The Remnant Rings as Compactifications

The remnant rings were defined in [6] by removing the point-filters from $\mathbb{N}_k$, respectively $\mathbb{N}_\infty$, and observing that the restrictions of the addition and multiplication extended continuously to $\mathbb{N}_k$, respectively $\mathbb{N}_\infty$, make these remnant structures compact topological rings. As in [6], these rings will be denoted $\mathbf{R}_k$, respectively $\mathbf{R}_\infty$.

This section makes extensive use of definitions, notations, and results from [5] and [6]. In particular, the sequence associated with a non-point ultrafilter in $\mathbb{N}_k$ (defined in R10.2.3) is repeatedly employed, as well as properties of these sequences: the recursive relationship of terms (R10.2.5), the recursive method of defining a non-point ultrafilter in $\mathbf{R}_k$ (R10.2.6), and the way in which these sequences describe the operations (R12.4.4). Some special examples: The additive identity of $\mathbf{R}_k$ is $\mathcal{O}_k$, which corresponds to the sequence $\{k^n\}$. For $m \in \mathbb{N}$, the corresponding element of $\mathbf{R}_k$ (which is $\hat{m} + \mathcal{O}_k$, where $\hat{m}$ is the point-filter of m) will be denoted $\mathcal{F}_k(m)$ when there are several possible values of k or, more simply, $\overline{m}$, when there is no ambiguity about k. By R12.5.9ii $\mathcal{F}_k(m)$, i.e. $\overline{m}$, corresponds to the sequence $\{t_n\}$, where $t_n \equiv m \bmod k^n$ for all n. By R12.4.4 $\overline{m} + \overline{n} = \overline{m+n}$ and $\overline{m} \cdot \overline{n} = \overline{mn}$ for m, n in $\mathbb{N}$. As the equations in the previous sentence illustrate, typical algebraic conventions are used throughout this section: The reader is expected to recognize the structures appropriate for the binary operations.

In R12.6.9 it was shown that $\{\overline{n} : n \in \mathbb{N}\}$ is a dense non-discrete subset of $\mathbf{R}_k$. As a result $\mathbf{R}_k$ with the map $n \mapsto \overline{n}$ is not a compactification of $\mathbb{N}$ with the discrete topology. Similar results hold for $\mathbf{R}_\infty$. In this section maps and topologies making the remnant rings compactifications will be described.

Because the ring structure is also of interest, the map $n \mapsto \overline{n}$ will first be extended to the ring of integers, $\mathbf{Z}$.

Lemma R16.1 Let $k, m \in \mathbb{N}$ with $k \geq 2$. Then

i) The negative of $\overline{m}$ corresponds to the sequence $\{t_n\}$, where $t_n \equiv -m \bmod k^n$ for all n.

ii) $\mathcal{O}_k$ corresponds to the sequence $\{s_n\}$, where $s_n \equiv 0 \bmod k^n$ for all n.

Proof: By R12.6.2 the negative of $\overline{m}$ corresponds to the sequence $\{t_n\}$, where $t_n \equiv k^n - m \bmod k^n$ for all n, and by R12.5.8 $\mathcal{O}_k$ corresponds to the sequence $\{s_n\}$, where $s_n \equiv k^n \bmod k^n$ for all n. Since $k^n \equiv 0 \bmod k^n$, i) and ii) are immediate.

The previous lemma allows the notation used to this point for natural numbers only to be extended to the integers: For $m \in \mathbf{Z}$ and $k \geq 2$ in $\mathbb{N}$, $\overline{m}$ corresponds to the sequence $\{t_n\}$, where $t_n \equiv m \bmod k^n$ for all n. By R16.1ii $\mathcal{O}_k = \overline{0}$. In what follows, when k is unambiguous, $\overline{0}$ will normally be used instead of $\mathcal{O}_k$.

Lemma R16.2 Let $k \in \mathbb{N}$ with $k \geq 2$, and let $\mathcal{F}, \mathcal{G}$ in $\mathbf{R}_k$ correspond to the sequences $\{x_n\}, \{y_n\}$ respectively. Then $\mathcal{F} - \mathcal{G}$ corresponds to the sequence $\{z_n\}$, where $z_n \equiv x_n - y_n \bmod k^n$ for all n.

