mathcal{U} \vee \mathcal{U}_m$ by R15.2.9. For $U \in \mathcal{U}$, U is also in $\mathcal{U} \vee \mathcal{U}_m$, and there exist $A_1, A_2, \ldots, A_n$ in $\mathcal{R}(\mathcal{U} \vee \mathcal{U}_m)$ such that $\cap_{i=1}^n E(A_i) \subseteq U$. By R15.2.14 and R15.2.16 $\cap_{i=1}^n E(A_i)$ is a zd-suitable equivalence relation on X. Thus the collection of all zd-suitable equivalence relations on X is a pseudo-basis for $\mathcal{U}$. For the converse, since R15.2.9 holds and $\mathcal{U}(\mathcal{R}(\mathcal{U} \vee \mathcal{U}_m))$ is automatically a subset of $\mathcal{U} \vee \mathcal{U}_m$, it is sufficient to show $\mathcal{U} \vee \mathcal{U}_m \subseteq \mathcal{U}(\mathcal{R}(\mathcal{U} \vee \mathcal{U}_m))$. For T co-finite in X, with $U_T = T \times T \cup \{(x,x) : x \notin T\}$ as above, it is easy to check that $U_T = \cap\{E(\{x\}) : x \notin T\}$, and, as a finite intersection of elements of $\mathcal{U}(\mathcal{R}(\mathcal{U} \vee \mathcal{U}_m))$, $U_T \in \mathcal{U}(\mathcal{R}(\mathcal{U} \vee \mathcal{U}_m))$. Since sets of the form U_T are basic in $\mathcal{U}_m$, $\mathcal{U}_m \subseteq \mathcal{U}(\mathcal{R}(\mathcal{U} \vee \mathcal{U}_m))$. Finally let $U \in \mathcal{U}$ and let E be a zd-suitable equivalence relation on X such that $E \subseteq U$. Let A be an equivalence class of E. Then there exist S co-finite in X and $V = V^{-1}$ in $\mathcal{U}$ such that $V[x] \cap S \subseteq A$ for all x in $A \cap S$. It is easy to check that $V \cap U_S \subseteq E(A)$, and so $E(A) \in \mathcal{U} \vee \mathcal{U}_m$. Thus $A \in \mathcal{R}(\mathcal{U} \vee \mathcal{U}_m)$, which yields $E(A) \in \mathcal{U}(\mathcal{R}(\mathcal{U} \vee \mathcal{U}_m))$. Let $C_1, \ldots, C_n$ denote the distinct equivalence classes of E. Since $\cap_{i=1}^n E(C_i) \subseteq E \subseteq U$, $U \in \mathcal{U}(\mathcal{R}(\mathcal{U} \vee \mathcal{U}_m))$. Thus $\mathcal{U} \subseteq \mathcal{U}(\mathcal{R}(\mathcal{U} \vee \mathcal{U}_m))$, as required for the conclusion.

The rest of this subsection will develop a necessary topological condition for $S_m(\mathcal{U})$ to determine a zero-dimensional compactification.

Lemma R15.2.19 Let $(X,\mathcal{U})$ be a totally bounded, separated uniform space, let $A \subseteq X$, and let $\mathcal{V}$ be the subspace uniformity induced by $\mathcal{U}$ on A. If $\mathcal{U}$ has a pseudo-basis of zd-suitable equivalence relations, then $\mathcal{V}$ also has a pseudo-basis of zd-suitable equivalence relations.

Proof: For $V \in \mathcal{V}$, $V = (A \times A) \cap U$ for some $U \in \mathcal{U}$. By hypothesis there is a zd-suitable equivalence relation E on X such that $E \subseteq U$. Then $(A \times A) \cap E$ is an equivalence relation on A contaned in V, and it is routine to check that $(A \times A) \cap E$ is zd-suitable relative to $\mathcal{V}$.

