

## The Remnant Rings as Compactifications

The remnant rings were defined in [6] by removing the point-filters from  $\mathbf{N}_k$ , respectively  $\mathbf{N}_\infty$ , and observing that the restrictions of the addition and multiplication extended continuously to  $\mathbf{N}_k$ , respectively  $\mathbf{N}_\infty$ , make these remnant structures compact topological rings. As in [6], these rings will be denoted  $\mathbf{R}_k$ , respectively  $\mathbf{R}_\infty$ .

This section makes extensive use of definitions, notations, and results from [5] and [6]. In particular, the sequence associated with a non-point ultrafilter in  $\mathbf{N}_k$  (defined in R10.2.3) is repeatedly employed, as well as properties of these sequences: the recursive relationship of terms (R10.2.5), the recursive method of defining a non-point ultrafilter in  $\mathbf{R}_k$  (R10.2.6), and the way in which these sequences describe the operations (R12.4.4). Some special examples: The additive identity of  $\mathbf{R}_k$  is  $\mathcal{O}_k$ , which corresponds to the sequence  $\{k^n\}$ . For  $m \in \mathbf{N}$ , the corresponding element of  $\mathbf{R}_k$  (which is  $\hat{m} + \mathcal{O}_k$ , where  $\hat{m}$  is the point-filter of  $m$ ) will be denoted  $\mathcal{F}_k(m)$  when there are several possible values of  $k$  or, more simply,  $\overline{m}$ , when there is no ambiguity about  $k$ . By R12.5.9ii  $\mathcal{F}_k(m)$ , i.e.  $\overline{m}$ , corresponds to the sequence  $\{t_n\}$ , where  $t_n \equiv m \pmod{k^n}$  for all  $n$ . By R12.4.4  $\overline{m} + \overline{n} = \overline{m+n}$  and  $\overline{m} \cdot \overline{n} = \overline{mn}$  for  $m, n$  in  $\mathbf{N}$ . As the equations in the previous sentence illustrate, typical algebraic conventions are used throughout this section: The reader is expected to recognize the structures appropriate for the binary operations.

In R12.6.9 it was shown that  $\{\overline{n} : n \in \mathbf{N}\}$  is a dense non-discrete subset of  $\mathbf{R}_k$ . As a result  $\mathbf{R}_k$  with the map  $n \mapsto \overline{n}$  is not a compactification of  $\mathbf{N}$  with the discrete topology. Similar results hold for  $\mathbf{R}_\infty$ . In this section maps and topologies making the remnant rings compactifications will be described.

Because the ring structure is also of interest, the map  $n \mapsto \overline{n}$  will first be extended to the ring of integers,  $\mathbf{Z}$ .

**Lemma R16.1** Let  $k, m \in \mathbf{N}$  with  $k \geq 2$ . Then

- i) The negative of  $\overline{m}$  corresponds to the sequence  $\{t_n\}$ , where  $t_n \equiv -m \pmod{k^n}$  for all  $n$ .
- ii)  $\mathcal{O}_k$  corresponds to the sequence  $\{s_n\}$ , where  $s_n \equiv 0 \pmod{k^n}$  for all  $n$ .

Proof: By R12.6.2 the negative of  $\overline{m}$  corresponds to the sequence  $\{t_n\}$ , where  $t_n \equiv k^n - m \pmod{k^n}$  for all  $n$ , and by R12.5.8  $\mathcal{O}_k$  corresponds to the sequence  $\{s_n\}$ , where  $s_n \equiv k^n \pmod{k^n}$  for all  $n$ . Since  $k^n \equiv 0 \pmod{k^n}$ , i) and ii) are immediate.

The previous lemma allows the notation used to this point for natural numbers only to be extended to the integers: For  $m \in \mathbf{Z}$  and  $k \geq 2$  in  $\mathbf{N}$ ,  $\overline{m}$  corresponds to the sequence  $\{t_n\}$ , where  $t_n \equiv m \pmod{k^n}$  for all  $n$ . By R16.1ii  $\mathcal{O}_k = \overline{0}$ . In what follows, when  $k$  is unambiguous,  $\overline{0}$  will normally be used instead of  $\mathcal{O}_k$ .