Proof: Let ${}^-\mathcal{G}$ be the element of $\mathbf{R}_k$ corresponding to $\{w_n\}$, where $w_n \equiv k^n - y_n \bmod k^n$. By R12.6.3 ${}^-\mathcal{G}$ is the additive inverse of $\mathcal{G}$. Since $\mathcal{F} - \mathcal{G} = \mathcal{F} + ({}^-\mathcal{G})$, by R12.4.4 $z_n \equiv x_n + (k^n - y_n) \bmod k^n$, i.e., $z_n \equiv x_n - y_n \bmod k^n$ for all n.

Lemma R16.3 Let $k \in \mathbb{N}$ with $k \geq 2$, and let $m, n \in \mathbf{Z}$. Then $\overline{m+n} = \overline{m} + \overline{n}$, $\overline{m-n} = \overline{m} - \overline{n}$, and $\overline{mn} = \overline{m} \cdot \overline{n}$.

Proof: For any integer z, $\overline{z}$ corresponds to the sequence $\{x_n\}$, where $x_n \equiv z \bmod k^n$ for all n. The assertions follow easily from this, R12.4.4, R16.2, and R10.2.4.

Lemma R16.4 Let $k \in \mathbb{N}$ with $k \geq 2$, let $m \in \mathbf{Z}$, and let $\overline{m}$ correspond to the sequence $\{x_n\}$. Let t be in $\mathbb{N}$ with $|m| < k^t$. If $m > 0$, then $x_t = m$. If $m \leq 0$, then $x_t = k^t + m$.

Proof: By definition $x_t \in \{1, 2, \dots, k^t\}$ and as noted above $x_t \equiv m \bmod k^t$. Since $|m| < k^t$, clearly $m > 0$ implies $x_t = m$. For $m \leq 0$, $-m = |m| < k^t$ and so $0 < k^t + m \leq k^t$. Thus $k^t + m$ is the unique element of $\{1, 2, \dots, k^t\}$ congruent to $m \bmod k^t$, i.e., $x_t = k^t + m$.

Lemma R16.5 Let $k \in \mathbb{N}$ with $k \geq 2$. The map from $\mathbf{Z}$ into $\mathbf{R}_k$ by $z \mapsto \overline{z}$ is one-to-one.

Proof: Let $z \neq w$ be in $\mathbf{Z}$, let $\overline{z}$ correspond to $\{x_n\}$, and let $\overline{w}$ correspond to $\{y_n\}$. Let t be a positive integer such that $\max\{|z|, |w|\} < k^t$. If z and w are both positive or both non-positive, it follows easily from R16.4 that $x_t \neq y_t$. In the other case, assume without loss of generality $z > 0$ and $w \leq 0$ so that $x_t = z$ and $y_t = k^t + w$ by R16.4. The lemma also applies at the next level, i.e., $x_{t+1} = z$ and $y_{t+1} = k^{t+1} + w$. Clearly either $x_t \neq y_t$ or $x_{t+1} \neq y_{t+1}$. In all cases the corresponding sequences differ in at least one position and so $\overline{z} \neq \overline{w}$ by the definition of the sequence corresponding to a non-point ultrafilter (R10.2.3).

Definition R16.6 Let $k \in \mathbb{N}$ with $k \geq 2$. f_k is the map from $\mathbf{Z}$ to $\mathbf{R}_k$ defined by $f_k(z) = \overline{z}$.

By R12.6.9 $\{\overline{n} : n \in \mathbb{N}\}$ is dense in $\mathbf{R}_k$, and so R16.5 shows that $(\mathbf{R}_k, f_k)$ is a T_2 compactification of $\mathbf{Z}$ with the weak topology induced by f_k. The next few results describe that topology intrinsically.

For $k \in \mathbb{N}$ with $k \geq 2$ and $m \in \mathbf{Z}$, $D_n^m(k)$ will denote the equivalence class in $\mathbf{Z}$ of m mod k^n. $\mathcal{B}_k = \{D_n^m(k) : m \in \mathbf{Z} \text{ and } n \in \mathbb{N}\}$.

