Algebraic Structure of the Remnant Rings

The remnant rings were defined in [8] by removing the point-filters from $\mathbb{N}_k$, respectively $\mathbb{N}_\infty$, and observing that the restrictions of the addition and multiplication extended continuously to $\mathbb{N}_k$, respectively $\mathbb{N}_\infty$, make the structures topological rings. As in [8], these rings will be denoted $\mathbf{R}_k$, respectively $\mathbf{R}_\infty$.

The main results describe the remnant rings as follows: If p is a prime, then $\mathbf{R}_p$ is isomorphic to the p-adic integers. For k composite, $\mathbf{R}_k$ is isomorphic to a finite direct product, with each distinct prime factor p of k contributing a single factor, $\mathbf{R}_p$, to the direct product. $\mathbf{R}_\infty$is isomorphic to the infinite direct product $\Pi\{\mathbf{R}_p : p \in \mathbb{N} \text{ is prime}\}$.

This section makes extensive use of definitions, notations, and results from [7] and [8]. In particular, the sequence associated with a non-point ultrafilter in $\mathbb{N}_k$ (defined in R10.2.3) is repeatedly employed, as well as properties of these sequences: the recursive relationship of terms (R10.2.5), the recursive method of defining a non-point ultrafilter in $\mathbb{N}_k$ (R10.2.6), and the way in which these sequences describe the operations (R12.4.4). An example: The additive identity of $\mathbf{R}_k$ is $\mathcal{O}_k$, which corresponds to the sequence $\{k^n\}$.

In [10] the map f_k from $\mathbf{Z}$ to $\mathbf{R}_k$ was defined (R16.6) and shown to be a continuous, one-to-one ring homomorphism with $f_k(1)$ equal to the multiplicative identity of $\mathbf{R}_k$. For any $z \in \mathbf{Z}$, by R12.5.9ii and R16.1 $f_k(z)$ corresponds to the sequence $\{x_n\}$, where $x_n \equiv z$ mod k^n for all n. When a discussion involves only one value of k, $\overline{z}$ will denote $f_k(z)$.

Algebraic Properties of $\mathbf{R}_k$

In R12.6.5 $\mathbf{R}_k$ was shown to be a commutative ring with unity for all natural numbers $k \geq 2$. (When $k = 1$, since $\mathbb{N}_1$ is the one-point compactification, the ring $\mathbf{R}_1$ is the zero ring. This case will be ignored in this subsection.) Furthermore, if $p \in \mathbb{N}$ is a prime, by R12.5.16 $\mathbf{R}_p$ is an integral domain.

Lemma R17.1.1 Let $k \in \mathbb{N}$ with $k \geq 2$, and let $\mathcal{F}$ in $\mathbf{R}_k$ correspond to the sequence $\{x_n\}$. Let x_1 be invertible mod k. Then x_n is invertible mod k^n for all n .

Proof: By induction on n. The statement for $n = 1$ is given and so assume x_n is invertible mod k^n. By R10.2.5 there is $j \in \{0, 1, \ldots, k-1\}$ such that $x_{n+1} = x_n + jk^n$. By the induction hypothesis there are integers c and d such that $cx_n = 1 + dk^n$. Then $cx_{n+1} = 1 + dk^n + jk^n$ and so x_{n+1} and k^n are relatively prime. It follows that x_{n+1} and k^{n+1} are also relatively prime, and so x_{n+1} must be invertible mod k^{n+1}.

Proposition R17.1.2 Let $k \in \mathbb{N}$ with $k \geq 2$, and let $\mathcal{F}$ in $\mathbf{R}_k$ correspond to the sequence $\{x_n\}$. Then $\mathcal{F}$ is invertible if and only if x_1 is invertible mod k.

Proof: If $\mathcal{F}$ is invertible, then there is $\mathcal{G}$ in $\mathbf{R}_k$ corresponding to $\{y_n\}$ such that $\mathcal{F}\mathcal{G} = \overline{1}$. By R12.4.4 $x_1 y_1 \equiv 1$ mod k, i.e., x_1 is invertible mod k. Conversely, let x_1 be invertible mod k, and, using the previous lemma, let y_n be the element of $\{1, 2, \ldots, k^n\}$ such that $x_n y_n \equiv 1$ mod k^n. If the sequence $\{y_n\}$ corresponds to some $\mathcal{G}$ in $\mathbf{R}_k$, then by R12.4.4 $\mathcal{F}\mathcal{G} = \overline{1}$. To see that $\mathcal{G}$ exists, R10.2.6 will be used. By choice $y_1 \in \{1, \ldots, k\}$, and so it is sufficient to show that, for all n, there is t in $\{0, 1, \ldots, k-1\}$ such that $y_{n+1} = y_n + tk^n$. By the choice of y_n, there is an integer r such that $x_n y_n = 1 + rk^n$ and by R10.2.5 there is s in $\{0, 1, \ldots, k-1\}$ such that $x_{n+1} = x_n + sk^n$. Let t be the element of $\{0, 1, \ldots, k-1\}$ such that $t \equiv (-r - sy_n)y_n$ mod k. Since $x_n y_n \equiv 1$ mod k, $x_n t \equiv (-r - sy_n)$ mod k, i.e.,

k divides $(x_n t + r + s y_n)$. Now consider

$$\begin{aligned} x_{n+1}(y_n + tk^n) &= (x_n + sk^n)(y_n + tk^n) \\ &= x_n y_n + (x_n t + s y_n)k^n + stk^{2n} \\ &= 1 + (r + x_n t + s y_n)k^n + stk^{2n}. \end{aligned}$$

Since k divides $r + x_n t + s y_n$ and $n \geq 1$ implies $2n \geq n+1$, $x_{n+1}(y_n + tk^n) \equiv 1 \bmod k^{n+1}$. Thus $y_n + tk^n$ is the unique element of $\{1, 2, \ldots, k^{n+1}\}$ which satisfies this equivalence, i.e., $y_{n+1} = y_n + tk^n$.

Corollary R17.1.3 Let $p \in \mathbb{N}$ be a prime, and let $\mathcal{F}$ in $\mathbf{R}_p$ correspond to the sequence $\{x_n\}$. Then $\mathcal{F}$ is non-invertible if and only if $x_1 = p$.

Proof: By the definition of the sequence associated with a non-point ultrafilter, x_1 is in $\{1, 2, \ldots, p\}$. Within that set, x_1 is not invertible mod p if and only if $x_1 = p$.

Recall that a commutative ring with unity is called a local ring provided it has a unique maximal ideal, which must be the set of non-invertible elements in the ring.

Corollary R17.1.4 Let $p \in \mathbb{N}$ be a prime. Then $\mathbf{R}_p$ is a local ring.

Proof: It is sufficient to show that the set of non-invertible elements in $\mathbf{R}_p$ is an ideal. Let $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$ in $\mathbf{R}_p$ correspond to the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ respectively. Assume $\mathcal{F}$ and $\mathcal{G}$ are non-invertible, and let $\mathcal{F} - \mathcal{G}$ correspond to the sequence $\{w_n\}$. By R16.2 $w_1 \equiv x_1 - y_1 \bmod p$, and so by the previous corollary $w_1 \equiv 0 \bmod p$. Because $1 \leq w_1 \leq p$, $w_1 = p$ and so $\mathcal{F} - \mathcal{G}$ is non-invertible. Similarly $\mathcal{HF}$ is non-invertible.

Example R17.1.5 Consider $\overline{2}$ and $\overline{3}$, which are both non-invertible in $\mathbf{R}_6$ by R17.1.2. However, $\overline{2} + \overline{3} = \overline{5}$, which is invertible in $\mathbf{R}_6$. Thus the set of non-invertible elements of $\mathbf{R}_6$ is not closed under addition and so is not an ideal. Therefore $\mathbf{R}_6$ is not a local ring.

In what follows $\mathbf{Z}$ denotes the ring of integers and $\mathbf{Z}_k$ the integers mod k.

Proposition R17.1.6 Let $p \in \mathbb{N}$ be a prime, and let M_p be the unique maximal ideal in $\mathbf{R}_p$. Then $\mathbf{R}_p / M_p$ is isomorphic to the field $\mathbf{Z}_p$.

Proof: It is sufficient to show that the field $\mathbf{R}_p / M_p$ has exactly p elements. First observe that $\overline{1} + M_p, \ldots, \overline{p} + M_p$ are all distinct: Let $r, s \in \{1, \ldots, p\}$ with $r < s$. Then $1 \leq s - r \leq p - 1$ and so by R17.1.3 $\overline{s} - \overline{r} = \overline{s - r} \notin M_p$, i.e., $\overline{s} + M_p \neq \overline{r} + M_p$. Next let $\mathcal{F}$ in $\mathbf{R}_p$ correspond to the sequence $\{x_n\}$, and let $t = x_1$. By definition $t \in \{1, \ldots, p\}$. If we let $\mathcal{F} - \overline{t}$ correspond to $\{y_n\}$, by R16.2 $y_1 = p$. R17.1.3 implies $\mathcal{F} - \overline{t} \in M_p$, i.e., $\mathcal{F} + M_p = \overline{t} + M_p$.

Definition R17.1.7 Let $k \in \mathbb{N}$ with $k \geq 2$, and let $\mathcal{F} \neq \overline{0}$ in $\mathbf{R}_k$ correspond to the sequence $\{x_n\}$. $v(\mathcal{F})$ is defined to be the smallest element of $\{m : x_m \neq k^m\}$.

Lemma R17.1.8 Let $k \in \mathbb{N}$ with $k \geq 2$, and let $\mathcal{F} \neq \overline{0}$ and $\mathcal{G} \neq \overline{0}$ in $\mathbf{R}_k$ correspond to the sequences $\{x_n\}, \{y_n\}$ respectively. Then

i) $v(\mathcal{F}) \geq 1$.

ii) If $\mathcal{F}$ is invertible, then $v(\mathcal{F}) = 1$.

iii) If $\mathcal{FG} \neq \overline{0}$, then $v(\mathcal{F}) \leq v(\mathcal{FG})$.

iv) If k is prime and $v(\mathcal{F}) = 1$, then $\mathcal{F}$ is invertible.