**Lemma R16.2** Let  $k \in \mathbf{N}$  with  $k \geq 2$ , and let  $\mathcal{F}, \mathcal{G}$  in  $\mathbf{R}_k$  correspond to the sequences  $\{x_n\}, \{y_n\}$  respectively. Then  $\mathcal{F} - \mathcal{G}$  corresponds to the sequence  $\{z_n\}$ , where  $z_n \equiv x_n - y_n \pmod{k^n}$  for all  $n$ .

Proof: Let  $-\mathcal{G}$  be the element of  $\mathbf{R}_k$  corresponding to  $\{w_n\}$ , where  $w_n \equiv k^n - y_n \pmod{k^n}$ . By R12.6.3  $-\mathcal{G}$  is the additive inverse of  $\mathcal{G}$ . Since  $\mathcal{F} - \mathcal{G} = \mathcal{F} + (-\mathcal{G})$ , by R12.4.4  $z_n \equiv x_n + (k^n - y_n) \pmod{k^n}$ , i.e.,  $z_n \equiv x_n - y_n \pmod{k^n}$  for all  $n$ .

**Lemma R16.3** Let  $k \in \mathbf{N}$  with  $k \geq 2$ , and let  $m, n \in \mathbf{Z}$ . Then  $\overline{m+n} = \overline{m} + \overline{n}$ ,  $\overline{m-n} = \overline{m} - \overline{n}$ , and  $\overline{mn} = \overline{m} \cdot \overline{n}$ .

Proof: For any integer  $z$ ,  $\bar{z}$  corresponds to the sequence  $\{x_n\}$ , where  $x_n \equiv z \pmod{k^n}$  for all  $n$ . The assertions follow easily from this, R12.4.4, R16.2, and R10.2.4.

**Lemma R16.4** Let  $k \in \mathbf{N}$  with  $k \geq 2$ , let  $m \in \mathbf{Z}$ , and let  $\bar{m}$  correspond to the sequence  $\{x_n\}$ . Let  $t$  be in  $\mathbf{N}$  with  $|m| < k^t$ . If  $m > 0$ , then  $x_t = m$ . If  $m \leq 0$ , then  $x_t = k^t + m$ .

Proof: By definition  $x_t \in \{1, 2, \dots, k^t\}$  and as noted above  $x_t \equiv m \pmod{k^t}$ . Since  $|m| < k^t$ , clearly  $m > 0$  implies  $x_t = m$ . For  $m \leq 0$ ,  $-m = |m| < k^t$  and so  $0 < k^t + m \leq k^t$ . Thus  $k^t + m$  is the unique element of  $\{1, 2, \dots, k^t\}$  congruent to  $m \pmod{k^t}$ , i.e.,  $x_t = k^t + m$ .

**Lemma R16.5** Let  $k \in \mathbf{N}$  with  $k \geq 2$ . The map from  $\mathbf{Z}$  into  $\mathbf{R}_k$  by  $z \mapsto \bar{z}$  is one-to-one.

Proof: Let  $z \neq w$  be in  $\mathbf{Z}$ , let  $\bar{z}$  correspond to  $\{x_n\}$ , and let  $\bar{w}$  correspond to  $\{y_n\}$ . Let  $t$  be a positive integer such that  $\max\{|z|, |w|\} < k^t$ . If  $z$  and  $w$  are both positive or both non-positive, it follows easily from R16.4 that  $x_t \neq y_t$ . In the other case, assume without loss of generality  $z > 0$  and  $w \leq 0$  so that  $x_t = z$  and  $y_t = k^t + w$  by R16.4. The lemma also applies at the next level, i.e.,  $x_{t+1} = z$  and  $y_{t+1} = k^{t+1} + w$ . Clearly either  $x_t \neq y_t$  or  $x_{t+1} \neq y_{t+1}$ . In all cases the corresponding sequences differ in at least one position and so  $\bar{z} \neq \bar{w}$  by the definition of the sequence corresponding to a non-point ultrafilter (R10.2.3).

**Definition R16.6** Let  $k \in \mathbf{N}$  with  $k \geq 2$ .  $f_k$  is the map from  $\mathbf{Z}$  to  $\mathbf{R}_k$  defined by  $f_k(z) = \bar{z}$ .

By R12.6.9  $\{\bar{n} : n \in \mathbf{N}\}$  is dense in  $\mathbf{R}_k$ , and so R16.5 shows that  $(\mathbf{R}_k, f_k)$  is a  $T_2$  compactification of  $\mathbf{Z}$  with the weak topology induced by  $f_k$ . The next few results describe that topology intrinsically.