Lemma R16.7 Let $k \in \mathbb{N}$ with $k \geq 2$. $\mathcal{B}_k$ is a basis for a topology on $\mathbf{Z}$.

Proof: Clearly the union of $\mathcal{B}_k$ is $\mathbf{Z}$. Let $x \in D_n^m(k) \cap D_t^j(k)$ and assume without loss of generality that $n \leq t$, so that $D_t^x(k) \subseteq D_n^x(k)$. Since $D_n^m(k) = D_n^x(k)$ and $D_t^j(k) = D_t^x(k)$, $x \in D_t^x(k) \subseteq D_n^m(k) \cap D_t^j(k)$, and so the intersection requirement also holds.

Definition R16.8 Let $k \in \mathbb{N}$ with $k \geq 2$. τ_k is the topology for $\mathbf{Z}$ with basis $\mathcal{B}_k$.

Lemma R16.9 Let $k \in \mathbb{N}$ with $k \geq 2$ and let $m \in \mathbf{Z}$. Then for all $n \in \mathbb{N}$ $D_n^m(k)$ is clopen relative to τ_k.

Proof: For any n $D_n^m(k)$ is a basic open set in τ_k. Its complement, the union of the other k^n-equivalence classes, is also τ_k-open.

Proposition R16.10 Let $k \in \mathbb{N}$ with $k \geq 2$. $(\mathbf{Z}, \tau_k)$ is a zero-dimensional $T_{3\frac{1}{2}}$ space.

Proof: R16.9 shows that $\mathcal{B}_k$ is a clopen basis for τ_k. Let $z \neq w$ be integers and assume without loss of generality that $z < w$. Pick a positive integer n such that $0 < w - z < k^n$. Then w is not equivalent to z mod k^n and so $D_n^w(k) \cap D_n^z(k) = \emptyset$, i.e., τ_k is T_2. A zero-dimensional Hausdorff topology is automatically $T_{3\frac{1}{2}}$.

In the next two results, notation from [5] will be used: For $k, n \in \mathbb{N}$, $E_n(k)$ will denote equivalence mod k^n restricted to $\mathbb{N}$, and $C_n^i(k)$, where $1 \leq i \leq k^n$, denotes the $\mathbb{N}$-equivalence class of i relative to $E_n(k)$. Of course, $C_n^i(k) = D_n^i(k) \cap \mathbb{N}$.

Lemma R16.11 Let $k \in \mathbb{N}$ with $k \geq 2$. Then f_k is continuous from $(\mathbf{Z}, \tau_k)$ to $\mathbf{R}_k$.

Proof: Let W be in $\mathcal{Z}_k$, the normal basis used to generate $\mathbb{N}_k$ in [5]. By P3.6 $\mathcal{Z}_k^\omega$ is a normal basis and so a basis for the closed subsets of $\mathbb{N}_k$. Thus it is sufficient to show that $H = f_k^{-1}[W^\omega \cap \mathbf{R}_k]$ is closed in $(\mathbf{Z}, \tau_k)$. By R10.1.3 there is $m \in \mathbb{N}$ such that $W \in \mathcal{Z}(E_m(k))$. By R5.3.2 W is associated with some $\Gamma \subseteq \{1, 2, \ldots, k^m\}$.

Let z be in $\mathbf{Z} - H$, and let $\overline{z}$ correspond to the sequence $\{x_n\}$. To show that the complement of H is τ_k-open, it suffices to verify that $D_m^z(k) \subseteq \mathbf{Z} - H$. Let $w \in D_m^z(k)$ with $\overline{w}$ corresponding to $\{y_n\}$.Since w, z, y_m, and x_m are congruent mod k^m and y_m, x_m are in $\{1, 2, \ldots, k^m\}$, $y_m = x_m$. By definition of a corresponding sequence, $\overline{z} \cap \mathcal{Z}(E_m(k)) = \{Z \in \mathcal{Z}(E_m(k)) : Z \text{is associated with } \Delta \subseteq \{1, \ldots, k^m\} - \{x_m\}\}$. Since $\overline{z} \notin W^\omega$, i.e., $W \notin \overline{z}$, $x_m \in \Gamma$. Thus $y_m \in \Gamma$ and, again by the definition of a corresponding sequence, $W \notin \overline{w}$. Thus $\overline{w} \notin W^\omega$ and so w is in $\mathbf{Z} - H$ as required.

