Metrizable Compactifications

Some compactifications are metrizable, e.g. the remnant rings (as subspaces of the metrizable spaces in R10.1.10iii) or the one-point compactification of the reals, while other examples show that a metrizable $T_{3\frac{1}{2}}$ space can have a non-metrizable compactification, e.g. the Stone-Čech compactification of a countable discrete space.

This section collects some results about metrizable compactifications. Most, maybe all, of these are known. Proofs are included for the convenience of the reader. Notation: In a space with metric $d, B_d(x, \epsilon)$ denotes the ϵ-ball centered at x. $\overline{A}$ denotes the closure of A whenever there is no ambiguity about the topology.

The first result is a simple generalization of R3.2.8 (for compactifications of a fixed $T_{3\frac{1}{2}}$ space (X, τ), a countable supremum of metrizable compactifications is metrizable) to mixed suprema.

Proposition R18.1 Let X be a set and let Δ be a countable non-empty set. For each α in Δ let τ_α be a $T_{3\frac{1}{2}}$ topology on X and let (Y_α, f_α) be a T_2 compactification of (X, τ_α). Let $\tau = \vee\{\tau_\alpha : \alpha \in \Delta\}$. Let (Y, f) be a T_2 compactification of (X, τ) which acts as a supremum of $\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$. If Y_α is metrizable for every $\alpha \in \Delta$, then Y is also metrizable.

Proof: By the results of R13.2 Y is homeomorphic to the inverse limit of an appropriate inverse spectrum. This inverse limit is a subspace of a countable product, each factor of which is metrizable since each Y_α is assumed metrizable. Thus Y is metrizable.

The next result provides a simple characterization of spaces with a metrizable compactification.

Theorem R18.2 Let (X, τ) be a $T_{3\frac{1}{2}}$ space. The following are equivalent.

i) (X, τ) has a metrizable compactification.

ii) X has a totally bounded metric generating τ.

iii) (X, τ) is second countable.

Proof: To see that i) implies ii), let (Y, f) be a metrizable compactification of (X, τ) and let ρ be a metric generating the topology of Y. Since Y is compact, ρ is totally bounded when restricted to any subspace of Y. The subspace metric on $f[X]$ induces the required metric on X. Next assume d is a totally bounded metric on X with $\tau = \tau_d$. For each natural number n there exist $x_1^n, x_2^n, \ldots, x_{j(n)}^n$ such that $X = \cup_{i=1}^{j(n)} B_d(x_i^n, \frac{1}{n})$. The collection $\{B_d(x_i^n, \frac{1}{n}) : n \in \mathbb{N} \text{ and } 1 \leq i \leq j(n)\}$ is a countable basis for τ. Finally, assume (X, τ) is second countable. The standard proof of Urysohn's metrization theorem constructs an embedding g of (X, τ) into $\Pi_{1=1}^{\infty}[0, 1]$, the Hilbert cube, which is compact and metrizable. Then $(\overline{g[X]}, g)$ is a metrizable compactification of (X, τ).

The previous result identifies a large class of metric spaces which have no metric compactification. The most straightforward of these is the following.

Corollary R18.3 An uncountable discrete space has no metric compactification.

Proof: Since each singleton is open, an uncountable discrete space cannot have a countable basis.

The theorem also points the way to the following characterization of metrizable finite point compactifications.

Proposition R18.4 Let (X, τ) be a non-compact locally compact T_2 space. The following are equivalent.

i) (X, τ) is second countable.

ii) Every finite point compactification of (X, τ) is metrizable.

iii) The one point compactification of (X, τ) is metrizable.

Proof: By R18.2 iii) implies i) and clearly ii) implies iii). To finish, assume i) holds and let (Z, g) be an n point compactification of (X, τ). By R5.1.1 there exists a pairwise disjoint collection of open sets in X, $\{G_1, \ldots, G_n\}$, such that $K = X - \cup_{i=1}^n G_i$ is compact and $K \cup G_i$ is non-compact for all i. By R5.1.2 (Z, g) is equivalent to (Y, f), where $Y = X \cup \{p_1, \ldots, p_n\}$ with $p_i \neq p_j$ for $i \neq j$ and $p_i \notin X$ for all i, $f : X \to Y$ by $f(x) = x$, and the topology on Y is $\sigma = \{O \subseteq Y : O \cap X$ is open and $p_i \in O$ implies $(X - O) \cap G_i$ has compact closure in $X\}$. Since (X, τ) is locally compact and second countable, there is a countable open basis $\{V_j : j \in \mathbb{N}\}$ for (X, τ) such that $\overline{V_j}$ is compact in X for all j. Now fix i. It is claimed that $\mathcal{S}_i = \{((X - \overline{V_j}) \cap G_i) \cup \{p_i\} : j \in \mathbb{N}\}$ is a local subbasis at p_i. It is easy to check that $((X - \overline{V_j}) \cap G_i) \cup \{p_i\}$ is open in σ for all j. Let $p_i \in O \in \sigma$. By compactness there is a finite set F of $\mathbb{N}$ such that the X-closure of $(X - O) \cap G_i$ is contained in $\cup\{V_j : j \in F\}$. It is easy to check that $\cap\{((X - \overline{V_j}) \cap G_i) \cup \{p_i\} : j \in F\} \subseteq O$, as required. The countable local subbasis $\mathcal{S}_i$ generates a countable local basis at p_i. $\{V_j : j \in \mathbb{N}\}$ together with the n local bases is a countable basis for (Y, σ). By Urysohn's metrization theorem Y is metrizable.