Definition R15.2.20 Let X be a set, let $T \subseteq X$, let $\mathcal{C}$ be a collection of subsets of X, and let $a, b \in X$. A T-link of length k from a to b in $\mathcal{C}$ is a map $l : \{1, 2, \ldots, k\} \to \mathcal{C}$ such that $a \in l(1), b \in l(k)$, and $l(i) \cap l(i+1) \cap T \neq \emptyset$ for $1 \leq i \leq k-1$.

Lemma R15.2.21 Let (X, τ) be a connected topological space, and let $a, b \in X$. If $\mathcal{G}$ is an open cover of X with $|\mathcal{G}| = n$ and $\emptyset \notin \mathcal{G}$, then there is a X-link from a to b in $\mathcal{G}$ of length at most n.

Proof: Induction on n will be used to show that there is a one-to-one X-link of length k from a to b. Since the link will be one-to-one, it will follow automatically that $k \leq n$. For $n = 1$ the conclusion is trivial. Now assume that the conclusion holds for any open cover with n non-empty elements, and let $\mathcal{G}$ be an open cover with $n+1$ non-empty elements. By connectedness any element of $\mathcal{G}$ must intersect some other element of $\mathcal{G}$, and so one can list $\mathcal{G}$ as $O_1, \ldots, O_n, O_{n+1}$ where $O_n \cap O_{n+1} \neq \emptyset$. Apply the induction hypothesis to the cover $\mathcal{G}_1 = \{O_1, \ldots, O_{n-1}, O_n \cup O_{n+1}\}$ to obtain a one-to-one X-link l of length k from a to b in $\mathcal{G}_1$. If $l(i) \in \{O_1, \ldots, O_{n-1}\}$ for all i, l is also the desired X-link in $\mathcal{G}$. Thus assume $l(i) = O_n \cup O_{n+1}$ and proceed by cases. First assume $i = 1$ and, without loss of generality, $a \in O_n$. If $O_n \cap l(2) \neq \emptyset$, then let $\tilde{l}(1) = O_n$ and $\tilde{l}(t) = l(t)$ for $2 \leq t \leq k$. If $O_n \cap l(2) = \emptyset$, by the definition of a link $O_{n+1} \cap l(2) \neq \emptyset$. Let $\tilde{l}(1) = O_n, \tilde{l}(2) = O_{n+1}$ and $\tilde{l}(t) = l(t-1)$ for $3 \leq t \leq k+1$. In either case $\tilde{l}$ is the required one-to-one X-link in $\mathcal{G}$. The second case when $i = k$ is similar. Finally suppose $2 \leq i \leq k-1$. If $O_n \cap l(i-1) \neq \emptyset$ and $O_n \cap l(i+1) \neq \emptyset$, let $\tilde{l}(t) = l(t)$ for $1 \leq t \leq i-1$, $\tilde{l}(i) = O_n$, and $\tilde{l}(t) = l(t)$ for $i+1 \leq t \leq k$. If $O_n \cap l(i-1) \neq \emptyset$ and $O_n \cap l(i+1) = \emptyset$, then $O_{n+1} \cap l(i+1) \neq \emptyset$. Here let $\tilde{l}(t) = l(t)$ for $1 \leq t \leq i-1$, $\tilde{l}(i) = O_n, \tilde{l}(i+1) = O_{n+1}$, and $\tilde{l}(t) = l(t-1)$ for $i+2 \leq t \leq k+1$. If $O_n \cap l(i-1) = \emptyset$, then $O_{n+1} \cap l(i-1) \neq \emptyset$. This leads to the two remaining subcases, $O_{n+1} \cap l(i+1) \neq \emptyset$ and $O_{n+1} \cap l(i+1) = \emptyset$ so that $O_n \cap l(i+1) \neq \emptyset$. In each $\tilde{l}$ is defined similarly to the above. In all these situations $\tilde{l}$ is the required one-to-one X-link in $\mathcal{G}$.

Corollary R15.2.22 Let (X, τ) be a connected T_1 space with $|X| \geq 2$, and let $a, b \in X$. Let T be a co-finite subset of X. If $\mathcal{G}$ is an open cover of X with $|\mathcal{G}| = n$ and $\emptyset \notin \mathcal{G}$, then there is a T-link from a to b in $\mathcal{G}$ of length at most n.

