Ordering the Remnant Rings

The remnant rings were defined in [7] by removing the point-filters from $\mathbb{N}_k$, respectively $\mathbb{N}_\infty$, and observing that the restrictions of the addition and multiplication extended continuously to $\mathbb{N}_k$, respectively $\mathbb{N}_\infty$, make the structures topological rings. As in [7], these rings will be denoted $\mathbf{R}_k$, respectively $\mathbf{R}_\infty$. As shown in R12.6.5 and R12.6.14, the remnant rings are zero-dimensional, compact, and metrizable.

Purisch [2] states that Lynn [1] proved that every zero-dimensional, separable metric space is orderable, i.e., the topology is the open interval topology induced by a linear order. In this section such linear orders will be identified for the remnant rings $\mathbf{R}_k$, with some partial results related to ordering $\mathbf{R}_\infty$.

For a set X with linear order $\leq$, the open interval topology has a subbasis consisting of all open left and right rays. Interval notation will be used in the standard way. For example, given $x \in X$, the open left ray determined by x, $\{t \in X : t < x\}$, will be denoted $(-\infty, x)$ and the closed left ray, $\{t \in X : t \leq x\}$, by $(-\infty, x]$.

This section makes extensive use of definitions, notations, and results from [6] and [7]. In particular, the sequence associated with a non-point ultrafilter in $\mathbb{N}_k$ (defined in R10.2.3) is repeatedly employed, as well as properties of these sequences: the recursive relationship of terms (R10.2.5) and the recursive method of defining a non-point ultrafilter in $\mathbb{N}_k$ (R10.2.6).

In [8] the map f_k from $\mathbf{Z}$ to $\mathbf{R}_k$ was defined (R16.6) and shown to be a continuous, one-to-one ring homomorphism with $f_k(1)$ equal to the multiplicative identity of $\mathbf{R}_k$. For any $z \in \mathbf{Z}$, by R12.5.9ii and R16.1 $f_k(z)$ corresponds to the sequence $\{x_n\}$, where $x_n \equiv z \mod k^n$ for all n.

Ordering of $\mathbf{R}_k$

Throughout this subsection k will denote a fixed natural number with $k \geq 2$. For an integer x and $a \in \mathbb{N}$, $x(a) = y$, where y is the unique element of $\{1, 2, \ldots, k^a\}$ such that $x \equiv y \mod k^a$. For $x \in \{1, 2, \ldots, k^{a+1}\}$, $c(x)$ will denote the unique coefficient in $\{0, 1, \ldots, k-1\}$ such that $x = x(a) + c(x)k^a$.

R10.2.5 can be interpreted as showing that the associated sequences of elements in $\mathbf{R}_k$ correspond to paths in one of k trees. The ordering required will be based on an ordering of these trees. As a first step the following inductive definition of orders at each level of those trees is needed.

Definition R19.1.1 The natural order on $\{1, 2, \ldots, k\}$ is r_1. Given r_j defined on $\{1, 2, \ldots, k^j\}$, the relation r_{j+1} on $\{1, 2, \ldots, k^{j+1}\}$ is defined as follows: Let $s, t \in \{1, 2, \ldots, k^{j+1}\}$. $s\ r_{j+1} t$ if and only if $s(j) \neq t(j)$ and $s(j) r_j t(j)$, or $s(j) = t(j)$ and $c(s) \leq c(t)$.

Lemma R19.1.2 For every $j \in \mathbb{N}$, r_j is a linear order on $\{1, 2, \ldots, k^j\}$.

Proof: By inducttion on j: The assertion is clearly true for r_1, and so assume r_j is a linear order on $\{1, 2, \ldots, k^j\}$. It is clear from the definition and the induction hypothesis that r_{j+1} is reflexive and that, for $s, t \in \{1, 2, \ldots, k^{j+1}\}$, either $s\ r_{j+1} t$ or $t\ r_{j+1} s$. Next assume $s\ r_{j+1} t$ and $t\ r_{j+1} s$. We cannot have $s(j) \neq t(j)$ because the antisymmetry of r_j and the definition would yield a contradiction. Thus $s(j) = t(j)$, and so by definition $c(s) \leq c(t)$ and $c(t) \leq c(s)$. $s(j) = t(j)$ and $c(s) = c(t)$ imply $s = t$. Finally assume

$u \in \{1, 2, \ldots, k^{j+1}\}$ and $s\ r_{j+1}t$ and $t\ r_{j+1}u$. If $s(j) = t(j) = u(j)$, the transitivity of $\leq$ yields $s\ r_{j+1}u$. If $s(j), t(j)$, and $u(j)$ are all distinct, the transitivity of r_j yields $s(j)r_ju(j)$ and so $s\ r_{j+1}u$. The remaining cases arise when the cardinality of $\{s(j), t(j), u(j)\}$ is 2. If $s(j) = t(j)$, then $s(j) \neq u(j)$ and by definition $t(j)r_ju(j)$ so that $s\ r_{j+1}u$. If $t(j) = u(j)$, then $s(j) \neq u(j)$ and by definition $s(j)r_jt(j)$ so that $s\ r_{j+1}u$. The remaining case, $s(j) = u(j) \neq t(j)$, cannot occur. For if it does, $s(j)r_jt(j)$ and $t(j)r_ju(j)$. By the antisymmetry of r_j, $u(j) = t(j)$, a contradiction.

Lemma R19.1.3 Let $\mathcal{F}, \mathcal{G}$ be in $\mathbf{R}_k$ with $\mathcal{F}$ associated with $\{x_n\}$ and $\mathcal{G}$ associated with $\{y_n\}$. If $x_j \neq y_j$ and $x_jr_jy_j$, then $x_nr_ny_n$ for every $n \geq j$.

Proof: Let $n \geq j$. $n = j + t$ and it will be shown by induction on t that $x_nr_ny_n$ and $x_n \neq y_n$. For $t = 0$ the claim holds by hypothesis and so assume it holds for some t and let $n = j + t$. By R10.2.5 $x_{n+1} = x_n + \alpha k^n$ and $y_{n+1} = y_n + \beta k^n$, where $\alpha, \beta \in \{0, 1, \ldots, k-1\}$. Since $x_n, y_n \in \{1, 2, \ldots, k^n\}$, clearly $x_{n+1}(n) = x_n$ and $y_{n+1}(n) = y_n$. By the induction hypothesis $x_n \neq y_n$ and $x_nr_ny_n$ so that $x_{n+1}r_{n+1}y_{n+1}$ by definition. In addition, $x_{n+1} \neq y_{n+1}$ since, otherwise, k^n would divide $|x_n - y_n|$ which is between 1 and $k^n - 1$.