Proof: i) is clear, ii) is immediate from R17.1.2, and iv) follows easily from R17.1.3. For iii), let $v(\mathcal{F}) = r$. If $r = 1$, the conclusion holds and so assume $r \geq 2$. Let $\mathcal{FG}$ correspond to $\{z_n\}$. For all $i < r, x_i = k^i$ and so $z_i \equiv y_i k^i \bmod k^i$, i.e., $z_i \equiv 0 \bmod k^i$ and, since $z_i \in \{1, 2, \ldots, k^i\}$, $z_i = k^i$. Thus $r \leq v(\mathcal{FG})$.

Lemma R17.1.9 Let $p \in \mathbb{N}$ be a prime, and let $\mathcal{F} \neq \overline{0}$ in $\mathbf{R}_p$ correspond to the sequence $\{x_n\}$. Assume $v(\mathcal{F}) \geq 2$. Then there is $\mathcal{G}$ in $\mathbf{R}_p$ such that $\mathcal{F} = \overline{p}\mathcal{G}$ and $v(\mathcal{G}) = v(\mathcal{F}) - 1$.

Proof: By R10.2.5 for all n there is j_n in $\{0, 1, \ldots, p-1\}$ such that $x_{n+1} = x_n + j_n p^n$. Let $v(\mathcal{F}) = m$ so that $x_m \neq p^m$ and for $1 \leq i \leq m-1, x_i = p^i$. It follows that $j_{m-1} \neq p-1$ and $1 \leq i \leq m-2 \Rightarrow j_i = p-1$. Let $y_1 = 1 + j_1$, and let $r_n = j_{n+1}$ and $y_{n+1} = y_n + r_n p^n$ for all n. Since $y_1 \in \{1, 2, \ldots, p\}$ and $r_n \in \{0, 1, \ldots, p-1\}$ for all n, by R10.2.6 there is $\mathcal{G}$ in $\mathbf{R}_p$ such that $\mathcal{G}$ corresponds to $\{y_n\}$. The values of $j_1, \ldots, j_{m-1}$ imply $y_{m-1} \neq p^{m-1}$ and, for $1 \leq i \leq m-2$, $y_i = p^i$. Thus $v(\mathcal{G}) = m-1$. Next induction will be used to show $py_n = x_{n+1}$ for all n. When $n = 1$, $py_1 = p(1+j_1) = x_1 + j_1 p = x_2$. Assuming $py_n = x_{n+1}$, $py_{n+1} = py_n + r_n p^{n+1} = x_{n+1} + j_{n+1}p^{n+1} = x_{n+2}$. The equation $py_n = x_{n+1}$ can also be written $py_n = x_n + j_n p^n$ and so $py_n \equiv x_n \mod p^n$ for all n. By R12.4.4 $\mathcal{F} = \overline{p}\mathcal{G}$.

Proposition R17.1.10 Let $p \in \mathbb{N}$ be a prime, and let $\mathcal{F} \neq \overline{0}$ in $\mathbf{R}_p$. Assume $v(\mathcal{F}) = m$. Then there is $\mathcal{G}$ in $\mathbf{R}_p$ such that $\mathcal{F} = \overline{p^{m-1}}\mathcal{G}$, where $\mathcal{G}$ is invertible.

Proof: By induction on m: When $m = 1$, $\mathcal{F}$ is invertible by R17.1.8iv and the conclusion holds with $\mathcal{G} = \mathcal{F}$. Now assume the corollary holds for any $\mathcal{H}$ with $v(\mathcal{H}) = m-1$. Here $m \geq 2$ and so by R17.1.9 $\mathcal{F} = \overline{p}\mathcal{G}_1$ for some $\mathcal{G}_1$ with $v(\mathcal{G}_1) = m-1$. By the induction hypothesis there is $\mathcal{G}$ invertible with $\mathcal{G}_1 = \overline{p^{m-2}}\mathcal{G}$. Since $\overline{p} \cdot \overline{p^{m-2}} = \overline{p^{m-1}}$, the desired conclusion follows by substitution.

Corollary R17.1.11 Let $p \in \mathbb{N}$ be a prime, and let $\mathcal{F} \neq \overline{0}$ and $\mathcal{G} \neq \overline{0}$ be in $\mathbf{R}_p$. Assume $v(\mathcal{F}) \leq v(\mathcal{G})$. Then there is $\mathcal{H}$ in $\mathbf{R}_p$ such that $\mathcal{G} = \mathcal{F}\mathcal{H}$.

Proof: Let $v(\mathcal{F}) = m$ and $v(\mathcal{G}) = r$. By the previous proposition $\mathcal{F} = \overline{p^{m-1}}\mathcal{F}_1$, where $\mathcal{F}_1$ is invertible, and $\mathcal{G} = \overline{p^{r-1}}\mathcal{G}_1$, where $\mathcal{G}_1$ is invertible. $\mathcal{H} = \overline{p^{r-m}}\mathcal{F}_1^{-1}\mathcal{G}_1$ satisfies the required equation.

Corollary R17.1.12 Let $p \in \mathbb{N}$ be a prime. Then v is a euclidean valuation making $\mathbf{R}_p$ a Euclidean ring.

Proof: As mentioned above, $\mathbf{R}_p$ is an integral domain. By definition v maps the non-zero elements of $\mathbf{R}_p$ into the positive integers, and R17.1.8iii implies the second requirement for a euclidean valuation. To verify the third requirement, let $\mathcal{G} \neq \overline{0}, \mathcal{F} \neq \overline{0}$ be in $\mathbf{R}_p$. If $v(\mathcal{G}) < v(\mathcal{F})$, then $\mathcal{G} = \mathcal{F} \cdot \overline{0} + \mathcal{G}$. If $v(\mathcal{G}) \geq v(\mathcal{F})$, then by R17.1.11 there is $\mathcal{H}$ in $\mathbf{R}_p$ such that $\mathcal{G} = \mathcal{F}\mathcal{H} + \overline{0}$.

Note that $\mathbf{R}_p$ differs from the familiar examples of Euclidean rings, because, up to multiplication by invertibles, $\overline{p}$ is the only $\mathbf{R}_p$-prime element. The Euclidean argument that there are infinitely many prime integers or irreducible polynomials cannot be used because $\overline{p} + \overline{1}$ is invertible.

Given x, an element of some ring, $< x >$ denotes the principal ideal generated by x.

Proposition R17.1.13 Let $p \in \mathbb{N}$ be a prime, and let M_p be the unique maximal ideal in $\mathbf{R}_p$. Then

i) $M_p =< \overline{p} >$.

ii) If I is a proper ideal in $\mathbf{R}_p$, then there is $m \in \mathbb{N}$ such that $I =< \overline{p^m} >$.

iii) The ideals of $\mathbf{R}_p$ form a chain.

iv) $\mathbf{R}_p$ satifies the ascending chain condition.

Proof: By R17.1.3 $\overline{p} \in M_p$ and so $< \overline{p} > \subseteq M_p$. Now let $\mathcal{F}$ be in M_p. $\mathcal{F}$ is non–invertible and so $v(\mathcal{F}) \geq 2$. By R17.1.9 $\mathcal{F}$ is in $< \overline{p} >$. Thus i) holds. For ii) let a proper ideal I be given, let $m+1$ be the smallest element of $\{v(\mathcal{F}) : \mathcal{F} \in I\}$, and let $\mathcal{F}$ be an element of I with $v(\mathcal{F}) = m+1$. Since I is proper, $I \subseteq M_p$ and so $m+1 \geq 2$, i.e., m is in $\mathbb{N}$. By R17.1.10 $\mathcal{F} = \overline{p^m}\mathcal{F}_1$, where $\mathcal{F}_1$ is invertible. Thus $\overline{p^m} = \mathcal{F}_1^{-1}\mathcal{F}$ is in I so that $< \overline{p^m} > \subseteq I$. Now let $\mathcal{G}$ be in I. $v(\overline{p^m}) = m+1 \leq v(\mathcal{G})$ and so by R17.1.11 $\mathcal{G} \in < \overline{p^m} >$. For iii) let $I = < \overline{p^r} >$ and $J = < \overline{p^t} >$ be ideals in $\mathbf{R}_p$. If $r \leq t$, then $J \subseteq I$. If $t \leq r$, then $I \subseteq J$. For iv) let $I = < \overline{p^r} >$ be the first in an increasing chain of ideals. There are only finitely many possible larger ideals, i.e., $< \overline{p^0} >, \ldots, < \overline{p^{r-1}} >$, and so the ACC holds.

Lemma R17.1.14 Let $p \in \mathbb{N}$ be a prime, and let $\mathcal{F}$ in $\mathbf{R}_p$. Then for all $j \in \mathbb{N}$ there is $a_j \in \mathbf{Z}$ such that $\mathcal{F} - \overline{a_j} \in < \overline{p^j} >$.

Proof: Let $\mathcal{F}$ correspond to $\{x_n\}$ and proceed by induction on j. For $j = 1$ let $a_1 = x_1$. By R16.2 and R17.1.3 $\mathcal{F} - \overline{a_1}$ is not invertible and so is in M_p, which is $< \overline{p} >$ by R17.1.13i. Now assume $\mathcal{F} - \overline{a_j} \in < \overline{p^j} >$. Then $\mathcal{F} - \overline{a_j} = \mathcal{G} \cdot \overline{p^j}$ for some $\mathcal{G}$ in $\mathbf{R}_p$. Let $\mathcal{G}$ correspond to $\{y_n\}$. As in the case $j = 1$, $\mathcal{G} - \overline{y_1} = \mathcal{H} \cdot \overline{p}$ for some $\mathcal{H}$ in $\mathbf{R}_p$. Let $a_{j+1} = a_j + y_1 \cdot p^j$. A routine calculation shows $\mathcal{F} - \overline{a_{j+1}} \in < \overline{p^{j+1}} >$ as required.

There is a possible notational confusion in the next proposition. For $z \in \mathbf{Z}$, $< z >$ denotes a principal ideal in $\mathbf{Z}$ while $< \overline{z} >$ denotes a principal ideal in the appropriate remnant ring.

Proposition R17.1.15 Let $p \in \mathbb{N}$ be a prime. Let m, r be integers with $0 \leq m < r$. Then $< \overline{p^m} > / < \overline{p^r} >$, a quotient space of ideals in $\mathbf{R}_p$, is isomorphic to $< p^m > / < p^r >$.