Proof: Let l be the X-link of length $k \leq n$ produced by the previous lemma. Since every non-empty open set in a connected T_1 space with at least two elements is infinite, $l(i) \cap l(i+1) \cap T \neq \emptyset$ for $1 \leq i \leq k-1$, i.e., l is also a T-link.

In the next few results the following notational convention is used: given $S \subseteq (X \times X)$, $S^2 = S \circ S, S^3 = S \circ S \circ S$, etc. As usual, for T co-finite in X, $U_T = (T \times T) \cup \{(x, x) : x \notin T\}$.

Lemma R15.2.23 Let X be a set and let $V \subseteq (X \times X)$ with $V = V^{-1}$. Let T be a co-finite subset of X. Assume $l : \{1, 2, \ldots, k\} \to \{V[t] : t \in T\}$ is a map with $l(i) \cap l(i+1) \cap T \neq \emptyset$ for $1 \leq i \leq k-1$, and let $a \in l(1) \cap T$. Then for every $x \in l(k) \cap T$, $(a, x) \in ((V \cap U_T) \circ (V \cap U_T))^k$.

Proof: For convenience write $V[t_i]$ for $l(i)$, where $t_i \in T$. The proof proceeds by induction on k: For $k = 1$, let $x \in V[t_1] \cap T$. Then (a, t_1) and (t_1, x) are in $V \cap U_T$ so that $(a, x) \in (V \cap U_T) \circ (V \cap U_T)$. Now assume the conclusion is true for k and let $x \in V[t_{k+1}] \cap T$. Pick $t \in V[t_k] \cap V[t_{k+1}] \cap T$. By the induction hypothesis $(a, t) \in ((V \cap U_T) \circ (V \cap U_T))^k$. Since (t, t_{k+1}) and (t_{k+1}, x) are both in $V \cap U_T$, (t, x) is in $(V \cap U_T) \circ (V \cap U_T)$. Thus

$(a,x) \in ((V \cap U_T) \circ (V \cap U_T))^{k+1}$.

Lemma R15.2.24 Let $(X,\mathcal{U})$ be a totally bounded, separated uniform space such that $(X,\tau(\mathcal{U}))$ is connected, and let $U \in \mathcal{U}$. Assume $|X| \geq 2$, and let T be a co-finite subset of X. Then there exist $t_1,\ldots,t_n \in T$ such that $X = \cup_{i=1}^{n} U[t_i]$.

Proof: Pick $V \in \mathcal{U}$ such that $V = V^{-1}$ and $V \subseteq U$. It is sufficient to verify the conclusion for V. T with the subspace uniformity from $\mathcal{U}$ is totally bounded and so there exist $t_1,\ldots,t_j \in T$ such that $T = \cup_{i=1}^{j}(T \times T) \cap V[t_i]$. Thus $T \subseteq \cup_{i=1}^{j} V[t_i]$. Let $X - T = \{x_{j+1},\ldots,x_{j+k}\}$. For $1 \leq i \leq k$, since $V[x_{j+i}]$ is a neighborhood of x_{j+i} and every non-empty open set of X is infinite, there is $t_{j+i} \in T \cap V[x_{j+i}]$. Since $V = V^{-1}$, $x_{j+i} \in V[t_{j+i}]$. It follows that $X = (\cup_{i=1}^{j} V[t_i]) \cup (\cup_{i=1}^{k} V[t_{j+i}])$.

Lemma R15.2.25 Let $(X,\mathcal{U})$ be a totally bounded, separated uniform space such that $(X,\tau(\mathcal{U}))$ is connected, and let $U \in \mathcal{U}$. Assume $|X| \geq 2$, and let T be a co-finite subset of X. Then there exists a positive integer n such that $(U \cap U_T)^n = U_T$.