Corollary R18.5 Every finite point compactification of a countably infinite discrete space is metrizable.

Proof: Every countable discrete space is second countable.

It is known (e.g., [3; p. 148]) that the Stone-Čech compactification cannnot be metrizable unless original space is compact. Thus every compactification of a $T_{3\frac{1}{2}}$ space (X, τ) is metrizable if and only if (X, τ) is compact and metrizable.

R18.4 and R18.1 show that a countable supremum of finite point compactifications of a countably infinite discrete space must be metrizable. This raises the question of whether every metrizable compactification of a countably infinite discrete space must be a countable supremum of finite point compactifications. The following results include an example that answers that question in the negative. They also describe the metrizable compactification associated with a totally bounded metric in terms of the map Ψ_0 defined in R1.3. Recall that $\mathcal{U}_d$ denotes the uniformity generated by the metric d.

Proposition R18.6 Let d be a totally bounded metric on X, let ρ be a complete metric on Y, and let $f : X \to Y$ be an isometry with $\overline{f[X]} = Y$. Then $\Psi_0(\mathcal{U}_d) = [(Y, f)]$.

Proof: The hypothesis means that $(Y, \mathcal{U}_\rho)$ is a separated completion of the totally bounded uniform space $(X, \mathcal{U}_d)$ with uniform embedding f. By definition $\Psi_0(\mathcal{U}_d) = [(Y, f)]$.

Corollary R18.7 If d is a totally bounded metric for X and $\Psi_0(\mathcal{U}_d) = [(Z, g)]$, then Z is metrizable.

Proof: Every metric space has a completion, and so R18.6 applies: $\Psi_0(\mathcal{U}_d) = [(Y, f)]$, where Y is metrizable. Z is also metrizable since equivalent compactifications are homeomorphic.

Lemma R18.8 Let $\mathcal{U}$ be a totally bounded uniformity for the $T_{3\frac{1}{2}}$ space (X, τ), and let $\Psi_0(\mathcal{U}) = [(Y, f)]$. If Y is metrizable, then $\mathcal{U}$ is metrizable.

Proof: The unique uniformity for Y must metrizable and f is a unimorphism from $(X,\mathcal{U})$ to $f[X]$ with the subspace uniformity.

Lemma R18.9 Let $\mathcal{U}$ and $\mathcal{V}$ be metrizable uniformities on a set X. Then $\mathcal{U} \vee \mathcal{V}$ is also a metrizable uniformity.

Proof: Let $\{U_i : i \in \mathbb{N}\}$ and $\{V_j : j \in \mathbb{N}\}$ be countable bases for $\mathcal{U}$ and $\mathcal{V}$ respectively. Then $\{U_i \cap V_j : i, j \in \mathbb{N}\}$ is a countable basis for $\mathcal{U} \vee \mathcal{V}$, which must therefore be pseudo-metrizable. Since $\mathcal{U} \vee \mathcal{V}$ contains a separated uniformity, it is also separated. Thus $\mathcal{U} \vee \mathcal{V}$ is metrizable.

Example R18.10 Let X be the set of rational numbers in $[0, 1]$, let $\mathcal{V}$ be the uniformity generated by the absolute value metric on X, let $\mathcal{U}_m$ be the smallest totally bounded uniformity for X with the discrete topology, and let $\mathcal{U} = \mathcal{V} \vee \mathcal{U}_m$. Since $\Psi_0(\mathcal{U}_m)$ is the class of the one point compactification of X with the discrete topology, by R18.5 and R18.8 $\mathcal{U}_m$ is metrizable. By R18.9, P2.13, and P2.14 $\mathcal{U}$ is a totally bounded metrizable uniformity for X with the discrete topology. Let $\Psi_0(\mathcal{U}) = [(Y, f)]$. By R18.7 Y is metrizable. However, Y is not zero dimensional. This can be verified as in R9.3.6 and R9.3.7. (The argument there works almost unchanged because X is an infinite dense subset of $[0, 1]$ with the usual topology.) By R9.3.3 (Y, f) is not a supremum of finite point compactifications.

