Metrizable Compactifications

Some compactifications are metrizable, e.g. the remnant rings (as subspaces of the metrizable spaces in R10.1.10iii) or the one-point compactification of the reals, while other examples show that a metrizable $T_{3\frac{1}{2}}$ space can have a non-metrizable compactification, e.g. the Stone-Čech compactification of a countable discrete space.

This section collects some results about metrizable compactifications. Most, maybe all, of these are known. Proofs are included for the convenience of the reader. Notation: In a space with metric $d, B_d(x, \epsilon)$ denotes the ϵ-ball centered at x. $\overline{A}$ denotes the closure of A whenever there is no ambiguity about the topology.

The first result is a simple generalization of R3.2.8 (for compactifications of a fixed $T_{3\frac{1}{2}}$ space (X, τ), a countable supremum of metrizable compactifications is metrizable) to mixed suprema.

Proposition R18.1 Let X be a set and let Δ be a countable non-empty set. For each α in Δ let τ_α be a $T_{3\frac{1}{2}}$ topology on X and let (Y_α, f_α) be a T_2 compactification of (X, τ_α). Let $\tau = \vee\{\tau_\alpha : \alpha \in \Delta\}$. Let (Y, f) be a T_2 compactification of (X, τ) which acts as a supremum of $\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$. If Y_α is metrizable for every $\alpha \in \Delta$, then Y is also metrizable.

Proof: By the results of R13.2 Y is homeomorphic to the inverse limit of an appropriate inverse spectrum. This inverse limit is a subspace of a countable product, each factor of which is metrizable since each Y_α is assumed metrizable. Thus Y is metrizable.

The next result provides a simple characterization of spaces with a metrizable compactification.

Theorem R18.2 Let (X, τ) be a $T_{3\frac{1}{2}}$ space. The following are equivalent.

i) (X, τ) has a metrizable compactification.

ii) X has a totally bounded metric generating τ.

iii) (X, τ) is second countable.

Proof: To see that i) implies ii), let (Y, f) be a metrizable compactification of (X, τ) and let ρ be a metric generating the topology of Y. Since Y is compact, ρ is totally bounded when restricted to any subspace of Y. The subspace metric on $f[X]$ induces the required metric on X. Next assume d is a totally bounded metric on X with $\tau = \tau_d$. For each natural number n there exist $x_1^n, x_2^n, \ldots, x_{j(n)}^n$ such that $X = \cup_{i=1}^{j(n)} B_d(x_i^n, \frac{1}{n})$. The collection $\{B_d(x_i^n, \frac{1}{n}) : n \in \mathbb{N} \text{ and } 1 \leq i \leq j(n)\}$ is a countable basis for τ. Finally, assume (X, τ) is second countable. The standard proof of Urysohn's metrization theorem constructs an embedding g of (X, τ) into $\Pi_{1=1}^{\infty}[0, 1]$, the Hilbert cube, which is compact and metrizable. Then $(\overline{g[X]}, g)$ is a metrizable compactification of (X, τ).

The previous result identifies a large class of metric spaces which have no metric compactification. The most straightforward of these is the following.

Corollary R18.3 An uncountable discrete space has no metric compactification.

Proof: Since each singleton is open, an uncountable discrete space cannot have a countable basis.

The theorem also points the way to the following characterization of metrizable finite point compactifications.

Proposition R18.4 Let (X,τ) be a non-compact locally compact T_2 space. The following are equivalent.

i) (X,τ) is second countable.

ii) Every finite point compactification of (X,τ) is metrizable.

iii) The one point compactification of (X,τ) is metrizable.

Proof: By R18.2 iii) implies i) and clearly ii) implies iii). To finish, assume i) holds and let (Z,g) be an n point compactification of (X,τ). By R5.1.1 there exists a pairwise disjoint collection of open sets in X, $\{G_1,\ldots,G_n\}$, such that $K = X - \cup_{i=1}^n G_i$ is compact and $K \cup G_i$ is non-compact for all i. By R5.1.2 (Z,g) is equivalent to (Y,f), where $Y = X \cup \{p_1,\ldots,p_n\}$ with $p_i \neq p_j$ for $i \neq j$ and $p_i \notin X$ for all i, $f : X \to Y$ by $f(x) = x$, and the topology on Y is $\sigma = \{O \subseteq Y : O \cap X$ is open and $p_i \in O$ implies $(X - O) \cap G_i$ has compact closure in $X\}$. Since (X,τ) is locally compact and second countable, there is a countable open basis $\{V_j : j \in \mathbb{N}\}$ for (X,τ) such that $\overline{V_j}$ is compact in X for all j. Now fix i. It is claimed that $\mathcal{S}_i = \{((X - \overline{V_j}) \cap G_i) \cup \{p_i\} : j \in \mathbb{N}\}$ is a local subbasis at p_i. It is easy to check that $((X - \overline{V_j}) \cap G_i) \cup \{p_i\}$ is open in σ for all j. Let $p_i \in O \in \sigma$. By compactness there is a finite set F of $\mathbb{N}$ such that the X-closure of $(X - O) \cap G_i$ is contained in $\cup\{V_j : j \in F\}$. It is easy to check that $\cap\{((X - \overline{V_j}) \cap G_i) \cup \{p_i\} : j \in F\} \subseteq O$, as required. The countable local subbasis $\mathcal{S}_i$ generates a countable local basis at p_i. $\{V_j : j \in \mathbb{N}\}$ together with the n local bases is a countable basis for (Y,σ). By Urysohn's metrization theorem Y is metrizable.