Lemma R16.12 Let $m, k \in \mathbb{N}$ with $k \geq 2$, and let $z \in \mathbf{Z}$. For all $t \in \{1, 2, \ldots, k^m\}$, $C_m^t(k) \in \overline{z}$ if and only if $z \equiv t \bmod k^m$.

Proof: Fix $t \in \{1, 2, \ldots, k^m\}$. As an element of $\mathcal{Z}(E_m(k))$, $C_m^t(k)$ is associated with a unique subset of $\{1, 2, \ldots, k^m\}$, specifically $\{1, 2, \ldots, k^m\} - \{t\}$. Let $\overline{z}$ correspond to $\{x_n\}$. By definition of a corresponding sequence, $C_m^t(k) \in \overline{z}$ if and only if $\{1, 2, \ldots, k^m\} - \{t\}$ is contained in $\{1, 2, \ldots, k^m\} - \{x_m\}$, i.e., if and only if $t = x_m$. Since $x_m \equiv z \bmod k^m$, the conclusion follows.

Lemma R16.13 Let $k \in \mathbb{N}$ and let $W \in \mathcal{Z}_k$. Then W^ω is clopen in $\mathbb{N}_k$.

Proof: By R10.1.3 and R9.1.7 $\mathbf{Z} - W$ is also in $\mathcal{Z}_k$. It is easy to check that W^ω and $(\mathbf{Z} - W)^\omega$, which are both closed in $\mathbb{N}_k$, are complements of each other.

Lemma R16.14 Let $k \in \mathbb{N}$ with $k \geq 2$. Then f_k is an open map from $(\mathbf{Z}, \tau_k)$ to its image, $f_k[\mathbf{Z}]$.

Proof: Let $m \in \mathbb{N}$ and let $z \in \mathbf{Z}$. Let t be the unique element of $\{1, 2, \ldots, k^m\}$ such that $z \equiv t \bmod k^m$. Since $C_m^t(k)$ is in $\mathcal{Z}_k$, by R16.13 it suffices to show that $f_k[D_m^z(k)] = f_k[\mathbf{Z}] \cap ([C_m^t(k)]^\omega \cap \mathbf{R}_k)$. Since $w \in D_m^z(k)$ if and only $w \equiv z \equiv t \bmod k^m$, the desired equation is immediate from R16.12.

Proposition R16.15 Let $k \in \mathbb{N}$ with $k \geq 2$. Then $(\mathbf{R}_k, f_k)$ is a T_2 compactification of $(\mathbf{Z}, \tau_k)$.

Proof: By R12.6.9 $\{\overline{n} : n \in \mathbb{N}\}$ is dense in $\mathbf{R}_k$, and so its superset, $f_k[\mathbf{Z}]$, is also dense. Since $\mathbf{R}_k$ is compact and T_2 and f_k is one-to-one, the conclusion follows immediately from R16.11 and R16.14.

In the remainder of this section, the goal is to describe a topology on $\mathbf{Z}$ and an embedding, which identify $\mathbf{R}_\infty$ as a compactification of $\mathbf{Z}$. Recall from [5] and [6] that $\mathcal{Z}_\infty = \cup\{\mathcal{Z}_k : k \in \mathbb{N}\}$ is a normal basis for $\mathbb{N}$ generating $\mathbb{N}_\infty$, $\mathcal{O}_\infty$ is the additive identity in $\mathbf{R}_\infty$, and for $j \in \mathbb{N}$, $\mathcal{F}_\infty(j) = \hat{j} + \mathcal{O}_\infty$ where $\hat{j}$ denotes the $\mathcal{Z}_\infty$ point filter of j.