Proof: Define $h :< p^m > \rightarrow < \overline{p^m} > / < \overline{p^r} >$ by $h(ap^m) = \overline{a} \cdot \overline{p^m} + < \overline{p^r} >$. Then $h(ap^m \cdot bp^m) = h((abp^m)p^m) = \overline{abp^m} \cdot \overline{p^m} + < \overline{p^r} >$. Using R16.3 and coset algebra, $h(ap^m) \cdot h(bp^m) = (\overline{a} \cdot \overline{p^m} + < \overline{p^r} >)(\overline{b} \cdot \overline{p^m} + < \overline{p^r} >) = h(ap^m \cdot bp^m)$. Similarly h preserves addition, and so h is a ring homomorphism. To see that h is onto, let $\mathcal{F} \cdot \overline{p^m}$ be in $< \overline{p^m} >$, and let $t = r - m$. By R17.1.14 there is an integer a such that $\mathcal{F} - \overline{a} \in < \overline{p^t} >$. Then $\mathcal{F} \cdot \overline{p^m} + < \overline{p^r} > = \overline{a} \cdot \overline{p^m} + < \overline{p^r} > = h(ap^m)$, i.e., h is onto. Finally it will be shown that the kernel of h is $< p^r >$. For bp^r in $< p^r >$, $h(bp^{r-m}p^m) = \overline{bp^{r-m}} \cdot \overline{p^m} + < \overline{p^r} > = \overline{0} + < \overline{p^r} >$. Given ap^m in the kernel of h, $\overline{a} \cdot \overline{p^m} = \mathcal{F}\overline{p^r}$ for some $\mathcal{F}$ in $\mathbf{R}_p$. Let $\mathcal{F}$ correspond to $\{x_n\}$. By R12.4.4 $ap^m \equiv x_r p^r \mod p^r$, and it follows easily that p^{r-m} divides a. Thus $ap^m \in < p^r >$.

In what follows the more fundamental elements used to define $\mathbb{N}_k$ for $k \in \mathbb{N}$ will be needed: $E_n(k)$ denotes equivalence mod k^n. For $i \in \mathbb{N}$ the $E_n(k)$-equivalence class of i is $C_n^i(k)$. As in [4] the normal basis determined by $E_n(k)$ is denoted $\mathcal{Z}(E_n(k))$. $\mathcal{Z}_k$, defined in [7] as $\cup\{\mathcal{Z}(E_n(k)) : n \in \mathbb{N}\}$, is the normal basis which yields $\mathbb{N}_k$.

This subsection will conclude by identifying $\mathbf{R}_p$ for a fixed prime p as the p-adic numbers, which can be described (see [1; p.24]) as the inverse limit of the inverse system $\{\mathbf{Z} / < p^i >; \phi_{ij}\}$ where the underlying directed set is $\mathbb{N}$ with the usual order and where, for $i \leq j$, $\phi_{ij}(z + < p^j >) = z + < p^i >$. Each finite ring $\mathbf{Z} / < p^i >$ is assumed to have the discrete topology.

Definition R17.1.16 Let $i \in \mathbb{N}$. $\pi_i : \mathbf{R}_p \rightarrow \mathbf{Z} / < p^i >$ by $\pi_i(\mathcal{F}) = x_i + < p^i >$, where $\mathcal{F}$ corresponds to the sequence $\{x_n\}$

Lemma R17.1.17 Let $i, j \in \mathbb{N}$ with $i \leq j$. Then

i) π_i is a surjective ring homomorphism.

ii) π_i is continuous.

iii) $\phi_{ij} \circ \pi_j = \pi_i$.

Proof: R12.4.4 easily shows that π_i is a ring homomorphism. For any $t \in \{1, 2, \ldots, p^i\}$, $\pi_i(\bar{t}) = t+ < p^i >$ and so π_i is surjective. Recall that $C_i^t(p)$ is in the normal basis $\mathcal{Z}_p$ and that $[C_i^t(p)]^\omega \cap \mathbf{R}_p$ is clopen in $\mathbf{R}_p$ by R16.13. It follows easily from the definition of the sequence corresponding to an ultrafilter in $\mathbf{R}_p$ that $\pi_i^{-1}[\{t+ < p^i >\}] = [C_i^t(p)]^\omega \cap \mathbf{R}_p$ and so ii) holds. For iii) let $\mathcal{F}$ in $\mathbf{R}_p$ correspond to $\{x_n\}$. By R10.2.5ii p^i divides $x_j - x_i$ and so $x_j+ < p^i >= x_i+ < p^i >$, i.e., iii) holds.

Lemma R17.1.18 Let $k \in \mathbb{N}$, and let $W \in \mathcal{Z}_k$. Assume $W \in \mathcal{Z}(E_n(k))$ and that, as such, it is associated with $\Delta \subseteq \{1, 2, \ldots, k^n\}$.
Then $W^\omega \cap \mathbf{R}_k = \cup\{(C_n^t(k))^\omega \cap \mathbf{R}_k : t \in \{1, 2, \ldots, k^n\} - \Delta\}$.

Proof: Let $\mathcal{F}$ be in $W^\omega \cap \mathbf{R}_k$. Then $W \in \mathcal{F}$. By definition of associated set (R5.3.1) $W \cap (\cup\{C_n^t(k) : t \in \Delta\})$ is finite and so, since $\mathcal{F}$ is not a point filter, $\cup\{C_n^t(k) : t \in \Delta\}$, a $\mathcal{Z}_k$-set, is not in $\mathcal{F}$. Its complement $\cup\{C_n^t(k) : t \in \{1, 2, \ldots, k^n\} - \Delta\}$,which is also a $\mathcal{Z}_k$-set, must therefore be in the $\mathcal{Z}_k$-ultrafilter $\mathcal{F}$. Since $\mathcal{F}$ is also a prime $\mathcal{Z}_k$-filter, $C_n^t(k) \in \mathcal{F}$, i.e., $\mathcal{F} \in (C_n^t(k))^\omega$, for some $t \in \{1, 2, \ldots, k^n\} - \Delta$. Conversely, let $\mathcal{F} \in (C_n^t(k))^\omega \cap \mathbf{R}_k$ for some $t \in \{1, 2, \ldots, k^n\} - \Delta$. By R9.1.7 $\mathbb{N} - W$ is also in $\mathcal{Z}(E_n(k))$. Either W or $\mathbb{N} - W$ must be in the $\mathcal{Z}_k$-ultrafilter $\mathcal{F}$. Since $(\mathbb{N} - W) \cap C_n^t(k)$ is finite and $C_n^t(k) \in \mathcal{F}$ and $\mathcal{F}$ is not a point filter, we must have $W \in \mathcal{F}$, i.e., $\mathcal{F} \in W^\omega$.

Theorem R17.1.19 Let $p \in \mathbb{N}$ be a prime. Then $\mathbf{R}_p$ is topologically isomorphic to the p-adic integers.

Proof: It will be shown that $\mathbf{R}_p$ and the maps $\{\pi_i\}$ have the universal property of the inverse limit of the inverse system $\{\mathbf{Z} / < p^i >; \phi_{ij}\}$, which immediately yields the conclusion. Let S be a topological ring and $\{q_i\}$ a sequence of continuous ring homomorphisms with $q_i : S \to \mathbf{Z} / < p^i >$ such that, whenever $i \leq j$, $\phi_{ij} \circ q_j = q_i$. By the categorical characterization of inverse limits, it is sufficient to show that there is a unique continuous ring homomorphism $\psi : S \to \mathbf{R}_p$ such that, for every i, $q_i = \pi_i \circ \psi$.

Given $s \in S$, let x_n be the element of $\{1, 2, \ldots, p^n\}$ such that $q_n(s) = x_n+ < p^n >$. Using $\phi_{ij} \circ q_j = q_i$ with $i = n$ and $j = n + 1$, we have $x_{n+1}+ < p^n >= x_n+ < p^n >$ and so $x_{n+1} = x_n + rp^n$ for some integer r. Because of the restrictions on the choice of x_{n+1} and x_n, r must be in $\{0, 1, \ldots, p - 1\}$. By R10.2.6 there is a unique $\mathcal{F}$ in $\mathbf{R}_p$, which corresponds to $\{x_n\}$. Define $\psi(s) = \mathcal{F}$. Clearly $q_i = \pi_i \circ \psi$ for every i. Now suppose ψ_1 also satisfies that equation, let $s \in S$, and let $\psi_1(s)$ correspond to $\{y_n\}$. For every n, $x_n+ < p^n >= y_n+ < p^n >$. Because both x_n and y_n must be in $\{1, 2, \ldots, p^n\}$, $x_n = y_n$. Thus $\psi(s) = \psi_1(s)$, i.e, ψ is unique.

To see that ψ is a ring homomorphism, let r, s be in S, let $\psi(r)$ correspond to $\{w_n\}$, and let $\psi(s)$ correspond to $\{x_n\}$, let $\psi(r) + \psi(s)$ correspond to $\{y_n\}$, and let $\psi(r + s)$ correspond to $\{z_n\}$. For every n, $q_n(r + s) = q_n(r) + q_n(s)$ so that $z_n \equiv w_n + x_n \mod p^n$. By R12.4.4 $y_n \equiv w_n + x_n \mod p^n$. Because of the restriction on the choice of z_n and y_n, $z_n = y_n$. By R10.2.4 $\psi(r + s) = \psi(r) + \psi(s)$. Similarly $\psi(rs) = \psi(r)\psi(s)$.

To see that ψ is continuous, let $W \in \mathcal{Z}_p$. Since $\{Z^\omega \cap \mathbf{R}_p : Z \in \mathcal{Z}_p\}$ is a clopen basis for $\mathbf{R}_p$, it is sufficient to show that $\psi^{-1}[W^\omega \cap \mathbf{R}_p]$ is open in S. For some n, $W \in \mathcal{Z}(E_n(k))$ and, as such, W corresponds to some $\Delta \subseteq \{1, 2, \ldots, p^n\}$. For $t \in \{1, 2, \ldots, p^n\} - \Delta$, as in the proof of R17.1.17ii, $\pi_n^{-1}[\{t+ < p^n >\}] = [C_n^t(p)]^\omega \cap \mathbf{R}_k$, and so, since $q_n = \pi_n \circ \psi$, we have $\psi^{-1}[[C_n^t(p)]^\omega \cap \mathbf{R}_k] = q_n^{-1}[\{t+ < p^n >\}]$, which is open in S. It now follows easily from R17.1.18 that $\psi^{-1}[W^\omega \cap \mathbf{R}_p]$ is open in S.