Proof: For any n, $(U \cap U_T)^n \subseteq (U_T)^n = U_T$, and so only the opposite containment will be verified. Since every uniformity has an open symmetric base, there is $V \in \mathcal{U}$ with V open in $X \times X$, $V = V^{-1}$, and $V \circ V \subseteq U$. By R15.2.24 there exist n and $t_1,\ldots,t_n$ in T such that $X = \cup_{i=1}^{n} V[t_i]$. Let $(a,b) \in U_T$. Without loss of generality, assume $a \neq b$ so that $a,b \in T$. Since each $V[t_i]$ is open, by R15.2.22 there is a T-link l of length $k \leq n$ from a to b in $\{V[t_i] : 1 \leq i \leq n\}$. By R15.2.23 $(a,b) \in ((V \cap U_T) \circ (V \cap U_T))^k \subseteq (U \cap U_T)^n$, as required.

Lemma R15.2.26 Let $(X,\mathcal{U})$ be a totally bounded, separated uniform space such that $(X,\tau(\mathcal{U}))$ is connected, and let $\Psi_0(S_m(\mathcal{U})) = [(Y,f)]$. If $|X| \geq 2$, then Y is not zero-dimensional.

Proof: Since $|X| \geq 2$ and $(X,\tau(\mathcal{U}))$ is connected and T_2, X is infinite. Since $x \notin X'$ implies $\{x\}$ is clopen, here $X' = X$. Now suppose Y is zero-dimensional. By R15.2.7 Y cannot be the one-point compactification of (X,δ), i.e., $S_m(\mathcal{U})$ is strictly larger than $\mathcal{U}_m$. Since $\mathcal{U}_m = \vee\{\mathcal{U}_{E(A)} : A$ is either finite or co-finite in $X\}$, by R15.2.9 there must be $A \in \mathcal{R}(\mathcal{U} \vee \mathcal{U}_m)$ such that A is neither finite nor co-finite. By R15.2.9 again there exist $U \in \mathcal{U}$ and T co-finite in X with $U \cap U_T \subseteq E(A)$. By R15.2.25 there is n such that $(U \cap U_T)^n = U_T$. Then $U_T = (U \cap U_T)^n \subseteq E(A)^n = E(A)$. But this implies $T \subseteq A$ or $T \subseteq (X - A)$, a contradiction.

Proposition R15.2.27 Let $(X,\mathcal{U})$ be a totally bounded, separated uniform space, and let $\Psi_0(S_m(\mathcal{U})) = [(Y,f)]$. If Y is zero-dimensional, then $(X,\tau(\mathcal{U}))$ is totally disconnected.

Proof: Let $x \in X$ and let C be the component of x relative to $(X,\tau(\mathcal{U}))$. Let $\mathcal{V}$ be the subspace uniformity induced by $\mathcal{U}$ on C, and let $\Psi_0(\mathcal{V}) = [(Z,g)]$. By R15.2.18 and R15.2.19 Z must be zero-dimensional. By R15.2.26 $|C| = 1$. Thus the components of X are all singletons, i.e., $(X,\tau(\mathcal{U}))$ is totally disconnected.

It is unknown whether the conclusion of the previous proposition can be strengthened to 'zero-dimensional.' Example R15.1.6 shows that the converse of R15.2.27 is false.

With S_m interpreted as a map of compactifications, the next corollary shows that the image of a Stone-Čech compactification need not be Stone-Čech.

Corollary R15.2.28 Let (X,τ) be a $T_{3\frac{1}{2}}$ space, and let $\mathcal{U}$ be the largest element of $\mathcal{TB}(X,\tau)$. If (X,τ) is not totally disconnected, then $S_m(\mathcal{U}) \neq \mathcal{U}_M$.

Proof: By R15.2.27 $\Psi_0(S_m(\mathcal{U}))$ is not the class of a zero-dimensional compactification. By R1.8 $\Psi_0(\mathcal{U}_M)$ is the class of the Stone-Čech compactification of (X, δ), which is zero-dimensional. The conclusion follows from R1.5.

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