For  $k \in \mathbf{N}$  with  $k \geq 2$  and  $m \in \mathbf{Z}$ ,  $D_n^m(k)$  will denote the equivalence class in  $\mathbf{Z}$  of  $m \pmod{k^n}$ .  $\mathcal{B}_k = \{D_n^m(k) : m \in \mathbf{Z} \text{ and } n \in \mathbf{N}\}$ .

**Lemma R16.7** Let  $k \in \mathbf{N}$  with  $k \geq 2$ .  $\mathcal{B}_k$  is a basis for a topology on  $\mathbf{Z}$ .

Proof: Clearly the union of  $\mathcal{B}_k$  is  $\mathbf{Z}$ . Let  $x \in D_n^m(k) \cap D_t^j(k)$  and assume without loss of generality that  $n \leq t$ , so that  $D_t^x(k) \subseteq D_n^m(k)$ . Since  $D_n^m(k) = D_n^x(k)$  and  $D_t^j(k) = D_t^x(k)$ ,  $x \in D_t^x(k) \subseteq D_n^m(k) \cap D_t^j(k)$ , and so the intersection requirement also holds.

**Definition R16.8** Let  $k \in \mathbf{N}$  with  $k \geq 2$ .  $\tau_k$  is the topology for  $\mathbf{Z}$  with basis  $\mathcal{B}_k$ .

**Lemma R16.9** Let  $k \in \mathbf{N}$  with  $k \geq 2$  and let  $m \in \mathbf{Z}$ . Then for all  $n \in \mathbf{N}$   $D_n^m(k)$  is clopen relative to  $\tau_k$ .

Proof: For any  $n$   $D_n^m(k)$  is a basic open set in  $\tau_k$ . Its complement, the union of the other  $k^n$ -equivalence classes, is also  $\tau_k$ -open.

**Proposition R16.10** Let  $k \in \mathbf{N}$  with  $k \geq 2$ .  $(\mathbf{Z}, \tau_k)$  is a zero-dimensional  $T_{3\frac{1}{2}}$  space.

Proof: R16.9 shows that  $\mathcal{B}_k$  is a clopen basis for  $\tau_k$ . Let  $z \neq w$  be integers and assume without loss of generality that  $z < w$ . Pick a positive integer  $n$  such that  $0 < w - z < k^n$ . Then  $w$  is not equivalent to  $z \pmod{k^n}$  and so  $D_n^w(k) \cap D_n^z(k) = \emptyset$ , i.e.,  $\tau_k$  is  $T_2$ . A zero-dimensional Hausdorff topology is automatically  $T_{3\frac{1}{2}}$ .

In the next two results, notation from [5] will be used: For  $k, n \in \mathbf{N}$ ,  $E_n(k)$  will denote equivalence mod  $k^n$  restricted to  $\mathbf{N}$ , and  $C_n^i(k)$ , where  $1 \leq i \leq k^n$ , denotes the  $\mathbf{N}$ -equivalence class of  $i$  relative to  $E_n(k)$ . Of course,  $C_n^i(k) = D_n^i(k) \cap \mathbf{N}$ .

**Lemma R16.11** Let  $k \in \mathbf{N}$  with  $k \geq 2$ . Then  $f_k$  is continuous from  $(\mathbf{Z}, \tau_k)$  to  $\mathbf{R}_k$ .

Proof: Let  $W$  be in  $\mathcal{Z}_k$ , the normal basis used to generate  $\mathbf{N}_k$  in [5]. By P3.6  $\mathcal{Z}_k^\omega$  is a normal basis and so a basis for the closed subsets of  $\mathbf{N}_k$ . Thus it is sufficient to

show that  $H = f_k^{-1}[W^\omega \cap \mathbf{R}_k]$  is closed in  $(\mathbf{Z}, \tau_k)$ . By R10.1.3 there is  $m \in \mathbf{N}$  such that  $W \in \mathcal{Z}(E_m(k))$ . By R5.3.2  $W$  is associated with some  $\Gamma \subseteq \{1, 2, \dots, k^m\}$ .