Lemma R19.1.4 Let $\mathcal{F}, \mathcal{G}$ be in $\mathbf{R}_k$ with $\mathcal{F}$ associated with $\{x_n\}$ and $\mathcal{G}$ associated with $\{y_n\}$. Either $x_nr_ny_n$ for all n or $y_nr_nx_n$ for all n.

Proof: If $\mathcal{F} = \mathcal{G}$, then $x_n = y_n$ for all n and the conclusion holds since every r_n is reflexive. If $\mathcal{F} \neq \mathcal{G}$, let j be the smallest element of the non-empty set $\{n : x_n \neq y_n\}$. Since r_j is linear, either $x_jr_jy_j$ or $y_jr_jx_j$. If $x_jr_jy_j$, by the preceding lemma $x_nr_ny_n$ for all $n \geq j$. For $n < j$ $x_n = y_n$ and, since r_n is reflexive, $x_nr_ny_n$. Similarly $y_jr_jx_j$ implies $y_nr_nx_n$ for all n.

Definition R19.1.5 Let $\mathcal{F}, \mathcal{G}$ be in $\mathbf{R}_k$ with $\mathcal{F}$ associated with $\{x_n\}$ and $\mathcal{G}$ associated with $\{y_n\}$. $\mathcal{F} \leq_k \mathcal{G}$ if and only if $x_nr_ny_n$ for all n.

Proposition R19.1.6 $\leq_k$ is a linear order on $\mathbf{R}_k$.

Proof: R19.1.4 shows that any two elements of $\mathbf{R}_k$ are comparable under $\leq_k$. Since r_j is reflexive, antisymmetric, and transitive for all j, it follows easily, using R10.2.4, that $\leq_k$ is also reflexive, antisymmetric, and transitive.

For the next proposition recall the normal basis $\mathcal{Z}_k$ for $\mathbb{N}$, defined in R10.1.3 as the union of the normal bases $\mathcal{Z}(E_n(k))$ over n, where $E_n(k)$ is equivalence mod k^n. The Wallman compactification induced by $\mathcal{Z}_k$ is $\mathbb{N}_k$, which has normal basis $\{Z^\omega : Z \in \mathcal{Z}_k\}$. As in [R10], for $n \in \mathbb{N}$ and $t \in \{1, 2, \ldots, k^n\}$, $C_n^t(k)$ (or simply C_n^t since k is fixed here) denotes the equivalence class in $\mathbb{N}$ of t mod k^n. C_n^t is in $\mathcal{Z}(E_n(k))$ and is associated with $\{1, 2 \ldots, k^n\} - \{t\}$. (Recall definition R5.3.1.)

Proposition R19.1.7 The topology of $\mathbf{R}_k$ is the order topology induced by $\leq_k$.

Proof: Let τ be the topology on $\mathbf{R}_k$ and let $\tau(\leq_k)$ be the order topology. Because $(\mathbf{R}_k ,\tau)$ is compact and Hausdorff, τ is minimal Hausdorff. Since an order topology must be Hausdorff, it is sufficient to show that the subbasis for $\tau(\leq_k)$ is contained in τ. Let $\mathcal{F}$ in $\mathbf{R}_k$ be associated with $\{x_n\}$. Suppose $\mathcal{G}$ associated with $\{y_n\}$ is not in $[\mathcal{F}, \infty)$. Since $\mathbf{R}_k$ is a subspace of $\mathbb{N}_k$, to show that $[\mathcal{F}, \infty)$ is τ-closed it is sufficient to find $Z \in \mathcal{Z}_k$ with $[\mathcal{F}, \infty) \subseteq Z^\omega$ and $\mathcal{G} \notin Z^\omega$. Since $\mathcal{F} \neq \mathcal{G}$, $\{n : x_n \neq y_n\}$ is non-empty and so has a smallest element j. Let $Z = \cup\{C_j^t : y_jr_jt \text{ and } y_j \neq t\}$, a non-empty union since x_j is one such t. As a member of $\mathcal{Z}(E_j(k))$, Z is associated with $\{s : s \in \{1, 2, \ldots, k^j\} \text{ and } s\ r_jy_j\}$, which will be called Γ. By definition of y_j (R10.2.3), $W \in \mathcal{G} \cap \mathcal{Z}(E_j(k))$ if and only W is associated with $\Delta \subseteq \{1, 2, \ldots, k^j\} - \{y_j\}$. Since $y_j \in \Gamma$, $Z \notin \mathcal{G}$, i.e., $\mathcal{G} \notin Z^\omega$. Now let $\mathcal{H} \in [\mathcal{F}, \infty)$ be associated with the sequence $\{z_n\}$. Since $\mathcal{F} \leq_k \mathcal{H}$, $x_j r_j z_j$. By transitivity and antisymmetry of r_j, $y_j r_j z_j$ and $y_j \neq z_j$. Thus $C_j^{z_j} \subseteq Z$ and, since $C_j^{z_j} \in \mathcal{H} \cap \mathcal{Z}(E_j(k))$ by definition of z_j, $Z \in \mathcal{H}$, or equivalently $\mathcal{H} \in Z^\omega$. Similarly $(-\infty, \mathcal{F}]$ is τ-closed.

This subsection concludes with some additional observations about the order $\leq_k$.

Proposition R19.1.8 In $\mathbf{R}_k$, relative to $\leq_k$, $f_k(1)$ is the smallest element and $f_k(0)$ is the largest element.

Proof: Let $\mathcal{F}$ in $\mathbf{R}_k$ correspond to $\{x_n\}$. For every n, $1 \; r_n x_n r_n k^n$. Since $f_k(1)$ corresponds to $\{1, 1, 1, \ldots\}$ and $f_k(0)$ corresponds to $\{k^n\}$, $f_k(1) \leq_k \mathcal{F} \leq_k f_k(0)$ by definition.

Definition R19.1.9 Let X be a set with linear order $\leq$. Let x, y be in X. x, y are a consecutive pair with x smaller provided $x < y$ and there is no $z \in X$ such that $x < z < y$. x and y are consecutive provided x, y are a consecutive pair with x smaller or x, y are a consecutive pair with y smaller.