Properties Related to Divisibility

In this subsection standard number theoretic notation will be used: $a|b$ for a divides b, (a, b) for the greatest common divisor of a and b, and $[a, b]$ for the least common multiple.

Lemma R17.2.1 Let $a, b \in \mathbb{N}$ with $a|b$. Let $i \in \mathbb{N}$. Then

i) $C_n^i(b) \subseteq C_n^i(a)$.

ii) $C_n^i(a) = \cup\{C_n^j(b) : j \equiv i \bmod a^n\}$.

Proof: Routine.

In R12.5.17i it was shown that, for $a, b \in \mathbb{N}$ with $a|b$, $\mathcal{Z}_a \subseteq \mathcal{Z}_b$. The next lemma is a refinement of R12.5.17ii.

Lemma R17.2.2 Let $a, b \in \mathbb{N}$ with $a|b$. If $\mathcal{F}$ is in $\mathbf{R}_b$, then $\mathcal{F} \cap \mathcal{Z}_a$ is in $\mathbf{R}_a$.

Proof: Let $\mathcal{F}$ be in $\mathbf{R}_b$, i.e., $\mathcal{F}$ is a non-point ultrafilter in $\mathbb{N}_b$. By R12.5.17ii $\mathcal{F} \cap \mathcal{Z}_a$ is in $\mathbb{N}_a$. If $\mathcal{F} \cap \mathcal{Z}_a$ were the point $\mathcal{Z}_a$-ultrafilter of x, then $\{x\}$ would be in $\mathcal{F}$, which would consequently have to be the point $\mathcal{Z}_b$-ultrafilter of x. Thus $\mathcal{F} \cap \mathcal{Z}_a$ is a non-point ultrafilter in $\mathbb{N}_a$, i.e., it is in $\mathbf{R}_a$.

Lemma R17.2.3 Let $a, b \in \mathbb{N}$ with $a|b$. Let $\mathcal{F}$ in $\mathbf{R}_b$ be associated with $\{x_n\}$ and let $\mathcal{F} \cap \mathcal{Z}_a$ be associated with $\{y_n\}$. Then $x_n \equiv y_n \bmod a^n$ for all n.

Proof: For any k, n in $\mathbb{N}$ and $i \in \{1, 2, \ldots, k^n\}$, recall that $C_n^i(k)$ is associated with $\{1, 2, \ldots, k^n\} - \{i\}$ and so $C_n^i(k) \in \mathcal{Z}(E_n(k))$. (See R5.3.1 and R5.3.2 for definitions.) As a result, since the normal basis $\mathcal{Z}(E_n(b))$ is closed under finite unions and R17.2.1ii holds, $C_n^i(a) \in \mathcal{Z}(E_n(b))$ for all n and $i \in \{1, 2, \ldots, a^n\}$. Now fix $n \in \mathbb{N}$. By definition of x_n, $\mathcal{F} \cap \mathcal{Z}(E_n(b)) = \{Z \in \mathcal{Z}(E_n(b)) : Z$ is associated with some $\Delta \subseteq \{1, 2, \ldots, b^n\} - \{x_n\}\}$. Thus $C_n^{x_n}(b) \in \mathcal{F}$ and , since $\mathcal{F}$ is closed under $\mathcal{Z}(E_n(b))$-supersets, $C_n^{x_n}(a) \in \mathcal{F}$. Therefore $C_n^{x_n}(a)$ is in $(\mathcal{F} \cap \mathcal{Z}_a) \cap \mathcal{Z}(E_n(a)) = \{Z \in \mathcal{Z}(E_n(a)) : Z$ is associated with some $\Delta \subseteq \{1, 2, \ldots, a^n\} - \{y_n\}\}$. Let i be the element of $\{1, 2, \ldots, a^n\}$ with $i \equiv x_n \bmod a^n$. By R5.3.5 $C_n^i(a) = C_n^{x_n}(a)$, as an element of $\mathcal{Z}(E_n(a))$, is associated with a unique subset of $\{1, 2, \ldots, a^n\}$. As above, that subset is $\{1, 2 \ldots, a^n\} - \{i\}$. Since $\{1, 2 \ldots, a^n\} - \{i\} \subseteq \{1, 2 \ldots, a^n\} - \{y_n\}$, $i = y_n$. Thus $x_n \equiv y_n \bmod a^n$.

Lemma R17.2.2 makes possible the next definition.

Definition R17.2.4 Let $a, b \in \mathbb{N}$ with $a|b$. ${}_ah_b : \mathbf{R}_b \to \mathbf{R}_a$ is defined by ${}_ah_b(\mathcal{F}) = \mathcal{F} \cap \mathcal{Z}_a$.

Proposition R17.2.5 Let $a, b \in \mathbb{N}$ with $a|b$. Then ${}_ah_b$ is a continuous, onto homomorhism from $\mathbf{R}_b$ to $\mathbf{R}_a$. Moreover, ${}_ah_b(f_b(z)) = f_a(z)$ for all z in $\mathbf{Z}$.

Proof: As in the proof of R9.1.1iii the map $\mathcal{F} \mapsto \mathcal{F} \cap \mathcal{Z}_a$ is continuous from $\mathbb{N}_b$ onto $\mathbb{N}_a$. The restriction of this map, ${}_ah_b$, is also continuous. Now let $\mathcal{F}, \mathcal{G}$, and $\mathcal{F} + \mathcal{G}$ in $\mathbf{R}_b$ correspond to $\{x_n\}, \{y_n\}, \{z_n\}$ respectively. By R12.4.4 $z_n \equiv x_n + y_n \bmod b^n$ for all n and so, since $a|b$, $z_n \equiv x_n + y_n \bmod a^n$ for all n. Assume ${}_ah_b(\mathcal{F})$, ${}_ah_b(\mathcal{G})$, and ${}_ah_b(\mathcal{F} + \mathcal{G})$ correspond to $\{x_n^*\}, \{y_n^*\}, \{z_n^*\}$ respectively. By R12.4.4 ${}_ah_b(\mathcal{F}) + {}_ah_b(\mathcal{G})$ corresponds to $\{w_n\}$, where $w_n \equiv x_n^* + y_n^* \bmod a^n$ for all n. By R17.2.3 $x_n^* \equiv x_n \bmod a^n$, $y_n^* \equiv y_n \bmod a^n$, and $z_n^* \equiv z_n \bmod a^n$ for all n. It follows that $w_n \equiv z_n^* \bmod a^n$ for all n. Since $w_n, z_n^* \in \{1, \ldots, a^n\}$ by definition of an associated sequence, $w_n = z_n^*$ for all n and so by R10.2.4 ${}_ah_b(\mathcal{F} + \mathcal{G}) = {}_ah_b(\mathcal{F}) + {}_ah_b(\mathcal{G})$. Similarly ${}_ah_b(\mathcal{F} \cdot \mathcal{G}) = {}_ah_b(\mathcal{F}) \cdot {}_ah_b(\mathcal{G})$. Next let z be in $\mathbf{Z}$, let $f_b(z)$ correspond to $\{t_n\}$, and let ${}_ah_b(f_b(z))$ correspond to $\{t_n^*\}$. By R16.1 $t_n \equiv z \bmod b^n$ and so, since $a|b$, $t_n \equiv z \bmod a^n$ for all n. By R17.2.3 $t_n^* \equiv z \bmod a^n$ for all n so that ${}_ah_b(f_b(z)) = f_a(z)$ by R16.1 and R10.2.4. Finally, by compactness and T_2, the image of ${}_ah_b$ is closed. Since it contains the dense $f_a[\mathbf{Z}]$, ${}_ah_b$ is onto.

Note that ${}_ah_b$ maps the multiplicative identity of $\mathbf{R}_b$, $f_b(1)$, to $f_a(1)$, the multiplicative identity of $\mathbf{R}_a$. The proposition also yields the following corollary for the compactifications identified in R16.15.

Corollary R17.2.6 Let $a, b \in \mathbb{N}$ with $a|b$. Then $[(\mathbf{R}_a, f_a)] \leq [(\mathbf{R}_b, f_b)]$.

Proof: R17.2.5 shows that ${}_ah_b$ is the map required by the definition of the ordering.

Lemma R17.2.7 Let $a, b, c \in \mathbb{N}$ with $a|b$ and $b|c$. Then ${}_ah_c = {}_ah_b \circ {}_bh_c$.

Proof: Let $\mathcal{F} \in \mathbf{R}_c$. ${}_ah_b \circ {}_bh_c(\mathcal{F}) = (\mathcal{F} \cap \mathcal{Z}_b) \cap \mathcal{Z}_a = \mathcal{F} \cap \mathcal{Z}_a = {}_ah_c(\mathcal{F})$ since $\mathcal{Z}_a \subseteq \mathcal{Z}_b$.

Lemma R17.2.8 Let T be a finite, non-empty subset of $\mathbb{N}$, and let c be the least common multiple of the elements of T. Let $\mathcal{F}, \mathcal{G}$ be in $\mathbf{R}_c$. If ${}_xh_c(\mathcal{F}) = {}_xh_c(\mathcal{G})$ for all $x \in T$, then $\mathcal{F} = \mathcal{G}$.

Proof: By induction on $|T|$: When $|T| = 1$, the statement is trivial. For the case $|T| = 2$, let $T = \{a, b\}$. Let $\mathcal{F}$ correspond to $\{x_n\}$ and $\mathcal{G}$ to $\{y_n\}$. Let ${}_ah_c(\mathcal{F})$ correspond to $\{x_n^*\}$ and ${}_ah_c(\mathcal{G})$ to $\{y_n^*\}$, and let ${}_bh_c(\mathcal{F})$ correspond to $\{x_n^{**}\}$ and ${}_bh_c(\mathcal{G})$ to $\{y_n^{**}\}$. By R17.2.3 $x_n \equiv x_n^* \bmod a^n$, $y_n \equiv y_n^* \bmod a^n$, $x_n \equiv x_n^{**} \bmod b^n$, and $y_n \equiv y_n^{**} \bmod b^n$ for all n. By hypothesis $x_n^* = y_n^*$ and $x_n^{**} = y_n^{**}$ for all n. Thus $x_n \equiv y_n \bmod a^n$ and mod b^n for all n. Because c is the LCM of a and b, these equivalences imply $x_n \equiv y_n \bmod c^n$ for all n. By definition x_n and y_n are both in $\{1, 2, \ldots, c^n\}$ and so $x_n = y_n$ for all n. It now follows from R10.2.4 that $\mathcal{F} = \mathcal{G}$. The induction step is derived in a routine way from the case $|T| = 2$.