Let  $z$  be in  $\mathbf{Z} - H$ , and let  $\bar{z}$  correspond to the sequence  $\{x_n\}$ . To show that the complement of  $H$  is  $\tau_k$ -open, it suffices to verify that  $D_m^z(k) \subseteq \mathbf{Z} - H$ . Let  $w \in D_m^z(k)$  with  $\bar{w}$  corresponding to  $\{y_n\}$ . Since  $w, z, y_m$ , and  $x_m$  are congruent mod  $k^m$  and  $y_m, x_m$  are in  $\{1, 2, \dots, k^m\}$ ,  $y_m = x_m$ . By definition of a corresponding sequence,  $\bar{z} \cap \mathcal{Z}(E_m(k)) = \{Z \in \mathcal{Z}(E_m(k)) : Z \text{ is associated with } \Delta \subseteq \{1, \dots, k^m\} - \{x_m\}\}$ . Since  $\bar{z} \notin W^\omega$ , i.e.,  $W \notin \bar{z}$ ,  $x_m \in \Gamma$ . Thus  $y_m \in \Gamma$  and, again by the definition of a corresponding sequence,  $W \notin \bar{w}$ . Thus  $\bar{w} \notin W^\omega$  and so  $w$  is in  $\mathbf{Z} - H$  as required.

**Lemma R16.12** Let  $m, k \in \mathbf{N}$  with  $k \geq 2$ , and let  $z \in \mathbf{Z}$ . For all  $t \in \{1, 2, \dots, k^m\}$ ,  $C_m^t(k) \in \bar{z}$  if and only if  $z \equiv t \pmod{k^m}$ .

Proof: Fix  $t \in \{1, 2, \dots, k^m\}$ . As an element of  $\mathcal{Z}(E_m(k))$ ,  $C_m^t(k)$  is associated with a unique subset of  $\{1, 2, \dots, k^m\}$ , specifically  $\{1, 2, \dots, k^m\} - \{t\}$ . Let  $\bar{z}$  correspond to  $\{x_n\}$ . By definition of a corresponding sequence,  $C_m^t(k) \in \bar{z}$  if and only if  $\{1, 2, \dots, k^m\} - \{t\}$  is contained in  $\{1, 2, \dots, k^m\} - \{x_m\}$ , i.e., if and only if  $t = x_m$ . Since  $x_m \equiv z \pmod{k^m}$ , the conclusion follows.

**Lemma R16.13** Let  $k \in \mathbf{N}$  and let  $W \in \mathcal{Z}_k$ . Then  $W^\omega$  is clopen in  $\mathbf{N}_k$ .

Proof: By R10.1.3 and R9.1.7  $\mathbf{Z} - W$  is also in  $\mathcal{Z}_k$ . It is easy to check that  $W^\omega$  and  $(\mathbf{Z} - W)^\omega$ , which are both closed in  $\mathbf{N}_k$ , are complements of each other.

**Lemma R16.14** Let  $k \in \mathbf{N}$  with  $k \geq 2$ . Then  $f_k$  is an open map from  $(\mathbf{Z}, \tau_k)$  to its image,  $f_k[\mathbf{Z}]$ .

Proof: Let  $m \in \mathbf{N}$  and let  $z \in \mathbf{Z}$ . Let  $t$  be the unique element of  $\{1, 2, \dots, k^m\}$  such that  $z \equiv t \pmod{k^m}$ . Since  $C_m^t(k)$  is in  $\mathcal{Z}_k$ , by R16.13 it suffices to show that  $f_k[D_m^z(k)] = f_k[\mathbf{Z}] \cap ([C_m^t(k)]^\omega \cap \mathbf{R}_k)$ . Since  $w \in D_m^z(k)$  if and only if  $w \equiv z \equiv t \pmod{k^m}$ , the desired equation is immediate from R16.12.

**Proposition R16.15** Let  $k \in \mathbf{N}$  with  $k \geq 2$ . Then  $(\mathbf{R}_k, f_k)$  is a  $T_2$  compactification of  $(\mathbf{Z}, \tau_k)$ .

Proof: By R12.6.9  $\{\bar{n} : n \in \mathbf{N}\}$  is dense in  $\mathbf{R}_k$ , and so its superset,  $f_k[\mathbf{Z}]$ , is also dense. Since  $\mathbf{R}_k$  is compact and  $T_2$  and  $f_k$  is one-to-one, the conclusion follows immediately from R16.11 and R16.14.