Lemma R19.1.10 Let $\mathcal{F}, \mathcal{G}$ be in $\mathbf{R}_k$ with $\mathcal{F}$ associated with $\{x_n\}$ and $\mathcal{G}$ associated with $\{y_n\}$. $\mathcal{F}$ and $\mathcal{G}$ are a consecutive pair with $\mathcal{F}$ smaller if and only if there is m in $\mathbb{N}$ such that $x_n = y_n$ for $n < m$, $y_m = x_m + 1$ if $m = 1$, there is $t \in \{0, 1, \ldots, k-2\}$ such that $x_m = x_{m-1} + tk^{m-1}$ and $y_m = y_{m-1} + (t+1)k^{m-1}$ if $m > 1$, and for $n \geq m$ $x_{n+1} = x_n + (k-1)k^n$ and $y_{n+1} = y_n$.

Proof: First assume the conditions hold and let m be the element of $\mathbb{N}$ guaranteed by this hypothesis. Then $x_m \neq y_m$ and $x_m r_m y_m$ by the definition of r_m so that $\mathcal{F} <_k \mathcal{G}$. Next suppose there is $\mathcal{H}$ in $\mathbf{R}_k$ such that $\mathcal{F} <_k \mathcal{H} <_k \mathcal{G}$. Let $\mathcal{H}$ be associated with $\{z_n\}$. Since $x_n r_n z_n r_n y_n$ for all n, $x_n = z_n = y_n$ for $n < m$ by the first condition. By the second and third z_m must be either x_m or y_m. If $z_m = x_m$, by induction and the fourth condition $x_n = z_n$ for all $n \geq m$ and $\mathcal{F} = \mathcal{H}$, a contradiction. If $z_m = y_m$, similarly $y_n = z_n$ for all $n \geq m$ and so $\mathcal{G} = \mathcal{H}$, a contradiction. Thus the requirements of R19.1.9 are satisfied. Conversely assume $\mathcal{F}, \mathcal{G}$ are a consecutive pair with $\mathcal{F}$ smaller. Let m be the smallest element of $\{n : x_n \neq y_n\}$. Clearly $x_n = y_n$ for $n < m$. If $m = 1$, $y_1 > x_1$. Suppose $y_1 > x_1 + 1$ and let $z_n = x_1 + 1$ for all n. By R10.2.6 there is a unique $\mathcal{H}$ in $\mathbf{R}_k$ such that $\mathcal{H}$ is associated with $\{z_n\}$. Since $x_1 r_1 z_1 r_1 y_1$ with no equalities, by definition $\mathcal{F} <_k \mathcal{H} <_k \mathcal{G}$, a contradiction. If $m > 1$, suppose $x_m = x_{m-1} + tk^{m-1}$ and $y_m = y_{m-1} + sk^{m-1}$ with $s > t + 1$. For $1 \leq n \leq m-1$, let $z_n = x_n$. Let $z_m = z_{m-1} + (t+1)k^{m-1}$ and, for $n \geq m$, $z_{n+1} = z_m$. By R10.2.6 there is a unique $\mathcal{H}$ in $\mathbf{R}_k$ such that $\mathcal{H}$ is associated with $\{z_n\}$. Since $x_m r_m z_m r_m y_m$ with no equalities, by definition $\mathcal{F} <_k \mathcal{H} <_k \mathcal{G}$, a contradiction. Finally, suppose there is $j \geq m$ such that $x_{j+1} < x_j + (k-1)k^j$ or $y_{j+1} > y_j$. If $x_{j+1} < x_j + (k-1)k^j$, let $z_n = x_n$ for $1 \leq n \leq j$ and $z_{n+1} = z_n + (k-1)k^n$ for $n \geq j$. By R10.2.6 there is a unique $\mathcal{H}$ in $\mathbf{R}_k$ such that $\mathcal{H}$ is associated with $\{z_n\}$. Since $z_m r_m y_m$ and $x_{j+1} r_{j+1} z_{j+1}$ with no equalities, by definition $\mathcal{F} <_k \mathcal{H} <_k \mathcal{G}$, a contradiction. If $y_{j+1} > y_j$, let $z_n = y_n$ for $1 \leq n \leq j$ and $z_{n+1} = z_n$ for $n \geq j$. By R10.2.6 there is a unique $\mathcal{H}$ in $\mathbf{R}_k$ such that $\mathcal{H}$ is associated with $\{z_n\}$. Since $x_m r_m z_m$ and $z_{j+1} r_{j+1} y_{j+1}$ with no equalities, by definition $\mathcal{F} <_k \mathcal{H} <_k \mathcal{G}$, a contradiction. Thus the four conditions hold as required.

Proposition R19.1.11 Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be in $\mathbf{R}_k$ with $\mathcal{F} <_k \mathcal{G} <_k \mathcal{H}$. Either $\mathcal{F}$ and $\mathcal{G}$ are not consecutive or $\mathcal{G}$ and $\mathcal{H}$ are not consecutive.

Proof: Let $\mathcal{F}$ be associated with $\{x_n\}$, $\mathcal{G}$ with $\{y_n\}$, and $\mathcal{H}$ with $\{z_n\}$. Assume both pairs are consecutive. Then $\mathcal{F}$ and $\mathcal{G}$ are a consecutive pair with $\mathcal{F}$ smaller so that, by R19.1.10, $y_{n+1} = y_n$ eventually. Also $\mathcal{G}$ and $\mathcal{H}$ are a consecutive pair with $\mathcal{G}$ smaller so that, by R19.1.10, $y_{n+1} = y_n + (k-1)k^n$ eventually, a contradiction.

Proposition R19.1.12 The smallest element $f_k(1)$ is not part of a consecutive pair. The largest element $f_k(0)$ is not part of a consecutive pair.

Proof: Let $f_k(1)$ be associated with $\{x_n\}$ and recall that $x_n = 1$ for all n. If $f_k(1)$ and $\mathcal{F}$ are consecutive, then $\mathcal{F}$ must be larger and by R19.1.10 $x_{n+1} = x_n + (k-1)k^n$ eventually, a contradiction. Similarly let $f_k(0)$ be associated with $\{y_n\}$ and recall that $y_n = k^n$ for all n. If $f_k(0)$ and $\mathcal{F}$ are consecutive, then $\mathcal{F}$ must be smaller and by R19.1.10 $y_{n+1} = y_n$ eventually, a contradiction.