Corollary R18.5 Every finite point compactification of a countably infinite discrete space is metrizable.

Proof: Every countable discrete space is second countable.

It is known (e.g., [3; p. 148]) that the Stone-Čech compactification cannnot be metrizable unless original space is compact. Thus every compactification of a $T_{3\frac{1}{2}}$ space (X,τ) is metrizable if and only if (X,τ) is compact and metrizable.

R18.4 and R18.1 show that a countable supremum of finite point compactifications of a countably infinite discrete space must be metrizable. This raises the question of whether every metrizable compactification of a countably infinite discrete space must be a countable supremum of finite point compactifications. The following results include an example that answers that question in the negative. They also describe the metrizable compactification associated with a totally bounded metric in terms of the map Ψ_0 defined in R1.3. Recall that $\mathcal{U}_d$ denotes the uniformity generated by the metric d.

Proposition R18.6 Let d be a totally bounded metric on X, let ρ be a complete metric on Y, and let $f : X \to Y$ be an isometry with $\overline{f[X]} = Y$. Then $\Psi_0(\mathcal{U}_d) = [(Y,f)]$.

Proof: The hypothesis means that $(Y,\mathcal{U}_\rho)$ is a separated completion of the totally bounded uniform space $(X,\mathcal{U}_d)$ with uniform embedding f. By definition $\Psi_0(\mathcal{U}_d) = [(Y,f)]$.

Corollary R18.7 If d is a totally bounded metric for X and $\Psi_0(\mathcal{U}_d) = [(Z,g)]$, then Z is metrizable.

Proof: Every metric space has a completion, and so R18.6 applies: $\Psi_0(\mathcal{U}_d) = [(Y,f)]$, where Y is metrizable. Z is also metrizable since equivalent compactifications are homeomorphic.

Lemma R18.8 Let $\mathcal{U}$ be a totally bounded uniformity for the $T_{3\frac{1}{2}}$ space (X,τ), and let $\Psi_0(\mathcal{U}) = [(Y,f)]$. If Y is metrizable, then $\mathcal{U}$ is metrizable.

Proof: The unique uniformity for Y must metrizable and f is a unimorphism from $(X,\mathcal{U})$ to $f[X]$ with the subspace uniformity.

Lemma R18.9 Let $\mathcal{U}$ and $\mathcal{V}$ be metrizable uniformities on a set X. Then $\mathcal{U} \vee \mathcal{V}$ is also a metrizable uniformity.

Proof: Let $\{U_i : i \in \mathbb{N}\}$ and $\{V_j : j \in \mathbb{N}\}$ be countable bases for $\mathcal{U}$ and $\mathcal{V}$ respectively. Then $\{U_i \cap V_j : i, j \in \mathbb{N}\}$ is a countable basis for $\mathcal{U} \vee \mathcal{V}$, which must therefore be pseudometrizable. Since $\mathcal{U} \vee \mathcal{V}$ contains a separated uniformity, it is also separated. Thus $\mathcal{U} \vee \mathcal{V}$ is metrizable.

Example R18.10 Let X be the set of rational numbers in $[0,1]$, let $\mathcal{V}$ be the uniformity generated by the absolute value metric on X, let $\mathcal{U}_m$ be the smallest totally bounded uniformity for X with the discrete topology, and let $\mathcal{U} = \mathcal{V} \vee \mathcal{U}_m$. Since $\Psi_0(\mathcal{U}_m)$ is the class of the one point compactification of X with the discrete topology, by R18.5 and R18.8 $\mathcal{U}_m$ is metrizable. By R18.9, P2.13, and P2.14 $\mathcal{U}$ is a totally bounded metrizable uniformity for X with the discrete topology. Let $\Psi_0(\mathcal{U}) = [(Y,f)]$. By R18.7 Y is metrizable. However, Y is not zero dimensional. This can be verified as in R9.3.6 and R9.3.7. (The argument there works almost unchanged because X is an infinite dense subset of $[0,1]$ with the usual topology.) By R9.3.3 (Y,f) is not a supremum of finite point compactifications.

Finally, let's mention without proof that every metric compactification is a Wallman compactification, i.e., that, given (Y,f) a metric compactification of (X,τ), there is a normal basis $\mathcal{Z}$ for the closed sets of X such that (Y,f) is equivalent to $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$. According to Chandler and Faulkner [2; p. 656] this result was discovered by J.M. Aarts [1].

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References

An asterisk indicates a reference not seen by me.

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4. This Website, P2: Uniform Spaces

5. This Website, R1:Existence of the Supremum via Uniform Space Theory

6. This Website, R3: Representation of Suprema

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8. This Website, R9: Directed Sets of Normal Bases

9. This Website, R10: Some Metric Compactifications of $\mathbb{N}$

10. This Website, R13: Mixed Suprema