The first assertion of the next proposition is analogous to R10.3.3 but does not seem to follow from it in any way. The usual somewhat loose notational convention for suprema and infima will be used.

Proposition R17.2.9 Let $a, b \in \mathbb{N}$. Then $(\mathbf{R}_a, f_a) \vee (\mathbf{R}_b, f_b)$ is equivalent to $(\mathbf{R}_{[a,b]}, f_{[a,b]})$.

Proof: Let $c = [a, b]$. Define H from $\mathbf{R}_c$ into $\mathbf{R}_a \times \mathbf{R}_b$ by $H(\mathcal{F}) = ({}_ah_c(\mathcal{F}), {}_bh_c(\mathcal{F}))$. By R17.2.8 H is one-to-one. By R17.2.5 H is continuous and $H \circ f_c(z) = (f_a(z), f_b(z))$. From the last equation and the density of $f_c[\mathbf{Z}]$ in $\mathbf{R}_c$ it follows in a routine way that $S = \{(f_a(z), f_b(z)) : z \in \mathbf{Z}\}$ is contained in the image of H, which is contained in $\overline{S}$, the closure of S in $\mathbf{R}_a \times \mathbf{R}_b$. By continuity, compactness, and T_2, H is a homeomorphism onto $\overline{S}$. By R13.2.1, if $g(z) = (f_a(z), f_b(z))$, $(\overline{S}, g)$ is a T_2 compactification of $\mathbf{Z}$ equivalent to $(\mathbf{R}_a, f_a) \vee (\mathbf{R}_b, f_b)$. By definition the map H makes $(\mathbf{R}_c, f_c)$ equivalent to $(\overline{S}, g)$.

The case of infima is more complicated. It is unknown to me whether the inequality in part ii) of the next result can be improved to equivalence. Recall from [9]: $TBS(\mathbf{Z})$ is the set of all totally bounded separated uniformities on $\mathbf{Z}$.

Proposition R17.2.10 Let $a, b \in \mathbb{N}$ with $(a, b) \geq 2$. Then

i) $(\mathbf{R}_a, f_a)$ and $(\mathbf{R}_b, f_b)$ have an infimum.

ii) $(\mathbf{R}_{(a,b)}, f_{(a,b)}) \leq (\mathbf{R}_a, f_a) \wedge (\mathbf{R}_b, f_b)$.

Proof: Let $c = (a, b)$, and let $\mathcal{V}_a, \mathcal{V}_b, \mathcal{V}_c$ be in $TBS(\mathbf{Z})$ such that $\Psi_0(\mathcal{V}_i) = [(\mathbf{R}_i, f_i)]$ for $i \in \{a, b, c\}$. By R17.2.6 and R13.1.2 $\mathcal{V}_c \subseteq \mathcal{V}_a \wedge \mathcal{V}_b$. Since $\mathcal{V}_a \wedge \mathcal{V}_b$ is a superset of a separated uniformity, it is separated and, since it is a subset of a totally bounded uniformity, it is totally bounded. R13.1.2 also shows that $\Psi_0(\mathcal{V}_a \wedge \mathcal{V}_b)$ is the infimum of $[(\mathbf{R}_a, f_a)]$ and $[(\mathbf{R}_b, f_b)]$ and that part ii) holds.

The next proposition identifies a case in which $(\mathbf{R}_a, f_a)$ and $(\mathbf{R}_b, f_b)$ do not have an infimum.

Proposition R17.2.11 Let $a, b \in \mathbb{N}$ with $a, b \geq 2$ and $(a, b) = 1$. Then $(\mathbf{R}_a, f_a)$ and $(\mathbf{R}_b, f_b)$ do not have a lower bound.

Proof: Let $\mathcal{V}_a, \mathcal{V}_b$ be in $TBS(\mathbf{Z})$ such that $\Psi_0(\mathcal{V}_i) = [(\mathbf{R}_i, f_i)]$ for $i \in \{a, b\}$, and deny the conclusion. Then there is $\mathcal{U}$ in $TBS(\mathbf{Z})$ such that $\Psi_0(\mathcal{U})$ is a lower bound. By R13.1.2 $\mathcal{U} \subseteq \mathcal{V}_a$ and $\mathcal{U} \subseteq \mathcal{V}_b$. This implies that $\tau(\mathcal{U}) \subseteq \tau(\mathcal{U}_a) \cap \tau(\mathcal{U}_b)$ and so $\tau(\mathcal{U}_a) \cap \tau(\mathcal{U}_b)$ is T_2. Let O, G be in $\tau(\mathcal{U}_a) \cap \tau(\mathcal{U}_b)$ with $0 \in O, 1 \in G$ and $O \cap G = \emptyset$. By R16.15 $\tau(\mathcal{U}_a)$ is the topology with basis consisting of the $\mathbf{Z}$-equivalence classes of powers of a. Thus there is $n \in \mathbb{N}$ with $\{wa^n : w \in \mathbf{Z}\} \subseteq O$. Similarly $\{1 + zb^m : z \in \mathbf{Z}\} \subseteq G$ for some $m \in \mathbb{N}$. Since $(a, b) = 1, (a^n, b^m) = 1$. Then there exist $c, d \in \mathbf{Z}$ such that $ca^n + db^m = 1$. But that implies $ca^n = 1 + (-d) \cdot b^m$ is in $O \cap G$, a contradiction.

This subsection will be concluded with results identifying an isomorphic image (and equivalent compactification) of $\mathbf{R}_k$ for k composite.

Lemma R17.2.12 Let $a, n \in \mathbb{N}$. Then $\mathcal{Z}_a = \mathcal{Z}_{a^n}$.

Proof: Since $a | a^n$, by R12.5.17i $\mathcal{Z}_a \subseteq \mathcal{Z}_{a^n}$. For $Z \in \mathcal{Z}_{a^n}$, there exists $m \in \mathbb{N}$ such that $Z \in \mathcal{Z}(E_m(a^n)) = \mathcal{Z}(E_{mn}(a))$. Thus $Z \in \mathcal{Z}_a$.

The previous lemma shows something stronger than the equivalence noted in R10.3.1ii: $\mathbb{N}_a$ and $\mathbb{N}_{a^n}$ are identical as sets with identical embeddings and identical point filters, so that $\mathbf{R}_a$ and $\mathbf{R}_{a^n}$ are identical as topological spaces. The next two lemmas show that $\mathbf{R}_a$ and $\mathbf{R}_{a^n}$ are also identical as rings and as compactifications. For $n \in \mathbb{N}$, $\hat{n}$ will denote the point filter of n.

Lemma R17.2.13 Let $a, n \in \mathbb{N}$. Then $\mathbf{R}_a$ and $\mathbf{R}_{a^n}$ have the same binary operations.

Proof: Let $+, +_1$, and $+_2$ denote the additions in $\mathbb{N}$, $\mathbb{N}_a$, and $\mathbb{N}_{a^n}$ respectively. Let $\mathcal{F}, \mathcal{G}$ be in $\mathbf{R}_a = \mathbf{R}_{a^n}$. Since these spaces are metrizable, there exist sequences $\{s_j\}, \{t_j\}$ in $\mathbb{N}$ such that $\hat{s_j} \to \mathcal{F}$ and $\hat{t_j} \to \mathcal{G}$. By continuity of the operations, $\hat{s_j} +_1 \hat{t_j} \to \mathcal{F} +_1 \mathcal{G}$ and $\hat{s_j} +_2 \hat{t_j} \to \mathcal{F} +_2 \mathcal{G}$. Since both operations extend ordinary addition, $\hat{s_j} +_1 \hat{t_j} = \widehat{s_j + t_j} = \hat{s_j} +_2 \hat{t_j}$. Thus $\widehat{s_j + t_j}$ converges to both $\mathcal{F} +_1 \mathcal{G}$ and $\mathcal{F} +_2 \mathcal{G}$. By uniqueness of limits in a T_2 space, $\mathcal{F} +_1 \mathcal{G} = \mathcal{F} +_2 \mathcal{G}$, i.e., $+_1 = +_2$. Similarly the multiplications are identical.

Lemma R17.2.14 Let $a, n \in \mathbb{N}$. Then $f_a = f_{a^n}$.

Proof: This will be done by applying R16.6 and R16.3 for $z \in \mathbf{Z}$ in the various cases. First let $z = 0$. Since there is a unique additive identity in a ring, $f_a(0) = \mathcal{O}_a = \mathcal{O}_{a^n} = f_{a^n}(0)$. Next let $z \geq 1$. $f_a(z) = \hat{z} + \mathcal{O}_a = \hat{z} + \mathcal{O}_{a^n} = f_{a^n}(z)$. Finally, let $z < 0$. Since additive inverses are unique, $f_a(z) = -f_a(-z) = -f_{a^n}(-z) = f_{a^n}(z)$.

These proofs avoided using the sequence associated with a non-point ultrafilter, because the corresponding sequences need not be identical. For example in $\mathbf{R}_2 = \mathbf{R}_4$ we have $\mathcal{O}_2 = \mathcal{O}_4$. As a member of $\mathbf{R}_2$, this filter corresponds to the sequence $\{2, 4, 8, 16, \ldots\}$, but in $\mathbf{R}_4$ it corresponds to $\{4, 16, 64, \ldots\}$.

The following notation will be used in the next few results: For $k \in \mathbb{N}$ with $k \geq 2$, let $p_1, p_2, \ldots, p_j$ denote all the distinct prime factors of k. As in R13.2.1 let the map $e_k : \mathbf{Z} \to \Pi_{i=1}^{j} \mathbf{R}_{p_i}$ be defined by $e_k(z) = (f_{p_1}(z), \ldots, f_{p_j}(z))$ and let rep$(\vee_{i=1}^{j} \mathbf{R}_{p_i})$ be the closure of $e_k[\mathbf{Z}]$ in the product. Note that by R16.6 and R16.3 e_k is a homomorphism into the direct product of the rings so that the closure of $e_k[\mathbf{Z}]$ is a subring of the direct product.