Proposition R19.1.13 Let $\mathcal{F}, \mathcal{G}$ be in $\mathbf{R}_k$ with $\mathcal{F} <_k \mathcal{G}$. If $\mathcal{F}$ and $\mathcal{G}$ are not consecutive, then there exist $2^{\aleph_0}$ many $\mathcal{H}$ in $\mathbf{R}_k$ with $\mathcal{F} <_k \mathcal{H} <_k \mathcal{G}$.

Proof: Let $\mathcal{F}$ be associated with $\{x_n\}$ and $\mathcal{G}$ be associated with $\{y_n\}$. By R10.2.5 there exist sequences $\{s_n\}$ and $\{t_n\}$ in $\{0, 1, \ldots, k-1\}$ such that $x_{n+1} = x_n + s_n k^n$ and $y_{n+1} = y_n + t_n k^n$ fot all n. Let m be the smallest element of $\{n : x_n \neq y_n\}$. As a first case, assume $m = 1$ and $y_1 - x_1 \geq 2$ or $m > 1$ and $t_{m-1} - s_{m-1} \geq 2$. If $m = 1$, let $z_1 = x_1 + 1$, and if $m > 1$, let $z_n = x_i$ for $1 \leq n \leq m-1$ and $z_m = z_{m-1} + (s_{m-1}+1)k^{m-1}$. For $n \geq m$ pick w_n in $\{0, 1, \ldots, k-1\}$ and let $z_{n+1} = z_n + w_n k^n$. By R10.2.6 there is a unique $\mathcal{H}$ in $\mathbf{R}_k$ such that $\mathcal{H}$ is associated with $\{z_n\}$. Since $x_m r_m z_m r_m y_m$ with no equalities, by definition $\mathcal{F} <_k \mathcal{H} <_k \mathcal{G}$. As a second case, assume $m = 1$ and $y_1 = x_1 + 1$ or $m > 1$ and $t_{m-1} = s_{m-1} + 1$. Since $\mathcal{F}$ and $\mathcal{G}$ are not consecutive, by R19.1.10 there exists $j \geq m$ such that $s_j < k-1$ or $t_j > 0$. If $s_j < k-1$, let $z_n = x_n$ for $n \leq j$ and $z_{n+1} = z_n + w_n k^n$ for $n \geq j$, where $w_j = s_j + 1$ and $w_n \in \{0, 1, \ldots, k-1\}$ for $n > j$. By R10.2.6 there is a unique $\mathcal{H}$ in $\mathbf{R}_k$ such that $\mathcal{H}$ is associated with $\{z_n\}$. Since $z_m r_m y_m$ and $x_{j+1} r_{j+1} z_{j+1}$ with no equalities, $\mathcal{F} <_k \mathcal{H} <_k \mathcal{G}$. If $t_j > 0$, let $z_n = y_n$ for $n \leq j$ and $z_{n+1} = z_n + w_n k^n$ for $n \geq j$, where $w_j = 0$ and $w_n \in \{0, 1, \ldots, k-1\}$ for $n > j$. By R10.2.6 there is a unique $\mathcal{H}$ in $\mathbf{R}_k$ such that $\mathcal{H}$ is associated with $\{z_n\}$. Since $x_m r_m z_m$ and $z_{j+1} r_{j+1} y_{j+1}$ with no equalities, $\mathcal{F} <_k \mathcal{H} <_k \mathcal{G}$. Finally, as in the comment following R10.2.3, each choice of $\{w_n\}$ determines a distinct $\mathcal{H}$. Since the product of countably many copies of $\{0, 1, \ldots, k-1\}$ has cardinality $2^{\aleph_0}$, the cardinality assertion holds.

Lemma R19.1.14 Let $\mathcal{F}$ be in $\mathbf{R}_k$ be associated with $\{x_n\}$. If $\mathcal{F} \neq f_k(1)$ and x_n is eventually constant, then $\mathcal{F}$ is the larger of a consecutive pair.

Proof: Let m be the smallest of $\{n : j \geq n \Rightarrow x_j = x_n\}$. If $m > 1$, by R10.2.5 $x_m = x_{m-1} + tk^{m-1}$ and, since $x_{m-1} \neq x_m$, $t \in \{1, 2, \ldots, k-1\}$. Let $y_n = x_n$ for $n \leq m-1$, $y_m = y_{m-1} + (t-1)k^{m-1}$, and $y_{n+1} = y_n + (k-1)k^n$ for $n \geq m$. By R10.2.6 there is $\mathcal{G}$ in $\mathbf{R}_k$ with $\mathcal{G}$ associated with $\{y_n\}$. Clearly $y_m \neq x_m$ and, by definition R19.1.1, $y_m r_m x_m$ so that $\mathcal{G} <_k \mathcal{F}$. By R19.1.10 $\mathcal{F}$ and $\mathcal{G}$ are a consecutive pair with $\mathcal{F}$ larger. If $m = 1$, $x_n = x_1$ for all n. Since $\mathcal{F} \neq f_k(1)$, $x_1 \neq 1$. Let $y_1 = x_1 - 1$ and $y_{n+1} = y_n + (k-1)k^n$ for $n \geq 1$. As in the prior case there is $\mathcal{G}$ in $\mathbf{R}_k$ with $\mathcal{G}$ associated with $\{y_n\}$. Clearly $y_1 \neq x_1$ and, by definition R19.1.1, $y_1 r_1 x_1$ so that $\mathcal{G} <_k \mathcal{F}$. By R19.1.10 $\mathcal{F}$ and $\mathcal{G}$ are a consecutive pair with $\mathcal{F}$ larger.

Corollary R19.1.15 Let $j \in \mathbb{N}$ with $j \neq 1$. Then $f_k(j)$ is the larger of a consecutive pair.

Proof: By R12.5.9ii and R16.6 the sequence corresponding to $f_k(j)$ is eventually constant.

Lemma R19.1.16 Let $\mathcal{F}$ be in $\mathbf{R}_k$ be associated with $\{x_n\}$. If $\mathcal{F} \neq f_k(0)$ and $x_{n+1} = x_n + (k-1)k^n$ eventually, then $\mathcal{F}$ is the smaller of a consecutive pair.