Definition R16.16 $f_\infty : \mathbf{Z} \to \mathbf{R}_\infty$ is defined by cases: If $z > 0$, then $f_\infty(z) = \mathcal{F}_\infty(z)$. If $z = 0$, then $f_\infty(z) = \mathcal{O}_\infty$. If $z < 0$, then $f_\infty(z) = {}^- \mathcal{F}_\infty(-z)$.

In R12.5.22 it was shown that, if $\mathcal{F} \in \mathbf{R}_\infty$, then $\mathcal{F} \cap \mathcal{Z}_k$ is in $\mathbf{R}_k$. That allows the following definition.

Definition R16.17 Let $k \in \mathbb{N}$ with $k \geq 2$. $\rho_k : \mathbf{R}_\infty \to \mathbf{R}_k$ is defined by $\rho_k(\mathcal{F}) = \mathcal{F} \cap \mathcal{Z}_k$.

Lemma R16.18 Let $k \in \mathbb{N}$ with $k \geq 2$, and let $\mathcal{F}, \mathcal{G}$ be in $\mathbf{R}_\infty$. Then $\rho_k(\mathcal{F} + \mathcal{G}) = \rho(\mathcal{F}) + \rho(\mathcal{G})$, $\rho_k(\mathcal{F} \cdot \mathcal{G}) = \rho(\mathcal{F}) \cdot \rho(\mathcal{G})$, and $\rho_k(\mathcal{F} - \mathcal{G}) = \rho(\mathcal{F}) - \rho(\mathcal{G})$.

Proof: The first two equations are a restatement of R12.2.6 in the notation of the current section. The third follows from $\rho_k(\mathcal{F}) = \rho_k((\mathcal{F} - \mathcal{G}) + \mathcal{G}) = \rho_k(\mathcal{F} - \mathcal{G}) + \rho_k(\mathcal{G})$.

Lemma R16.19 Let $k \in \mathbb{N}$ with $k \geq 2$. Then $\rho_k \circ f_\infty = f_k$.

Proof: Let $z \in \mathbf{Z}$. If $z = 0$, by R12.5.18 and R12.5.20, $\mathcal{O}_\infty \cap \mathcal{Z}_k = \mathcal{O}_k$, i.e., $\rho_k \circ f_\infty(0) = f_k(0)$. In the case $z > 0$, recall that $\mathcal{Z}_k$ intersected with the $\mathcal{Z}_\infty$ point filter of z is the $\mathcal{Z}_k$ point filter of z. That, the equation $\mathcal{O}_\infty \cap \mathcal{Z}_k = \mathcal{O}_k$, and R12.2.6i show that $\rho_k \circ f_\infty(z) = f_k(z)$. For the case $z < 0$, by R12.6.11 and R12.5.18 $(^-\mathcal{F}_\infty(-z)) \cap \mathcal{Z}_k$ is $^-(\mathcal{F}_\infty(-z) \cap \mathcal{Z}_k)$. By the previous case, $\mathcal{F}_\infty(-z) \cap \mathcal{Z}_k = f_k(-z)$. By R16.3 $f_k(-z) = -f_k(z)$. The double negative yields $\rho_k \circ f_\infty(z) = f_k(z)$.

Lemma R16.20 Let $\mathcal{F}, \mathcal{G}$ be in $\mathbf{R}_\infty$. If $\rho_k(\mathcal{F}) = \rho_k(\mathcal{G})$ for all $k \geq 2$, then $\mathcal{F} = \mathcal{G}$.

Proof: Since $\mathcal{Z}_\infty = \cup\{\mathcal{Z}_k : k \in \mathbb{N}\}$ and $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$, $\mathcal{F} = \cup\{\mathcal{F} \cap \mathcal{Z}_k : k \in \mathbb{N}$ and $k \geq 2\}$ and $\mathcal{G} = \cup\{\mathcal{G} \cap \mathcal{Z}_k : k \in \mathbb{N}$ and $k \geq 2\}$. The conclusion is immediate from these equations and the definition of ρ_k.