In the remainder of this section, the goal is to describe a topology on  $\mathbf{Z}$  and an embedding, which identify  $\mathbf{R}_\infty$  as a compactification of  $\mathbf{Z}$ . Recall from [5] and [6] that  $\mathcal{Z}_\infty = \cup\{\mathcal{Z}_k : k \in \mathbf{N}\}$  is a normal basis for  $\mathbf{N}$  generating  $\mathbf{N}_\infty$ ,  $\mathcal{O}_\infty$  is the additive identity in  $\mathbf{R}_\infty$ , and for  $j \in \mathbf{N}$ ,  $\mathcal{F}_\infty(j) = \hat{j} + \mathcal{O}_\infty$  where  $\hat{j}$  denotes the  $\mathcal{Z}_\infty$  point filter of  $j$ .

**Definition R16.16**  $f_\infty : \mathbf{Z} \rightarrow \mathbf{R}_\infty$  is defined by cases: If  $z > 0$ , then  $f_\infty(z) = \mathcal{F}_\infty(z)$ . If  $z = 0$ , then  $f_\infty(z) = \mathcal{O}_\infty$ . If  $z < 0$ , then  $f_\infty(z) = {}^- \mathcal{F}_\infty(-z)$ .

In R12.5.22 it was shown that, if  $\mathcal{F} \in \mathbf{R}_\infty$ , then  $\mathcal{F} \cap \mathcal{Z}_k$  is in  $\mathbf{R}_k$ . That allows the following definition.

**Definition R16.17** Let  $k \in \mathbf{N}$  with  $k \geq 2$ .  $\rho_k : \mathbf{R}_\infty \rightarrow \mathbf{R}_k$  is defined by  $\rho_k(\mathcal{F}) = \mathcal{F} \cap \mathcal{Z}_k$ .

**Lemma R16.18** Let  $k \in \mathbf{N}$  with  $k \geq 2$ , and let  $\mathcal{F}, \mathcal{G}$  be in  $\mathbf{R}_\infty$ . Then  $\rho_k(\mathcal{F} + \mathcal{G}) = \rho_k(\mathcal{F}) + \rho_k(\mathcal{G})$ ,  $\rho_k(\mathcal{F} \cdot \mathcal{G}) = \rho_k(\mathcal{F}) \cdot \rho_k(\mathcal{G})$ , and  $\rho_k(\mathcal{F} - \mathcal{G}) = \rho_k(\mathcal{F}) - \rho_k(\mathcal{G})$ .

Proof: The first two equations are a restatement of R12.2.6 in the notation of the current section. The third follows from  $\rho_k(\mathcal{F}) = \rho_k((\mathcal{F} - \mathcal{G}) + \mathcal{G}) = \rho_k(\mathcal{F} - \mathcal{G}) + \rho_k(\mathcal{G})$ .

**Lemma R16.19** Let  $k \in \mathbf{N}$  with  $k \geq 2$ . Then  $\rho_k \circ f_\infty = f_k$ .

Proof: Let  $z \in \mathbf{Z}$ . If  $z = 0$ , by R12.5.18 and R12.5.20,  $\mathcal{O}_\infty \cap \mathcal{Z}_k = \mathcal{O}_k$ , i.e.,  $\rho_k \circ f_\infty(0) = f_k(0)$ . In the case  $z > 0$ , recall that  $\mathcal{Z}_k$  intersected with the  $\mathcal{Z}_\infty$  point filter of  $z$  is the  $\mathcal{Z}_k$  point filter of  $z$ . That, the equation  $\mathcal{O}_\infty \cap \mathcal{Z}_k = \mathcal{O}_k$ , and R12.2.6i show that  $\rho_k \circ f_\infty(z) = f_k(z)$ . For the case  $z < 0$ , by R12.6.11 and R12.5.18  $(\neg \mathcal{F}_\infty(-z)) \cap \mathcal{Z}_k$  is  $\neg(\mathcal{F}_\infty(-z) \cap \mathcal{Z}_k)$ . By the previous case,  $\mathcal{F}_\infty(-z) \cap \mathcal{Z}_k = f_k(-z)$ . By R16.3  $f_k(-z) = -f_k(z)$ . The double negative yields  $\rho_k \circ f_\infty(z) = f_k(z)$ .

**Lemma R16.20** Let  $\mathcal{F}, \mathcal{G}$  be in  $\mathbf{R}_\infty$ . If  $\rho_k(\mathcal{F}) = \rho_k(\mathcal{G})$  for all  $k \geq 2$ , then  $\mathcal{F} = \mathcal{G}$ .