Proof: By R10.2.5 $x_1 \in \{1, 2, \ldots, k\}$ and $x_{n+1} = x_n + t_n k^n$, where $t_n \in \{0, 1, \ldots, k-1\}$ for all n. Let m be the smallest of $\{n : i \geq n \Rightarrow t_i = k-1\}$, which is non-empty by hypothesis. If $m = 1$, since $\mathcal{F} \neq f_k(0)$, $x_1 \in \{1, 2, \ldots, k-1\}$. Let $\mathcal{G} = f_k(x_1 + 1)$. By R12.5.9ii $\mathcal{G}$ is associated with $\{y_n\}$, where $y_n = x_1 + 1$ for all n. Since $x_1 \neq y_1$ and $x_1 r_1 y_1$, $\mathcal{F} <_k \mathcal{G}$. By R19.1.10 $\mathcal{F}$ and $\mathcal{G}$ are consecutive with $\mathcal{F}$ smaller. If $m > 1$, then $t_{m-1} < k-1$. Let $s_n = t_n$ for $1 \leq i < m-1$, $s_{m-1} = t_{m-1} + 1$, and $s_n = 0$ for $n \geq m$. Let $y_1 = x_1$ and $y_{n+1} = y_n + s_n k^n$ for all n. By R10.2.6 There is $\mathcal{G}$ in $\mathbf{R}_k$ such that $\mathcal{G}$ is associated with $\{y_n\}$. Since $x_n = y_n$ for $1 \leq n \leq m-1$, $x_m \neq y_m$, and $x_m r_m y_m$, $\mathcal{F} <_k \mathcal{G}$. By R19.1.10 $\mathcal{F}$ and $\mathcal{G}$ are consecutive with $\mathcal{F}$ smaller.

Corollary R19.1.17 Let $j \in \mathbb{N}$. Then $f_k(-j)$ is the smaller of a consecutive pair.

Proof: $f_k(-j)$ is associated with $\{x_n\}$, where $x_n \equiv -j \bmod k^n$ for every n. Pick m such that $j < k^m$. Then $x_n = k^n - j$ for all $n \geq m$ since $k^n - j$ is the unique element of $\{1, 2, \ldots, k^n\}$ congruent to $-j \bmod k^n$. Note that $k^{n+1} - j = k^n - j + (k-1)k^n$. Since $j \geq 1$, the conclusion holds by R19.1.16.

Lemma R19.1.18 Let $\mathcal{F}, \mathcal{G}$ be in $\mathbf{R}_k$ with $\mathcal{F}$ associated with $\{x_n\}$ and $\mathcal{G}$ associated with $\{y_n\}$. Assume there is m in $\mathbb{N}$ such that $x_n = y_n$ for $n < m$, $y_m = x_m + 1$ if $m = 1$, there is $t \in \{0, 1, \ldots, k-2\}$ such that $x_m = x_{m-1} + tk^{m-1}$ and $y_m = y_{m-1} + (t+1)k^{m-1}$ if $m > 1$, and for $n \geq m$ $x_{n+1} = x_n + (k-1)k^n$ and $y_{n+1} = y_n$. Then $\mathcal{F} = f_k(x_m - k^m)$ and $\mathcal{G} = f_k(y_m)$. Moreover $k^m - x_m + y_m = k^m + k^{m-1}$.

Proof: For $n \geq m$, $y_n = y_m$ so that $y_n \equiv y_m \bmod k^n$. If $n < m$, $y_n \equiv y_m \bmod k^n$ by R10.2.5ii. Thus $\mathcal{G} = f_k(y_m)$. Next note that $t \leq k-2$ implies $x_m < k^m$ so that $k^m - x_m \geq 1$. For $n \geq m$ write n as $m+i$ where $i \geq 0$. $k^n - x_n = k^m - x_m$ will be shown by induction on i. This is clear for $i = 0$ and so assume it is true for some value of i. By hypothesis $x_{m+i+1} = x_{m+i} + (k-1)k^{m+i}$ so that $k^{m+i+1} - x_{m+i+1} = k^{m+i} - x_{m+i} = k^m - x_m$ as required. Thus $x_n \equiv x_m - k^m \bmod k^n$ for all $n \geq m$. For $n < m$ again by R10.2.5ii $x_m - x_n = sk^n$ for some integer s so that $x_m - k^m = x_n - k^m + sk^n$, i.e., $x_n \equiv x_m - k^m \bmod k^n$. Thus $\mathcal{F} = f_k(x_m - k^m)$. For the final assertion, if $m = 1$, $y_1 = x_1 + 1$ so that $k - x_1 + y_1 = k + 1$. If $m > 1$, then $y_m = y_{m-1} + (t+1)k^{m-1} = x_{m-1} + tk^{m-1} + k^{m-1} = x_m + k^{m-1}$ so that $k^m - x_m + y_m = k^m + k^{m-1}$.

Corollary R19.1.19 Let $\mathcal{F}$ and $\mathcal{G}$ be a consecutive pair in $\mathbf{R}_k$ with $\mathcal{F}$ smaller. Then there are j, l in $\mathbb{N}$ with $l \geq 2$ such that $\mathcal{F} = f_k(-j)$ and $\mathcal{G} = f_k(l)$.

Proof: Let $\mathcal{F}$ be associated with $\{x_n\}$ and $\mathcal{G}$ with $\{y_n\}$. By R19.1.10 there m with the properties assumed in R19.1.18. Let $l = y_m$ and $j = k^m - x_m$. By R19.1.18 $\mathcal{F} = f_k(-j)$ and $\mathcal{G} = f_k(l)$, and, as noted in the proof, $j \geq 1$. Since $x_m \geq 1$ and $y_m > x_m$, $l \geq 2$.

Proposition R19.1.20 Let j, l in $\mathbb{N}$ with $l \geq 2$. Assume $j + l = k^m + k^{m-1}$ for some $m \in \mathbb{N}$ with $j < k^m$ and $l \leq k^m$. Then $f_k(-j)$ and $f_k(l)$ are consecutive with $f_k(-j)$ smaller.