Lemma R16.21 Let $z, w \in \mathbf{Z}$. Then $f_\infty(z + w) = f_\infty(z) + f_\infty(w)$, $f_\infty(z - w) = f_\infty(z) - f_\infty(w)$, and $f_\infty(zw) = f_\infty(z) \cdot f_\infty(w)$.

Proof: Let $k \in \mathbb{N}$ with $k \geq 2$. $\rho_k(f_\infty(z + w)) = f_k(z + w) = f_k(z) + f_k(w)$ by R16.19 and R16.3. Also $\rho_k(f_\infty(z) + f_\infty(w)) = \rho_k(f_\infty(z)) + \rho_k(f_\infty(w)) = f_k(z) + f_k(w)$ by R16.18 and R16.19. The additive equation now follows from R16.20. Similar arguments yield the other two equations.

Lemma R16.22 f_∞ is one-to-one.

Proof: For any $k \geq 2$, $\rho_k \circ f_\infty = f_k$ by R16.19. By R16.5 f_k is one-to-one, and so f_∞ must also be one-to-one.

The last lemma implies that the weak uniformity induced by f_∞ on $\mathbf{Z}$ makes f_∞ a uniform embedding. By R12.6.18 $f_\infty[\mathbf{Z}]$ is dense in $\mathbf{R}_\infty$, and so by R1.6a $(\mathbf{R}_\infty, f_\infty)$ is a T_2 compactification of $\mathbf{Z}$ with the appropriate topology. Recall from [7] that $TBS(X)$ is the collection of all totally bounded separable uniformities on X.

Definition R16.23 $\mathcal{V}_\infty$ is the element of $TBS(\mathbf{Z})$ such that $\Psi_0(\mathcal{V}_\infty) = [(\mathbf{R}_\infty, f_\infty)]$.

The uniformities at each level k will also be needed.

Definition R16.24 Let $k \in \mathbb{N}$ with $k \geq 2$. $\mathcal{V}_k$ is the element of $TBS(\mathbf{Z})$ such that $\Psi_0(\mathcal{V}_k) = [(\mathbf{R}_k, f_k)]$.

Lemma R16.25 $\vee\{\mathcal{V}_k : k \geq 2\} \subseteq \mathcal{V}_\infty$.

Proof: For every $k \geq 2$, by R10.2.8 and the proof of R9.1.1iii the map $\mathcal{F} \mapsto \mathcal{F} \cap \mathcal{Z}_k$ from $\mathbb{N}_\infty$ to $\mathbb{N}_k$ is a continuous surjection, and so its restriction, ρ_k, is also continuous and onto. Thus by definition $\rho_k \circ f_\infty = f_k$ implies $[(\mathbf{R}_\infty, f_\infty)] \geq [(\mathbf{R}_k, f_k)]$. By R13.1.2 $\mathcal{V}_k \subseteq \mathcal{V}_\infty$ and so $\vee\{\mathcal{V}_k : k \geq 2\} \subseteq \mathcal{V}_\infty$.

For notational convenience f_∞ and f_k will be treated as maps into $\mathbb{N}_\infty$, respectively $\mathbb{N}_k$, in what follows. It will also be necessary to use temporarily the following cumbersome notation: For $W \in \mathcal{Z}_k \subseteq \mathcal{Z}_\infty$, ${}^\infty W^\omega = \{\mathcal{F} \in \omega(\mathcal{Z}_\infty) : W \in \mathcal{F}\}$ and ${}^k W^\omega = \{\mathcal{F} \in \omega(\mathcal{Z}_k) : W \in \mathcal{F}\}$.

Lemma R16.26 Let $W \in \mathcal{Z}_k \subseteq \mathcal{Z}_\infty$. Then $f_\infty^{-1}[{}^\infty W^\omega] = f_k^{-1}[{}^k W^\omega]$.

Proof: By definition of ρ_k and R16.19, for any z in $\mathbf{Z}$, $f_k(z) = \mathcal{Z}_k \cap f_\infty(z)$. It follows that $W \in f_k(z)$ if and only if $W \in f_\infty(z)$, which easily yields the conclusion.