Proposition R17.2.15 Let $k \in \mathbb{N}$ with $k \geq 2$. Let $p_1, p_2, \ldots, p_j$ denote all the distinct prime factors of k. Then $(\mathbf{R}_k, f_k)$ and $(\mathrm{rep}(\vee_{i=1}^{j}\mathbf{R}_{p_i}), e_k)$ are equivalent compactifications.

Proof: Let $k = \Pi_{i=1}^{j} p_i^{t_i}$. By a routine induction using R17.2.9. $(\mathbf{R}_k, f_k)$ is equivalent to $\vee_{i=1}^{j}(\mathbf{R}_{p_i^{t_i}}, f_{p_i^{t_i}})$. Since $(\mathbf{R}_{p_i^{t_i}}, f_{p_i^{t_i}})$ is identical to $(\mathbf{R}_{p_i}, f_{p_i})$, $(\mathbf{R}_k, f_k)$ is equivalent to $\vee_{i=1}^{j}(\mathbf{R}_{p_i}, f_{p_i})$. By R13.2.1 this last is equivalent to $(\mathrm{rep}(\vee_{i=1}^{j}\mathbf{R}_{p_i}), e_k)$.

Lemma R17.2.16 Let $k \in \mathbb{N}$ with $k \geq 2$. Let $\{\mathcal{F}_i\}$ be a sequence in $\mathbf{R}_k$ with $\mathcal{F}_i$ corresponding to $\{{}^i x_n\}$. Assume $\mathcal{F}$ is in $\mathbf{R}_k$ with $\mathcal{F}$ corresponding to $\{y_n\}$. Then $\{\mathcal{F}_i\}$ converges to $\mathcal{F}$ if and only if for every n there exists j such that $i \geq j$ implies ${}^i x_n = y_n$.

Proof: Assume convergence and fix $n \in \mathbb{N}$. $C_n^{y_n}(k)$, as an element of $\mathcal{Z}(E_n(k)) \subseteq \mathcal{Z}_k$, is associated with $\{1, 2, \ldots, k^n\} - \{y_n\}$ and so by definition of y_n, $C_n^{y_n}(k) \in \mathcal{F}$, i.e., $\mathcal{F}$ is in $[C_n^{y_n}(k)]^\omega$. This last is clopen in $\mathbb{N}_k$ as noted in R16.13, and so there exists j such that $i \geq j$ implies $\mathcal{F}_i$ is in $[C_n^{y_n}(k)]^\omega$. For $i \geq j$, by the definition of ${}^i x_n$, $\{1, 2, \ldots, k^n\} - \{y_n\} \subseteq \{1, 2, \ldots, k^n\} - \{{}^i x_n\}$, i.e., ${}^i x_n = y_n$. For the converse let $W \in \mathcal{Z}_k$ with $\mathcal{F} \in W^\omega$. Since $\{Z^\omega : Z \in \mathcal{Z}_k\}$ is a clopen basis for $\mathbb{N}_k$, it is sufficient to show that $\mathcal{F}_i$ is eventually in W^ω. Pick n such that $W \in \mathcal{Z}(E_n(k))$. By hypothesis there exists j such that $i \geq j$ implies ${}^i x_n = y_n$. $W \in \mathcal{F} \cap \mathcal{Z}(E_n(k))$ and so by definition y_n is not in the unique $\Delta \subseteq \{1, 2, \ldots, k^n\}$ associated with W. Now let $i \geq j$. Since ${}^i x_n = y_n$ is not in Δ, $W \in \mathcal{F}_i \cap \mathcal{Z}(E_n(k))$ by definition of ${}^i x_n$. Thus $\mathcal{F}_i \in W^\omega$.

Proposition R17.2.17 Let $k \in \mathbb{N}$ with $k \geq 2$. Let $p_1, p_2, \ldots, p_j$ denote all the distinct prime factors of k. Then $\mathrm{rep}(\vee_{i=1}^{j}\mathbf{R}_{p_i}) = \Pi_{i=1}^{j}\mathbf{R}_{p_i}$.

Proof: For $1 \leq r \leq j$ let J_r be the set of all $\mathcal{F}$ in $\mathbf{R}_{p_r}$ such that the point $q_{\mathcal{F}}^r$ is in $\mathrm{rep}(\vee_{i=1}^{j}\mathbf{R}_{p_i})$, where $q_{\mathcal{F}}^r$ has its r-th component equal to $\mathcal{F}$ and its t-th component $(t \neq r)$ equal to $\mathcal{O}_{p_t}$. It will be shown that J_r is an ideal. Each J_r contains $\mathcal{O}_{p_r}$ since $e_k(0)$ is in $\mathrm{rep}(\vee_{i=1}^{j}\mathbf{R}_{p_i})$. Given $\mathcal{F}, \mathcal{G}$ in J_r, let $\{w_n\}, \{z_n\}$ be seqences in $\mathbf{Z}$ such that $e_k(w_n) \to q_{\mathcal{F}}^r$ and $e_k(z_n) \to q_{\mathcal{G}}^r$. Since e_k is a homomorphism and addition in the direct product is continuous, $e_k(w_n + z_n)$ converges to $q_{\mathcal{F}}^r + q_{\mathcal{G}}^r$, which clearly is $q_{\mathcal{F}+\mathcal{G}}^r$. Thus $\mathcal{F} + \mathcal{G}$ is in J_r. Similarly $-\mathcal{F}$ and $\mathcal{F} \cdot \mathcal{G}$ are in J_r. Now let $\mathcal{H}$ be in $\mathbf{R}_{p_r}$. Let $\{s_n\}$ be a sequence in $\mathbf{Z}$ such that $f_{p_r}(s_n) \to \mathcal{H}$. Using compactness and metrizability of the direct product, one sees that $\{s_n\}$ has a subsequence, which for notational convenience will be labeled $\{y_n\}$, such that $e_k(y_n)$ converges to a point p in $\mathrm{rep}(\vee_{i=1}^{j}\mathbf{R}_{p_i})$. Clearly the r-th component of p is $\mathcal{H}$. $e_k(y_n \cdot w_n)$ converges to $p \cdot q_{\mathcal{F}}^r$, which equals $q_{\mathcal{H}\cdot\mathcal{F}}^r$. Thus $\mathcal{H} \cdot \mathcal{F}$ is also in J_r.

Next fix r in $\{1, \ldots. j\}$, let $\mathcal{F}$ be in $\mathbf{R}_{p_r}$, and let $\{w_i\}$ be a sequence in $\mathbf{Z}$ such that $f_{p_r}(w_i) \to \mathcal{F}$. Let $a = (\Pi_{t=1}^{j} p_t)/p_r$ and let $z_i = a^i w_i$. Let $t \neq r$. It will be shown that $\{f_{p_t}(z_i)\}$ converges to $\mathcal{O}_{p_t}$ in $\mathbf{R}_{p_t}$. Let $f_{p_t}(z_i)$ correspond to $\{{}^i x_n\}$. Fix n and let $i \geq n$. By R16.1 ${}^i x_n \equiv z_i \bmod p_t^n$. Since $i \geq n$, $p_t^n | z_i$ and so ${}^i x_n \equiv 0 \bmod p_t^n$. Since ${}^i x_n \in \{1, \ldots, p_t^n\}$, ${}^i x_n = p_t^n$. In $\mathbf{R}_{p_t}$ $\mathcal{O}_{p_t}$ corresponds to the sequence $\{p_t^n\}$, and so by R17.2.16 $\{f_{p_t}(z_i)\}$ converges to $\mathcal{O}_{p_t}$ in $\mathbf{R}_{p_t}$, as claimed.

By the compactness of $\mathbf{R}_{p_r}$ $\{f_{p_r}(a^i)\}$ has a convergent subsequence. It can be described as $\{f_{p_r}(a^{T(i)})\}$, where T is an increasing map from $\mathbb{N}$ to $\mathbb{N}$. Let $f_{p_r}(a^{T(i)}) \to \mathcal{G}$ in $\mathbf{R}_{p_r}$ and let $\mathcal{G}$ correspond to $\{g_n\}$. It will be shown that $\mathcal{G}$ is invertible. Let $f_{p_r}(a^{T(i)})$ correspond to $\{{}^i s_n\}$. By R17.2.16 with $n = 1$, there is m such that $i \geq m$ implies ${}^i s_1 = g_1$. Since ${}^i s_1 \equiv a^{T(i)} \bmod p_r$ and $(a^{T(i)}, p_r) = 1$, ${}^i s_1 = g_1$ is invertible mod p_r. By R17.1.2 $\mathcal{G}$ is invertible in $\mathbf{R}_{p_r}$.

Now for $t \neq r$, $f_{p_t}(z_{T(i)})$ converges to $\mathcal{O}_{p_t}$, and $f_{p_r}(z_{T(i)}) = f_{p_r}(a^{T(i)}) \cdot f_{p_r}(w_{T(i)})$ converges to $\mathcal{G} \cdot \mathcal{F}$. Thus $e_k(z_{T(i)})$ converges to $q^r_{\mathcal{G} \cdot \mathcal{F}}$ so that $\mathcal{G} \cdot \mathcal{F}$ is in the ideal J_r. Since $\mathcal{G}$ is invertible, $\mathcal{F} \in J_r$.

Finally let $(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_j)$ be in $\Pi^j_{i=1}\mathbf{R}_{p_i}$. For each t with $1 \leq t \leq j$ the point $q^t_{\mathcal{F}_t}$ is in $\text{rep}(\vee^j_{i=1}\mathbf{R}_{p_i})$. Since the latter is a subring of the direct product, $(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_j) = \Sigma^j_{i=1} q^i_{\mathcal{F}_i}$ is also in $\text{rep}(\vee^j_{i=1}\mathbf{R}_{p_i})$.

The next definition describes a version of the mapping used in the proof of R17.2.9.

Definition R17.2.18 Let $k \in \mathbb{N}$ with $k \geq 2$. Let $p_1, p_2, \ldots, p_j$ denote all the distinct prime factors of k. Define $H_k : \mathbf{R}_k \to \Pi^j_{i=1}\mathbf{R}_{p_i}$ by $H_k(\mathcal{F}) = ({}_{p_1}h_k(\mathcal{F}), \ldots, {}_{p_j}h_k(\mathcal{F}))$.