Proof: $f_k(-j)$ is associated with $\{x_n\}$, where $x_n \equiv -j \bmod k^n$ for every n, and $f_k(l)$ is associated with $\{y_n\}$, where $y_n \equiv l \bmod k^n$ for every n. Since $j < k^m$, $x_n = k^n - j$ for all $n \geq m$, and, since $l \leq k^m$, $y_n = l$ for all $n \geq m$. Note that, for $n \geq m$, $x_{n+1} = x_n + (k-1)k^n$

and $y_{n+1} = y_n$. By hypothesis $k^m - j = l - k^{m-1}$ so that $x_m \equiv y_m \bmod k^n$ if $n < m$. For $n < m$, by R10.2.5ii $x_n \equiv x_m \bmod k^n$ and $y_n \equiv y_m \bmod k^n$. Since $x_n, y_n \in \{1, 2, \ldots, k^n\}$, $x_n = y_n$. If $m = 1$, $y_1 = l = k - j + 1 = x_1 + 1$. If $m > 1$, by R10.2.5i $x_m = x_{m-1} + tk^{m-1}$ and $y_m = y_{m-1} + sk^{m-1}$ for some $t, s \in \{0, 1, \ldots, k-1\}$. Since $x_{m-1} = y_{m-1}$, $k^m - j = y_{m-1} + tk^{m-1}$, i.e., $l - k^{m-1} = y_{m-1} + tk^{m-1}$, i.e., $y_m = y_{m-1} + (t+1)k^{m-1}$ so that $s = t + 1$. Since the four requirements of R19.1.10 are satisfied, $f_k(-j)$ and $f_k(l)$ are consecutive with $f_k(-j)$ smaller.

Corollary R19.1.21 Let $l \in \mathbb{N}$ with $l \geq 2$. Let m be the smallest natural number such that $l \leq k^m$. Let $j = k^m + k^{m-1} - l$. Then $f_k(-j)$ and $f_k(l)$ are consecutive with $f_k(-j)$ smaller.

Proof: Since $l \neq 1$, $l > k^{m-1}$ so that $j < k^m$. The conclusion now follows from R19.1.20.

Corollary R19.1.22 Let $j \in \mathbb{N}$. Let m be the natural number with $k^{m-1} \leq j < k^m$. Let $l = k^m + k^{m-1} - j$. Then $f_k(-j)$ and $f_k(l)$ are consecutive with $f_k(-j)$ smaller.

Proof: $k^{m-1} \leq j$ implies $l \leq k^m$. $j < k^m$ implies $l \geq 2$. The conclusion now follows from R19.1.20.

Proposition R19.1.23 Let $\mathcal{F}$ in $\mathbf{R}_k$ be associated with $\{x_n\}$. Let $m \in \mathbb{N}$, let $f_k(x_m)$ be associated with $\{y_n\}$, and let $f_k(x_m - k^m)$ be associated with $\{z_n\}$. Then, for $i \leq m$, $y_i = x_i = z_i$.

Proof: As usual $y_n \equiv x_m \bmod k^n$ and $z_n \equiv x_m - k^m \bmod k^n$ for all n. Given $i \leq m$, by R10.2.5ii $x_i \equiv x_m \bmod k^i$. Thus y_i, x_i, and z_i are all congruent mod k^i. Since all three must be in $\{1, 2, \ldots, k^i\}$, $y_i = x_i = z_i$.

Proposition R19.1.24 Let $\mathcal{F}$ in $\mathbf{R}_k$ be associated with $\{x_n\}$, and assume $\mathcal{F} \neq f_k(1)$. If $\mathcal{F}$ is not the larger of a consecutive pair, then for every $m \in \mathbb{N}$ there is $\mathcal{G}$ in $\mathbf{R}_k$ associated with $\{y_n\}$ such that $\mathcal{G} <_k \mathcal{F}$ and $y_n = x_n$ for $n \leq m$.

Proof: Fix $m \in \mathbb{N}$. By R19.1.14, x_n is not eventually constant and so there is $j > m$ such that $x_j \neq x_{j+1}$. By R10.2.5 $x_{j+1} = x_j + tk^j$ and, since $x_{j+1} \neq x_j$, $t \in \{1, 2, \ldots, k-1\}$. Let $y_n = x_n$ for all $n \leq j$, $y_{j+1} = y_j + (t-1)k^j$, and $y_{n+1} = y_n$ for $n \geq j+1$. By R10.2.6 there is $\mathcal{G}$ in $\mathbf{R}_k$ with $\mathcal{G}$ associated with $\{y_n\}$. Since $y_{j+1} r_{j+1} x_{j+1}$ by definition R19.1.1 and $y_{j+1} \neq x_{j+1}$, $\mathcal{G} <_k \mathcal{F}$. The second requirement holds by the definition of $\{y_n\}$.

Proposition R19.1.25 Let $\mathcal{F}$ in $\mathbf{R}_k$ be associated with $\{x_n\}$, and assume $\mathcal{F} \neq f_k(0)$. If $\mathcal{F}$ is not the smaller of a consecutive pair, then for every $m \in \mathbb{N}$ there is $\mathcal{G}$ in $\mathbf{R}_k$ associated with $\{y_n\}$ such that $\mathcal{F} <_k \mathcal{G}$ and $y_n = x_n$ for $n \leq m$.

Proof: By R10.2.5 $x_{n+1} = x_n + t_n k^n$ where $t_n \in \{0, 1, \ldots, k-1\}$ for all n. Now fix $m \in \mathbb{N}$. By R19.1.16 and the hypothesis, t_n is not eventually $k-1$ and so there is $j > m$ such that $t_j \leq k-2$. Let $y_n = x_n$ for all $n \leq j$, $y_{j+1} = y_j + (t+1)k^j$, and $y_{n+1} = y_n$ for $n \geq j+1$. By R10.2.6 there is $\mathcal{G}$ in $\mathbf{R}_k$ with $\mathcal{G}$ associated with $\{y_n\}$. Since $x_{j+1} r_{j+1} y_{j+1}$ by definition R19.1.1 and $y_{j+1} \neq x_{j+1}$, $\mathcal{F} <_k \mathcal{G}$. The second requirement holds by the definition of $\{y_n\}$.

Ordering of $\mathbf{R}_\infty$

A specific ordering for $\mathbf{R}_\infty$ as guaranteed by Lynn [1] has not been identified, but this subsection contains a few negative results. R17.3.16, which states that every element of $\Pi\{\mathbf{R}_p : p \text{ is prime}\}$ determines a unique element of $\mathbf{R}_\infty$, suggests that such an ordering will not be related in a straightforward way to the orders $\leq_k$.