Lemma R16.27 Let U be in the unique uniformity for $\mathbb{N}_\infty$. Then there exist $W_1, \ldots, W_j$ in $\mathcal{Z}_\infty$ such that $\cup_{i=1}^j {}^\infty W_i^\omega \times {}^\infty W_i^\omega$ is in the unique uniformity for $\mathbb{N}_\infty$ and $\cup_{i=1}^j {}^\infty W_i^\omega \times {}^\infty W_i^\omega \subseteq U$.

Proof: By an argument similar to that of R16.13 ${}^\infty W^\omega$ is clopen in $\mathbb{N}_\infty$ for all W in

$\mathcal{Z}_\infty$. Thus $\{{}^\infty W^\omega : W \in \mathcal{Z}_\infty\}$ is a clopen basis for $\mathbb{N}_\infty$. U must be an open neighborhood of the diagonal of $\mathbb{N}_\infty$. Using the clopen basis and the compactness of $\mathbb{N}_\infty$, we can find $W_1, \ldots, W_j$ in $\mathcal{Z}_\infty$ such that ${}^\infty W_1^\omega, \ldots, {}^\infty W_j^\omega$ cover $\mathbb{N}_\infty$ and $\cup_{i=1}^j \, {}^\infty W_i^\omega \times {}^\infty W_i^\omega \subseteq U$. Since the unique uniformity for $\mathbb{N}_\infty$ consists of all neighborhoods of its diagonal, it includes $\cup_{i=1}^j \, {}^\infty W_i^\omega \times {}^\infty W_i^\omega$.

Proposition R16.28 $\vee\{\mathcal{V}_k : k \geq 2\} = \mathcal{V}_\infty$.

Proof: Let $V \in V_\infty$. Since f_∞ is a uniform embedding into $\mathbf{R}_\infty$ and, by compactnes of both $\mathbf{R}_\infty$and $\mathbb{N}_\infty$, the unique uniformity of $\mathbf{R}_\infty$is the subspace uniformity from the unique uniformity of $\mathbb{N}_\infty$, $f_\infty \times f_\infty[V] = (f_\infty[\mathbf{Z}\,] \times f_\infty[\mathbf{Z}\,]) \cap U$ for some U in the unique uniformity for $\mathbb{N}_\infty$. Pick $W_1, \ldots, W_j$ in $\mathcal{Z}_\infty$ as in lemma R16.27. By R12.5.17i there is $m \geq 2$ in $\mathbb{N}$ such that $W_i \in \mathcal{Z}_m$ for $1 \leq i \leq j$. Some routine set algebra and lemma R16.26 yield

$$(f_\infty \times f_\infty)^{-1}[\cup_{i=1}^j \, {}^\infty W_i^\omega \times {}^\infty W_i^\omega] = (f_m \times f_m)^{-1}[\cup_{i=1}^j \, {}^m W_i^\omega \times {}^m W_i^\omega].$$

The left component of that equation is a subset of V and the right component is in $\mathcal{V}_m$ by the uniform continuity of f_m. Thus $V \in \mathcal{V}_m \subseteq \vee\{\mathcal{V}_k : k \geq 2\}$. The opposite containment is R16.25.

Corollary R16.29 $(\mathbf{R}_\infty, f_\infty)$ is equivalent to $\vee\{(\mathbf{R}_k\,, f_k) : k \geq 2\}$.

Proof: As in the proof of R13.1.6, the equivalence class of $\vee\{(\mathbf{R}_k\,, f_k) : k \geq 2\}$ is $\Psi_0(\vee\{\mathcal{V}_k : k \geq 2\})$. The conclusion follows from R16.28 and the definition of $\mathcal{V}_\infty$.

Definition R16.30 τ_∞ is the topology for $\mathbf{Z}$ such that $(\mathbf{R}_\infty, f_\infty)$ is a T_2 compactification of $(\mathbf{Z}\,, \tau_\infty)$.

Corollary R16.31 $\tau_\infty = \vee\{\tau_k : k \geq 2\}$.

Proof: Since $\tau(\mathcal{V}_k) = \tau_k$ and $\tau(\mathcal{V}_\infty) = \tau_\infty$, this follows from R16.28 and P2.14.

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http://www.susanjkleinart.com/compactification/

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