Proof: Since  $\mathcal{Z}_\infty = \cup\{\mathcal{Z}_k : k \in \mathbf{N}\}$  and  $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$ ,  $\mathcal{F} = \cup\{\mathcal{F} \cap \mathcal{Z}_k : k \in \mathbf{N} \text{ and } k \geq 2\}$  and  $\mathcal{G} = \cup\{\mathcal{G} \cap \mathcal{Z}_k : k \in \mathbf{N} \text{ and } k \geq 2\}$ . The conclusion is immediate from these equations and the definition of  $\rho_k$ .

**Lemma R16.21** Let  $z, w \in \mathbf{Z}$ . Then  $f_\infty(z + w) = f_\infty(z) + f_\infty(w)$ ,  $f_\infty(z - w) = f_\infty(z) - f_\infty(w)$ , and  $f_\infty(zw) = f_\infty(z) \cdot f_\infty(w)$ .

Proof: Let  $k \in \mathbf{N}$  with  $k \geq 2$ .  $\rho_k(f_\infty(z + w)) = f_k(z + w) = f_k(z) + f_k(w)$  by R16.19 and R16.3. Also  $\rho_k(f_\infty(z) + f_\infty(w)) = \rho_k(f_\infty(z)) + \rho_k(f_\infty(w)) = f_k(z) + f_k(w)$  by R16.18 and R16.19. The additive equation now follows from R16.20. Similar arguments yield the other two equations.

**Lemma R16.22**  $f_\infty$  is one-to-one.

Proof: For any  $k \geq 2$ ,  $\rho_k \circ f_\infty = f_k$  by R16.19. By R16.5  $f_k$  is one-to-one, and so  $f_\infty$  must also be one-to-one.

The last lemma implies that the weak uniformity induced by  $f_\infty$  on  $\mathbf{Z}$  makes  $f_\infty$  a uniform embedding. By R12.6.18  $f_\infty[\mathbf{Z}]$  is dense in  $\mathbf{R}_\infty$ , and so by R1.6a  $(\mathbf{R}_\infty, f_\infty)$  is a  $T_2$  compactification of  $\mathbf{Z}$  with the appropriate topology. Recall from [7] that  $TBS(X)$  is the collection of all totally bounded separable uniformities on  $X$ .

**Definition R16.23**  $\mathcal{V}_\infty$  is the element of  $TBS(\mathbf{Z})$  such that  $\Psi_0(\mathcal{V}_\infty) = [(\mathbf{R}_\infty, f_\infty)]$ .

The uniformities at each level  $k$  will also be needed.

**Definition R16.24** Let  $k \in \mathbf{N}$  with  $k \geq 2$ .  $\mathcal{V}_k$  is the element of  $TBS(\mathbf{Z})$  such that  $\Psi_0(\mathcal{V}_k) = [(\mathbf{R}_k, f_k)]$ .

**Lemma R16.25**  $\vee\{\mathcal{V}_k : k \geq 2\} \subseteq \mathcal{V}_\infty$ .

Proof: For every  $k \geq 2$ , by R10.2.8 and the proof of R9.1.1iii the map  $\mathcal{F} \mapsto \mathcal{F} \cap \mathcal{Z}_k$  from  $\mathbf{N}_\infty$  to  $\mathbf{N}_k$  is a continuous surjection, and so its restriction,  $\rho_k$ , is also continuous and onto. Thus by definition  $\rho_k \circ f_\infty = f_k$  implies  $[(\mathbf{R}_\infty, f_\infty)] \geq [(\mathbf{R}_k, f_k)]$ . By R13.1.2  $\mathcal{V}_k \subseteq \mathcal{V}_\infty$  and so  $\vee\{\mathcal{V}_k : k \geq 2\} \subseteq \mathcal{V}_\infty$ .

For notational convenience  $f_\infty$  and  $f_k$  will be treated as maps into  $\mathbf{N}_\infty$ , respectively  $\mathbf{N}_k$ , in what follows. It will also be necessary to use temporarily the following cumbersome notation: For  $W \in \mathcal{Z}_k \subseteq \mathcal{Z}_\infty$ ,  ${}^\infty W^\omega = \{\mathcal{F} \in \omega(\mathcal{Z}_\infty) : W \in \mathcal{F}\}$  and  ${}^k W^\omega = \{\mathcal{F} \in \omega(\mathcal{Z}_k) : W \in \mathcal{F}\}$ .

**Lemma R16.26** Let  $W \in \mathcal{Z}_k \subseteq \mathcal{Z}_\infty$ . Then  $f_\infty^{-1}[{}^\infty W^\omega] = f_k^{-1}[{}^k W^\omega]$ .