Recall the map $\rho_i : \mathbf{R}_\infty \to \mathbf{R}_i$ defined in R16.17 by $\rho_i(\mathcal{F}) = \mathcal{F} \cap \mathcal{Z}_i$. In R16.18 and R17.3.1 each ρ_i was shown to be a continuous ring homomorphism. The map $f_\infty : \mathbf{Z} \to \mathbf{R}_\infty$ is the embedding map defined in R16.16. By R16.19 $\rho_k \circ f_\infty = f_k$ for all $k \geq 2$.

Definition R19.2.1 Let $\leq_X$ and $\leq_Y$ be linear orders on sets X and Y respectively, and let $f : X \to Y$. f is order-preserving provided $a \leq_X b \Rightarrow f(a) \leq_Y f(b)$ and order-reversing provided $a \leq_X b \Rightarrow f(a) \geq_Y f(b)$. f is an order map provided f is either order-preserving or order-reversing.

Proposition R19.2.2 Let $\preceq$ be a linear order on $\mathbf{R}_\infty$. Then there are infinitely many i such that ρ_i is not an order map.

Proof: Let $k \in \mathbb{N}$ with $k \geq 4$. Let $\mathcal{F} = f_\infty(k-1)$, $\mathcal{G} = f_\infty(k)$, and $\mathcal{H} = f_\infty(k+1)$. Since f_∞ is one-to-one, $\mathcal{F}, \mathcal{G}$, and $\mathcal{H}$ are three distinct elements of $\mathbf{R}_\infty$. By R16.19 $\rho_i(\mathcal{F}) = f_i(k-1)$, $\rho_i(\mathcal{G}) = f_i(k)$, and $\rho_i(\mathcal{H}) = f_i(k+1)$ for all $i \geq 2$. For $k-1 \leq i \leq k+1$, the i-level order is determined is determined by the appropriate r_1 as follows:

$$f_{k-1}(k) <_{k-1} f_{k-1}(k+1) <_{k-1} f_{k-1}(k-1) \tag{1}$$

$$f_k(k+1) <_k f_k(k-1) <_k f_k(k) \tag{2}$$

$$f_{k+1}(k-1) <_{k+1} f_{k+1}(k) <_{k+1} f_{k+1}(k+1) \tag{3}$$

If $\mathcal{F}$ is the smallest of the three relative to $\preceq$, use (2): $\rho_k(\mathcal{H}) <_k \rho_k(\mathcal{F})$ shows that ρ_k is not order-preserving and $\rho_k(\mathcal{F}) <_k \rho_k(\mathcal{G})$ shows that ρ_k is not order-reversing. Similarly, if $\mathcal{G}$ is the smallest, then (3) shows that ρ_{k+1} is not an order map and, if $\mathcal{H}$ is the smallest, then (1) shows that ρ_{k-1} is not an order map.

The isomorphic representation of $\mathbf{R}_\infty$ in R17.3.13 might suggest a dictionary order imposed by the orders in $\mathbf{R}_p$ and the natural order of the primes in $\mathbb{N}$, although in general the product topology of ordered spaces is not the same as the dictionary order topology. In the rest of this subsection, this approach is examined, with some attention to connections to the topology of $\mathbf{R}_\infty$. We start with the following lemma, which is at least implicit in [9].

Lemma R19.2.3 For $\mathcal{F}, \mathcal{G}$ in $\mathbf{R}_\infty$, $\mathcal{F} = \mathcal{G}$ if and only if $\mathcal{F} \cap \mathcal{Z}_p = \mathcal{G} \cap \mathcal{Z}_p$ for every prime p.

Note: This almost duplicates R17.3.15, although the proof given here is different.

Proof: Assume $\mathcal{F} \cap \mathcal{Z}_p = \mathcal{G} \cap \mathcal{Z}_p$ for every prime p. By R17.3.7 $\mathcal{F} \cap \mathcal{Z}_k = \mathcal{G} \cap \mathcal{Z}_k$ for every $k \geq 2$. Since $\mathcal{Z}_\infty$, the normal basis generating $\mathbb{N}_\infty$, is $\cup\{\mathcal{Z}_k : k \geq 2\}$, $\mathcal{F} = \mathcal{G}$.

Definition R19.2.4 Let $\mathcal{F}, \mathcal{G}$ be in $\mathbf{R}_\infty$. $\mathcal{F} \leq_\infty \mathcal{G}$ if and only if $\mathcal{F} = \mathcal{G}$ or $\mathcal{F} \neq \mathcal{G}$ and $\mathcal{F} \cap \mathcal{Z}_q \leq_q \mathcal{G} \cap \mathcal{Z}_q$, where q is the smallest element of $\{p : p \text{ is prime and } \mathcal{F} \cap \mathcal{Z}_p \neq \mathcal{G} \cap \mathcal{Z}_p\}$.

Proposition R19.2.5 $\leq_\infty$ is a linear order on $\mathbf{R}_\infty$.

Proof: Reflexivity is clear from the definition, and antisymmetry is immediate because each $\leq_p$ is antisymmetric. Likewise, any two filters in $\mathbf{R}_\infty$ must be related somehow because each $\leq_p$ is linear. For transitivity, let $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$ be in $\mathbf{R}_\infty$ with $\mathcal{F} \leq_\infty \mathcal{G}$ and $\mathcal{G} \leq_\infty \mathcal{H}$. If the filters are not all distinct, one easiy sees that $\mathcal{F} \leq_\infty \mathcal{H}$. Thus assume three distinct filters and let q be the smallest element of $\{p : \mathcal{F} \cap \mathcal{Z}_p \neq \mathcal{G} \cap \mathcal{Z}_p\}$, r the smallest element of $\{p : \mathcal{G} \cap \mathcal{Z}_p \neq \mathcal{H} \cap \mathcal{Z}_p\}$, and s is the smallest element of $\{p : \mathcal{F} \cap \mathcal{Z}_p \neq \mathcal{H} \cap \mathcal{Z}_p\}$. If $r < q$, then $s = r$ and $\mathcal{F} \cap \mathcal{Z}_s = \mathcal{G} \cap \mathcal{Z}_s \leq_s \mathcal{H} \cap \mathcal{Z}_s$ and so $\mathcal{F} \leq_\infty \mathcal{H}$ by definition. If $r = q$, then $s = r = q$ and $\mathcal{F} \leq_\infty \mathcal{H}$ follows from the transitivity of $\leq_s$. If $r > q$, then $s = q$ and $\mathcal{F} \cap \mathcal{Z}_s \leq_s \mathcal{G} \cap \mathcal{Z}_s = \mathcal{H} \cap \mathcal{Z}_s$ and so $\mathcal{F} \leq_\infty \mathcal{H}$ by definition.