Finally, let's mention without proof that every metric compactification is a Wallman compactification, i.e., that, given (Y, f) a metric compactification of (X, τ), there is a normal basis $\mathcal{Z}$ for the closed sets of X such that (Y, f) is equivalent to $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$. According to Chandler and Faulkner [2; p. 656] this result was discovered by J.M. Aarts [1].

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http://www.susanjkleinart.com/compactification/

References

An asterisk indicates a reference not seen by me.

1* Aarts, J.M., Every Metric Compactification is a Wallman-type Compactification, Proc. Internat. Sympos. on Topology and its Applications, Herceg-Novi, 1968, 29-34.

2. Chandler, R.E. and Faulkner, G.D., Hausdorff Compactifications: A Retrospective, in: A Handbook of the History of General Topology, vol. 2, C.E. Aull and R. Lowen, editors, Kulwer Academic Publishers, 1998.

3. Wilansky, A., Topology for Analysis, Ginn and Co., 1970.

4. This Website, P2: Uniform Spaces

5. This Website, R1:Existence of the Supremum via Uniform Space Theory

6. This Website, R3: Representation of Suprema

7. This Website, R5: Finite-Point Compactifications

8. This Website, R9: Directed Sets of Normal Bases

9. This Website, R10: Some Metric Compactifications of $\mathbb{N}$

10. This Website, R13: Mixed Suprema

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This note identifies another equivalent condition, which involves the Stone-Čech compactification, that can be added to the list in R18.2. First a modest extention of R18.9 is presented.

Lemma R18.Add.1 Let X be a set, let $\mathcal{U}$ be a metrizable uniformity for X, and let d be a pseudo-metric for X. Then $\mathcal{U} \vee \mathcal{U}_d$ is a metrizable uniformity.

Proof: By hypothesis $\mathcal{U}$ is separable and has a countable base, and $\mathcal{U}_d$ has a countable base. As in the proof of R18.9, $\mathcal{U} \vee \mathcal{U}_d$ has a countable base. Since it contains $\mathcal{U}$, it is also separable and so metrizable.

Proposition R18.Add.2 Let (X, τ) be a $T_{3\frac{1}{2}}$ space. Then (X, τ) is second countable if and only if its Stone-Čech compactification is the supremum of a non-empty family of metrizable compactifications of (X, τ).

Proof: The existence of a metrizable compactification implies that (X, τ) is second countable by R18.2. Thus the condition is sufficient. For necessity, assume (X, τ) is second countable and let $\mathcal{U}_M$ be the largest totally bounded uniformity generating τ. By R18.2 there exists ρ, a totally bounded metric generating τ. Then $\mathcal{U}_\rho$ is a totally bounded uniformity generating τ and so is contained in $\mathcal{U}_M$. Let $V \in \mathcal{U}_M$. By R7.2.13 there is a pseudo-metric d on X with $V \in \mathcal{U}_d \subseteq \mathcal{U}_M$. Then $\mathcal{U}_d$ is totally bounded, as is $\mathcal{U} \vee \mathcal{U}_d$. Clearly $\mathcal{U} \vee \mathcal{U}_d \subseteq \mathcal{U}_M$ and $\tau(\mathcal{U} \vee \mathcal{U}_d) = \tau$. By R18.Add.1 $\mathcal{U} \vee \mathcal{U}_d$ is metrizable. Thus $\mathcal{U}_M$ is the supremum of a non-empty family of totally bounded, metrizable uniformities generating τ. By R18.7 each totally bounded, metrizable uniformity corresponds to a class of metrizable compactifications and by R1.8 $\mathcal{U}_M$ corresponds to the class of the Stone-Čech compactification. Since the complete upper semi-lattices are order isomorphic (R8.2), the Stone-Čech compactification is the supremum of a non-empty family of metrizable compactifications of (X, τ).

Two metrics for a set are equivalent provided they generate the same topology. That notion can be refined as follows.

Definition R18.Add.3 Let d, e be metrics for a set X. Then d is uniformly equivalent to e if and only if $\mathcal{U}_d = \mathcal{U}_e$.

Corollary R18.Add.4 Let (X, τ) be a second countable, non-compact $T_{3\frac{1}{2}}$ space. Then there are uncountably many equivalent metrics for X, no two of which are uniformly equivalent.

Proof: By hypothesis the Stone-Čech compactification of (X, τ) is not metrizable. By R18.Add.2 it is the supremum of a non-empty family of metrizable compactifications of (X, τ). By R18.1 the family must be uncountable. Two non-equivalent metric compactificatins of (X, τ) determine two metrics for X which are equivalent but not uniformly equivalent.

Additional References

11. This Website, R7: Uniform Continuity and Extension of Maps

12. This Website, R8: Lattice and Semi-lattice Properties