Proof: By definition of  $\rho_k$  and R16.19, for any  $z$  in  $\mathbf{Z}$ ,  $f_k(z) = \mathcal{Z}_k \cap f_\infty(z)$ . It follows that  $W \in f_k(z)$  if and only if  $W \in f_\infty(z)$ , which easily yields the conclusion.

**Lemma R16.27** Let  $U$  be in the unique uniformity for  $\mathbf{N}_\infty$ . Then there exist  $W_1, \dots, W_j$  in  $\mathcal{Z}_\infty$  such that  $\cup_{i=1}^j {}^\infty W_i^\omega \times {}^\infty W_i^\omega$  is in the unique uniformity for  $\mathbf{N}_\infty$  and  $\cup_{i=1}^j {}^\infty W_i^\omega \times {}^\infty W_i^\omega \subseteq U$ .

Proof: By an argument similar to that of R16.13  ${}^\infty W^\omega$  is clopen in  $\mathbf{N}_\infty$  for all  $W$  in

$\mathcal{Z}_\infty$ . Thus  $\{\infty W^\omega : W \in \mathcal{Z}_\infty\}$  is a clopen basis for  $\mathbf{N}_\infty$ .  $U$  must be an open neighborhood of the diagonal of  $\mathbf{N}_\infty$ . Using the clopen basis and the compactness of  $\mathbf{N}_\infty$ , we can find  $W_1, \dots, W_j$  in  $\mathcal{Z}_\infty$  such that  $\infty W_1^\omega, \dots, \infty W_j^\omega$  cover  $\mathbf{N}_\infty$  and  $\cup_{i=1}^j \infty W_i^\omega \times \infty W_i^\omega \subseteq U$ . Since the unique uniformity for  $\mathbf{N}_\infty$  consists of all neighborhoods of its diagonal, it includes  $\cup_{i=1}^j \infty W_i^\omega \times \infty W_i^\omega$ .

**Proposition R16.28**  $\vee\{\mathcal{V}_k : k \geq 2\} = \mathcal{V}_\infty$ .

Proof: Let  $V \in \mathcal{V}_\infty$ . Since  $f_\infty$  is a uniform embedding into  $\mathbf{R}_\infty$  and, by compactness of both  $\mathbf{R}_\infty$  and  $\mathbf{N}_\infty$ , the unique uniformity of  $\mathbf{R}_\infty$  is the subspace uniformity from the unique uniformity of  $\mathbf{N}_\infty$ ,  $f_\infty \times f_\infty[V] = (f_\infty[\mathbf{Z}] \times f_\infty[\mathbf{Z}]) \cap U$  for some  $U$  in the unique uniformity for  $\mathbf{N}_\infty$ . Pick  $W_1, \dots, W_j$  in  $\mathcal{Z}_\infty$  as in lemma R16.27. By R12.5.17i there is  $m \geq 2$  in  $\mathbf{N}$  such that  $W_i \in \mathcal{Z}_m$  for  $1 \leq i \leq j$ . Some routine set algebra and lemma R16.26 yield

$$(f_\infty \times f_\infty)^{-1}[\cup_{i=1}^j \infty W_i^\omega \times \infty W_i^\omega] = (f_m \times f_m)^{-1}[\cup_{i=1}^j {}^m W_i^\omega \times {}^m W_i^\omega].$$

The left component of that equation is a subset of  $V$  and the right component is in  $\mathcal{V}_m$  by the uniform continuity of  $f_m$ . Thus  $V \in \mathcal{V}_m \subseteq \vee\{\mathcal{V}_k : k \geq 2\}$ . The opposite containment is R16.25.

**Corollary R16.29**  $(\mathbf{R}_\infty, f_\infty)$  is equivalent to  $\vee\{(\mathbf{R}_k, f_k) : k \geq 2\}$ .

Proof: As in the proof of R13.1.6, the equivalence class of  $\vee\{(\mathbf{R}_k, f_k) : k \geq 2\}$  is  $\Psi_0(\vee\{\mathcal{V}_k : k \geq 2\})$ . The conclusion follows from R16.28 and the definition of  $\mathcal{V}_\infty$ .

**Definition R16.30**  $\tau_\infty$  is the topology for  $\mathbf{Z}$  such that  $(\mathbf{R}_\infty, f_\infty)$  is a  $T_2$  compactification of  $(\mathbf{Z}, \tau_\infty)$ .