Proposition R17.2.19 Let $k \in \mathbb{N}$ with $k \geq 2$. Let $p_1, p_2, \ldots, p_j$ denote all the distinct prime factors of k. Then H_k is a homeomorphism and isomorphism from $\mathbf{R}_k$ onto the direct product $\Pi^j_{i=1}\mathbf{R}_{p_i}$.

Proof: By R17.2.5 H_k is a continuous homomorphism from $\mathbf{R}_k$ into the direct product and $H_k \circ f_k = e_k$. The image of H_k is closed and contains $e_k[\mathbf{Z}]$. Thus $H_k[\mathbf{R}_k]$ contains the closure of $e_k[\mathbf{Z}]$, which is $\text{rep}(\vee^j_{i=1}\mathbf{R}_{p_i})$. By R17.2.17 H_k is onto. Now let $\mathcal{F}, \mathcal{G}$ be in $\mathbf{R}_k$ with $H_k(\mathcal{F}) = H_k(\mathcal{G})$. Since $k = \Pi^j_{i=1} p_i^{\alpha_i}$ where each exponent is an integer with $\alpha_i \geq 1$, k is the least common multiple of $p_1^{\alpha_1}, \ldots, p_j^{\alpha_j}$. By R17.2.12 and the definition of ${}_xh_k$, R17.2.8 applies so that $\mathcal{F} = \mathcal{G}$, i.e., H_k is one-to-one. Since the domain is compact and the image T_2, H_k^{-1} is also continuous.

Corollary R17.2.20 Let $k \geq 2$ be a composite number in $\mathbb{N}$. Then $\mathbf{R}_k$ has divisors of zero.

Proof: Let $p_1, p_2, \ldots, p_j$ denote all the distinct prime factors of k. By hypothesis $j \geq 2$. By R17.2.19, $\mathbf{R}_k$ is isomorphic to the direct product $\Pi^j_{i=1}\mathbf{R}_{p_i}$. The latter clearly has divisors of zero, e.g., $(f_{p_1}(1), \mathcal{O}_{p_2}, \ldots, \mathcal{O}_{p_j})$ and $(\mathcal{O}_{p_1}, f_{p_2}(1), \ldots, \mathcal{O}_{p_j})$.

Algebraic Properties of $\mathbf{R}_\infty$

$\mathbf{R}_\infty$ will first be identified as an inverse limit. The underlying directed set will be $D = \{k \in \mathbb{N} : k \geq 2\}$ with the divisibility order, i.e., $a \preceq b$ if and only if $a|b$. Note that R17.2.5 and R17.2.7 show that $\{\mathbf{R}_i ;\ {}_ih_j\}$ is an inverse system of topological rings.

The maps $\rho_i : \mathbf{R}_\infty \to \mathbf{R}_i$ were defined in R16.17 by $\rho_i(\mathcal{F}) = \mathcal{F} \cap \mathcal{Z}_i$. In R16.18 each ρ_i was shown to be a ring homomorphism. For the next results recall that the compactification $\mathbb{N}_\infty$ is generated from the normal basis $\mathcal{Z}_\infty = \cup\{\mathcal{Z}_n : n \in \mathbb{N}\} = \cup\{\mathcal{Z}_n : n \in D\}$.

Lemma R17.3.1 Each ρ_i is continuous.

Proof: $\mathcal{Z}_i \subseteq \mathcal{Z}_\infty$, and by R10.2.8 and the proof of R9.1.1iii the map $\mathcal{F} \mapsto \mathcal{F} \cap \mathcal{Z}_i$ is continuous from $\mathbb{N}_\infty$ to $\mathbb{N}_i$. ρ_i is the restriction of this map to the subspace $\mathbf{R}_\infty$.

Lemma R17.3.2 Let $i, j \in D$ with $i|j$. Then ${}_ih_j \circ \rho_j = \rho_i$.

Proof: By R12.5.17i $\mathcal{Z}_i \subseteq \mathcal{Z}_j$ and so ${}_ih_j \circ \rho_j(\mathcal{F}) = (\mathcal{F} \cap \mathcal{Z}_j) \cap \mathcal{Z}_i = \mathcal{F} \cap \mathcal{Z}_i = \rho_i(\mathcal{F})$.

The following notational refinement will be needed temporarily: For $W \in \mathcal{Z}_k$, ${}^\infty W^\omega = \{\mathcal{F} \in \omega(\mathcal{Z}_\infty) : W \in \mathcal{F}\}$ and ${}^kW^\omega = \{\mathcal{F} \in \omega(\mathcal{Z}_k) : W \in \mathcal{F}\}$

Lemma R17.3.3 Let $W \in \mathcal{Z}_k$, where $k \in D$. Then $\rho_k^{-1}[{}^kW^\omega \cap \mathbf{R}_k] = {}^\infty W^\omega \cap \mathbf{R}_\infty$.

Proof: If $\mathcal{F} \in \rho_k^{-1}[{}^kW^\omega \cap \mathbf{R}_k]$, then $\mathcal{F}$ is in $\mathbf{R}_\infty$ and $\mathcal{F} \cap \mathcal{Z}_k$ is a $\mathcal{Z}_k$-ultrafilter with $W \in \mathcal{F}$, so that $\mathcal{F} \in {}^\infty W^\omega \cap \mathbf{R}_\infty$. If $\mathcal{F} \in {}^\infty W^\omega \cap \mathbf{R}_\infty$, then $W \in \mathcal{F}$ and $\rho_k(\mathcal{F}) = \mathcal{F} \cap \mathcal{Z}_k$ is in $\mathbf{R}_k$ and contains W, i.e., $\mathcal{F} \in \rho_k^{-1}[{}^kW^\omega \cap \mathbf{R}_k]$.

The following simple variation of R12.5.18 is needed because attention is restricted to $k \geq 2$.

Lemma R17.3.4 Assume that for every $k \in \mathbb{N}$ with $k \geq 2$, there is $\mathcal{F}_k \in \mathbf{R}_k$ such that $k|m$ implies $\mathcal{F}_k \subseteq \mathcal{F}_m$. Let $\mathcal{F} = \cup_{k=2}^{\infty} \mathcal{F}_k$. Then $\mathcal{F}$ is in $\mathbf{R}_\infty$and $\mathcal{F} \cap \mathcal{Z}_k = \mathcal{F}_k$ for all $k \geq 2$.

Proof: Let $\mathcal{F}_1$ be the unique non-point ultrafilter in $\mathbb{N}_1$. For every k, $\mathcal{Z}_1 \subseteq \mathcal{Z}_k$ by R12.5.17 and, since $\mathcal{F}_k$ is a non-point ultrafilter, $\mathcal{F}_k \cap \mathcal{Z}_1$ is also non-point, i.e., it is $\mathcal{F}_1$ so that $\mathcal{F}_1 \subseteq \mathcal{F}_k$. Thus $\mathcal{F} = \cup_{k=1}^{\infty} \mathcal{F}_k$. By R12.5.18, $\mathcal{F}$ is in $\mathbb{N}_\infty$ and the second half of the conclusion holds. By R10.2.8 $\mathcal{F}$ is non-point.

Proposition R17.3.5 The inverse limit of $\{\mathbf{R}_i\,;\ {}_ih_j\}$ is topologically isomorphic to $\mathbf{R}_\infty$.

Proof: It will be shown that $\mathbf{R}_\infty$ and the maps $\{\rho_i\}$ have the universal property of the inverse limit of the inverse system $\{\mathbf{R}_i;\ {}_ih_j\}$, which immediately yields the conclusion. Let S be a topological ring and $\{q_i : i \geq 2\}$ a sequence of continuous ring homomorphisms with $q_i : S \to \mathbf{R}_i$ such that, whenever $i|j$, ${}_ih_j \circ q_j = q_i$. By the categorical characterization of inverse limits, it is sufficient to show that there is a unique continuous ring homomorphism $\psi : S \to \mathbf{R}_\infty$ such that, for every $i \geq 2$, $q_i = \rho_i \circ \psi$.

First, ψ will be defined and shown unique. Given $s \in S$, if $i|j$, $q_i(s) = {}_ih_j(q_j(s)) = q_j(s) \cap \mathcal{Z}_i$, so that $q_i(s) \subseteq q_j(s)$. By R17.3.4 $\cup\{q_n(s) : n \in D\}$ is an element of $\mathbf{R}_\infty$and $\cup\{q_n(s) : n \in D\} \cap \mathcal{Z}_i = q_i(s)$. Let $\psi(s) = \cup\{q_n(s) : n \in D\}$. We have, for every $i \geq 2$, $q_i = \rho_i \circ \psi$. If $\psi_1 : S \to \mathbf{R}_\infty$and $q_i = \rho_i \circ \psi_1$ for all $i \geq 2$, $\psi_1(s) = \cup\{\psi_1(s) \cap \mathcal{Z}_n : n \in D\} = \cup\{q_n(s) : n \in D\} = \psi(s)$.

To see that ψ is a ring homomorphism, let $s, t \in S$. By R12.2.7i, $\psi(s) + \psi(t) = (\psi(s) \cap \mathcal{Z}_1 + \psi(t) \cap \mathcal{Z}_1) \cup [\cup_{n=2}^{\infty}(\rho_n(\psi(s)) + \rho_n(\psi(t)))]$. Since $(\psi(s) \cap \mathcal{Z}_1 + \psi(t) \cap \mathcal{Z}_1)$ is contained in $\rho_2(\psi(s)) + \rho_2(\psi(t))$, the initial term can be omitted. Thus,

$$\begin{aligned}
\psi(s) + \psi(t) &= \cup_{n=2}^{\infty}(\rho_n(\psi(s)) + \rho_n(\psi(t))) \\
&= \cup_{n=2}^{\infty}(q_n(s) + q_n(t)) \\
&= \cup_{n=2}^{\infty} q_n(s + t) \\
&= \psi(s + t).
\end{aligned}$$

By using R12.2.7ii in a similar way, one also obtains $\psi(st) = \psi(s)\psi(t)$.