**Corollary R16.31**  $\tau_\infty = \vee\{\tau_k : k \geq 2\}$ .

Proof: Since  $\tau(\mathcal{V}_k) = \tau_k$  and  $\tau(\mathcal{V}_\infty) = \tau_\infty$ , this follows from R16.28 and P2.14.

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<http://www.susanjkleinart.com/compactification/>

## References

1. This Website, P2: Uniform Spaces
2. This Website, P3: Normal Bases
3. This Website, R5: Finite-Point Compactifications
4. This Website, R9: Directed Sets of Normal Bases
5. This Website, R10: Some Metric Compactifications of the Natural Numbers
6. This Website, R12: Extension of Arithmetic Operations
7. This Website, R13: Mixed Suprema

**Added 2021**

It follows easily from R16.31 and R16.8 that  $\cup\{\mathcal{B}_k : k \geq 2\}$  is a subbase for  $\tau_\infty$ . This added comment notes that  $\cup\{\mathcal{B}_k : k \geq 2\}$  is also a base for  $\tau_\infty$ .

**Lemma R16.Add.1** Let  $k \in \mathbf{N}$  with  $k \geq 2$ . For any  $m \in \mathbf{Z}$  and any  $n \in \mathbf{N}$ ,  $D_{n+1}^m(k) \subseteq D_n^m(k)$ .

Proof: Pick  $m \in \mathbf{Z}$  and  $n \in \mathbf{N}$ , and let  $t \in D_{n+1}^m(k)$ . Then  $k^{n+1}$  divides  $t - m$  so that  $k^n$  divides  $t - m$ , i.e.,  $t \in D_n^m(k)$ .

**Lemma R16.Add.2** Let  $m$  be in  $\mathbf{Z}$  and  $n$  in  $\mathbf{N}$ . Let  $k_1, k_2, \dots, k_t$  be in  $\mathbf{N}$ , where  $t$  is a positive integer, with  $k_i \geq 2$  for each  $i$ . Then  $\cap_{i=1}^t D_n^m(k_i) = D_n^m(\text{LCM}\{k_1, \dots, k_t\})$ , where LCM means least common multiple.

Proof: Let  $j = \text{LCM}\{k_1, \dots, k_t\}$  and recall that  $j^n = \text{LCM}\{k_1^n, \dots, k_t^n\}$ . If  $x$  is in  $D_n^m(j)$ , since  $j^n$  divides  $x - m$ ,  $k_i^n$  divides  $x - m$  for each  $i$ , i.e.,  $x \in \cap_{i=1}^t D_n^m(k_i)$ . If  $x \in \cap_{i=1}^t D_n^m(k_i)$ , then  $k_i^n$  divides  $x - m$  for each  $i$ . Thus the LCM of  $\{k_1^n, \dots, k_t^n\}$ , which is  $j^n$ , divides  $x - m$ , i.e.,  $x \in D_n^m(j)$ .

**Proposition R16.Add.3** The set  $\cup\{\mathcal{B}_k : k \geq 2\}$  is a base for  $\tau_\infty$ .

Proof: Let  $x \in O \in \tau_\infty$ . As noted above,  $\cup\{\mathcal{B}_k : k \geq 2\}$  is a subbase for  $\tau_\infty$ , and so  $x \in \cap_{i=1}^t D_{n_i}^{m_i}(k_i) \subseteq O$ , where  $t, n_1, \dots, n_t, k_1, k_2, \dots, k_t$  in  $\mathbf{N}$  with  $k_i \geq 2$  for all  $i$  and  $m_1, \dots, m_t$  are in  $\mathbf{Z}$ . Since the subbasic sets are equivalence classes containing  $x$ ,  $D_{n_i}^{m_i}(k_i) = D_{n_i}^x(k_i)$  for each  $i$ . Let  $n = \max\{n_1, \dots, n_t\}$ . It follows from R16.Add.1 that  $D_n^x(k_i) \subseteq D_{n_i}^x(k_i)$  for each  $i$ . Thus we have  $x \in \cap_{i=1}^t D_n^x(k_i) \subseteq O$ . By R16.Add.2  $\cap_{i=1}^t D_n^x(k_i) = D_n^x(\text{LCM}\{k_1, \dots, k_t\})$ , which is in  $\cup\{\mathcal{B}_k : k \geq 2\}$ . The conclusion follows.