Finally, for continuity of ψ, let $W \in \mathcal{Z}_\infty$. Since $\{{}^\infty Z^\omega : Z \in \mathcal{Z}_\infty\}$ is a clopen basis for $\mathbb{N}_\infty$, it is sufficient to to show that $\psi^{-1}[{}^\infty W^\omega \cap \mathbf{R}_\infty]$ is open in S. Pick $k \geq 2$ such that $W \in \mathcal{Z}_k$. As $q_k^{-1}[{}^k W^\omega \cap \mathbf{R}_k]$ is open in S and $q_k^{-1}[{}^k W^\omega \cap \mathbf{R}_k] = \psi^{-1}[\rho_k^{-1}[{}^k W^\omega \cap \mathbf{R}_k]]$, the desired conclusion follows from R17.3.3.

Definition R17.3.6 Let $k \in \mathbb{N}$ with $k \geq 2$. $T(k)$ is the set of primes which divide k.

Lemma R17.3.7 Let $k \in \mathbb{N}$ with $k \geq 2$. Let $\mathcal{F}, \mathcal{G}$ be in $\mathbf{R}_k$ with $\mathcal{F} \cap \mathcal{Z}_p = \mathcal{G} \cap \mathcal{Z}_p$ for every $p \in T(k)$. Then $\mathcal{F} = \mathcal{G}$.

Proof: $k = \Pi_{i=1}^{j} p_i^{\alpha_i}$, where $p_1, \ldots, p_j$ are the distinct primes in $T(k)$ and $\alpha_i \geq 1$ for each i. Since $\mathcal{Z}_{p_i^{\alpha_i}} = \mathcal{Z}_{p_i}$, $\mathcal{F} \cap \mathcal{Z}_{p_i^{\alpha_i}} = \mathcal{G} \cap \mathcal{Z}_{p_i^{\alpha_i}}$ for $1 \leq i \leq j$. Since k is the least common multiple of $p_1^{\alpha_1}, \ldots, p_j^{\alpha_j}$, by R17.2.8 $\mathcal{F} = \mathcal{G}$.

Definition R17.3.8 Let F be a non-empty subset of $\{p \in \mathbb{N} : p \text{ is prime}\}$. π_F is the natural projection from the infinite direct product $\Pi\{\mathbf{R}_p : p \text{ is prime}\}$ to the direct product $\Pi\{\mathbf{R}_p : p \in F\}$.

Note that each π_F is a continuous surjective homomorphism.

Definition R17.3.9 Let $k \in \mathbb{N}$ with $k \geq 2$. σ_k from $\Pi\{\mathbf{R}_p : p \text{ is prime}\}$ to $\mathbf{R}_k$ is defined by $\sigma_k = H_k^{-1} \circ \pi_{T(k)}$.

Note that each σ_k is a continuous surjective homomorphism by R17.2.19.

Lemma R17.3.10 Let $k, m \in \mathbb{N}$ with $k \geq 2$ and $k|m$. Then ${}_k h_m \circ \sigma_m = \sigma_k$.

Proof: Let x be in $\Pi\{\mathbf{R}_p : p \text{ is prime}\}$, and let ${}_k h_m \circ \sigma_m(x) = \mathcal{F}$ and $\sigma_k(x) = \mathcal{G}$. Let $\mathcal{H} = \sigma_m(x)$. By definition of ${}_k h_m$, $\mathcal{H} \cap \mathcal{Z}_k = \mathcal{F}$.

Now let $p \in T(k)$. Since $k|m$, $T(k) \subseteq T(m)$ so that $\pi_{T(k)}(x)$ and $\pi_{T(m)}(x)$ both have a value at p and $\pi_{T(k)}(x)(p) = \pi_{T(m)}(x)(p)$. By definition of σ_k and σ_m, $\pi_{T(k)}(x)(p) = {}_p h_k(\mathcal{G}) = \mathcal{G} \cap \mathcal{Z}_p$ and $\pi_{T(m)}(x)(p) = {}_p h_m(\mathcal{H}) = \mathcal{H} \cap \mathcal{Z}_p$ so that $\mathcal{G} \cap \mathcal{Z}_p = \mathcal{H} \cap \mathcal{Z}_p$. Since $\mathcal{Z}_p \subseteq \mathcal{Z}_k$, $\mathcal{F} \cap \mathcal{Z}_p = (\mathcal{H} \cap \mathcal{Z}_k) \cap \mathcal{Z}_p = \mathcal{H} \cap \mathcal{Z}_p = \mathcal{G} \cap \mathcal{Z}_p$. By R17.3.7 $\mathcal{F} = \mathcal{G}$.

Lemma R17.3.11 Let $k \in \mathbb{N}$ with $k \geq 2$, and let $z \in \mathbf{Z}$. Let q_z be the point of $\Pi\{\mathbf{R}_p : p \text{ is prime}\}$ described by $q_z(p) = f_p(z)$. Then $\sigma_k(q_z) = f_k(z)$.

Proof: Let $p \in T(k)$. Note that $H_p = {}_p h_p$ is the identity map on $\mathbf{R}_p$ and, since $T(p) = \{p\}$, $\pi_{T(p)}$ is simply projection onto the pth component. Thus $\sigma_p(q_z) = f_p(z)$, which by R17.2.5 equals $f_k(z) \cap \mathcal{Z}_p$. By R17.3.10 $\sigma_k(q_z) \cap \mathcal{Z}_p = {}_p h_k \circ \sigma_k(q_z) = \sigma_p(q_z)$. By R17.3.7 $\sigma_k(q_z) = f_k(z)$.

Lemma R17.3.12 Let $z \in \mathbf{Z}$. Then $f_\infty(z) = \cup_{i=2}^{\infty} f_i(z)$.

Proof: $f_\infty(z) = \cup_{i=1}^{\infty}(f_\infty(z) \cap \mathcal{Z}_i) = (f_\infty(z) \cap \mathcal{Z}_1) \cup (\cup_{i=2}^{\infty}(f_\infty(z) \cap \mathcal{Z}_i)$. By R16.19 $f_\infty(z) \cap \mathcal{Z}_i = f_i(z)$ for $i \geq 2$. Since $(f_\infty(z) \cap \mathcal{Z}_1) \subseteq (f_\infty(z) \cap \mathcal{Z}_2)$, the conclusion holds.

Theorem R17.3.13 The direct product $\Pi\{\mathbf{R}_p : p \text{ is prime}\}$ is topologically isomorphic to $\mathbf{R}_\infty$.

Proof: Because of R17.3.10, the universal property of the inverse limit $\mathbf{R}_\infty$ applies: there is a unique continuous homomorphism ψ from $\Pi\{\mathbf{R}_p : p \text{ is prime}\}$ to $\mathbf{R}_\infty$ such that $\rho_i \circ \psi = \sigma_i$ for all $i \geq 2$. As in the proof of R17.3.10 $\psi(q) = \cup_{i=2}^{\infty} \sigma_i(q)$ for all q in the direct product. By R17.3.11 and R17.3.12 $\psi(q_z) = f_\infty(z)$ for all $z \in \mathbf{Z}$, where $q_z(p) = f_p(z)$ for every prime p. Thus the image of ψ, which is closed as the continuous image from a compact domain to a T_2 space, contains the dense $f_\infty[\mathbf{Z}]$, i.e., ψ is onto. To see that ψ is one-to-one, let q be in the kernel of the homomorphism ψ, and let p be a prime. As in the proof of R17.3.11, $\sigma_p(q) = q(p)$. In addition, $\sigma_p(q) = \rho_p(\psi(q)) = \psi(q) \cap \mathcal{Z}_p = \mathcal{O}_\infty \cap \mathcal{Z}_p$, which is $\mathcal{O}_p$ by R12.5.21 and R12.5.18. Thus q is the additive identity of the direct product, and so ψ is one-to-one. Finally, a one-to-one continuous surjection from a compact domain to a Hausdorff space must be a homeomorphism.

Corollary R17.3.14 $\mathbf{R}_\infty$ has divisors of zero.

Proof: The isomorphic $\Pi\{\mathbf{R}_p : p \text{ is prime}\}$ has divisors of zero, e.g., $(f_2(1), \mathcal{O}_3, \mathcal{O}_5, \dots)$ and $(\mathcal{O}_2, f_3(1), \mathcal{O}_5, \dots)$.

The next corollary can also be easily derived from R17.3.7.

Corollary R17.3.15 Let $\mathcal{F}, \mathcal{G}$ be in $\mathbf{R}_\infty$ with $\mathcal{F} \cap \mathcal{Z}_p = \mathcal{G} \cap \mathcal{Z}_p$ for every prime p. Then $\mathcal{F} = \mathcal{G}$.

Proof: Let ψ be as in the proof of R17.3.13 and let q_1, q_2 be in $\Pi\{\mathbf{R}_p : p \text{ is prime}\}$ such that $\psi(q_1) = \mathcal{F}$ and $\psi(q_2) = \mathcal{G}$. For each prime p, $\mathcal{F} \cap \mathcal{Z}_p = \rho_p(\psi(q_1)) = \sigma_p(q_1) = q_1(p)$ and similarly $\mathcal{G} \cap \mathcal{Z}_p = q_2(p)$. The hypothesis yields $q_1 = q_2$ and so $\mathcal{F} = \mathcal{G}$.

Corollary R17.3.16 For every prime p assume $\mathcal{F}_p$ is in $\mathbf{R}_p$. Then there is a unique $\mathcal{F}$ in $\mathbf{R}_\infty$ such that $\mathcal{F} \cap \mathcal{Z}_p = \mathcal{F}_p$ for every prime p.

Proof: Let ψ be as in the proof of R17.3.13 and let q in $\Pi\{\mathbf{R}_p : p \text{ is prime}\}$ be defined by $q(p) = \mathcal{F}_p$. Let $\mathcal{F} = \psi(q)$. For each prime p, $\mathcal{F} \cap \mathcal{Z}_p = \rho_p(\psi(q)) = \sigma_p(q) = q(p) = \mathcal{F}_p$. The uniqueness of $\mathcal{F}$ follows from R17.3.15.

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3. This Website, P3: Normal Bases

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Added Comment 2010

In Robert's book [11] various representations of the p-adic integers are described, including the representation as an appropriate projective limit. While the exact construction used in this website and the modest generalization to $\mathbf{R}_\infty$ and $\mathbf{R}_k$ for non-prime k are not presented, most of the general facts in this section about $\mathbf{R}_p$ for prime p can be found there.

Additional Reference

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