The next proposition is similar to R19.1.8.

Lemma R19.2.6 In $\mathbf{R}_\infty$, relative to $\leq_\infty$, $f_\infty(1)$ is the smallest element and $f_\infty(0)$ is the largest element.

Proof: Let $\mathcal{F}$ be in $\mathbf{R}_\infty$. For every prime p, $f_\infty(1)\cap\mathcal{Z}_p = f_p(1)$ and $f_\infty(0)\cap\mathcal{Z}_p = f_p(0)$. If $\mathcal{F} \neq f_\infty(1)$, let q be the smallest in $\{p \text{ prime} : f_\infty(1) \cap \mathcal{Z}_p \neq \mathcal{F} \cap \mathcal{Z}_p\}$. By R19.1.8 $f_q(1) <_q \mathcal{F} \cap \mathcal{Z}_q$. By definition $f_\infty(1) \leq_\infty \mathcal{F}$. Similarly $\mathcal{F} \leq_\infty f_\infty(0)$.

Proposition R19.2.7 Let $\mathcal{F}, \mathcal{G}$ be in $\mathbf{R}_\infty$. $\mathcal{F}$ and $\mathcal{G}$ are consecutive with $\mathcal{F}$ smaller if and only if there is a prime q such that $\mathcal{F}\cap\mathcal{Z}_p = \mathcal{G}\cap\mathcal{Z}_p$ for every prime $p < q$, $\mathcal{F}\cap\mathcal{Z}_q$ and $\mathcal{G}\cap\mathcal{Z}_q$ are consecutive in $\mathbf{R}_q$ with $\mathcal{F}\cap\mathcal{Z}_p$ smaller, and for every prime $p > q$ $\mathcal{F}\cap\mathcal{Z}_p = f_p(0)$ and $\mathcal{G} \cap \mathcal{Z}_p = f_p(1)$.

Proof: First, assume the three condions hold. Clearly q is the smallest of the set $\{p : p \text{ is prime and } \mathcal{F}\cap\mathcal{Z}_p \neq \mathcal{G}\cap\mathcal{Z}_p\}$. Since $\mathcal{F}\cap\mathcal{Z}_q$ and $\mathcal{G}\cap\mathcal{Z}_q$ are consecutive with $\mathcal{F}\cap\mathcal{Z}_p$ smaller in $\mathbf{R}_q$, $\mathcal{F} <_\infty \mathcal{G}$. Suppose $\mathcal{I}$ is in $\mathbf{R}_\infty$ with $\mathcal{F} <_\infty \mathcal{I} <_\infty \mathcal{G}$. Let r be the smallest prime in $\{p : p \text{ is prime and } \mathcal{F}\cap\mathcal{Z}_p \neq \mathcal{I}\cap\mathcal{Z}_p\}$. By hypothesis and R19.1.8 $r \leq q$. r cannot be smaller than q since, otherwise $\mathcal{F} \cap \mathcal{Z}_r = \mathcal{G} \cap \mathcal{Z}_r$ and the assumed ordering would be contradicted. Thus $r = q$ and so, since $\mathcal{F}\cap\mathcal{Z}_q$ and $\mathcal{G}\cap\mathcal{Z}_q$ are consecutive, $\mathcal{I}\cap\mathcal{Z}_q \geq_q \mathcal{G}\cap\mathcal{Z}_q$. Since $f_p(1) \leq_p \mathcal{I}\cap\mathcal{Z}_p$ for all $p > q$, $\mathcal{I} <_\infty \mathcal{G}$ leads to a contradiction. Thus no such $\mathcal{I}$ exists, as required for the conclusion. Conversely, assume $\mathcal{F}$ and $\mathcal{G}$ are consecutive with $\mathcal{F}$ smaller. Let q be the smallest of the set $\{p : p \text{ is prime and } \mathcal{F}\cap\mathcal{Z}_p \neq \mathcal{G}\cap\mathcal{Z}_p\}$. The first requirement clearly holds and $\mathcal{F}\cap\mathcal{Z}_q <_q \mathcal{G}\cap\mathcal{Z}_q$. Suppose $\mathcal{I}_q$ is in $\mathbf{R}_q$ with $\mathcal{F}\cap\mathcal{Z}_q <_q \mathcal{I}_q <_q \mathcal{G}\cap\mathcal{Z}_q$. Let $\mathcal{I}_p = \mathcal{F}\cap\mathcal{Z}_p$ for each prime $p \neq q$. By R17.3.16 there is $\mathcal{I}$ in $\mathbf{R}_\infty$ such that $\mathcal{I}\cap\mathcal{Z}_p = \mathcal{I}_p$ for each prime p. By definition $\mathcal{F} <_\infty \mathcal{I} <_\infty \mathcal{G}$, contradicting the assumption. Thus $\mathcal{F} \cap \mathcal{Z}_q$ and $\mathcal{G} \cap \mathcal{Z}_q$ are consecutive in $\mathbf{R}_q$ with $\mathcal{F} \cap \mathcal{Z}_p$ smaller. For the third condition, let r be a prime greater than q and assume $\mathcal{F} \cap \mathcal{Z}_r \neq f_r(0)$. Let $\mathcal{I}_p = \mathcal{F} \cap \mathcal{Z}_p$ for each prime $p \neq r$ and $\mathcal{I}_r = f_r(0)$. By R17.3.16 there is $\mathcal{I}$ in $\mathbf{R}_\infty$ such that $\mathcal{I} \cap \mathcal{Z}_p = \mathcal{I}_p$ for each prime p. By definition $\mathcal{F} <_\infty \mathcal{I}$ and, since $\mathcal{I}_q <_q \mathcal{G} \cap \mathcal{Z}_q$, $\mathcal{I} <_\infty \mathcal{G}$, a contradiction. For the case $\mathcal{G} \cap \mathcal{Z}_r \neq f_r(1)$, proceed similarly with $\mathcal{I}_r = f_r(1)$ and $\mathcal{I}_p = \mathcal{G} \cap \mathcal{Z}_p$ for each prime $p \neq r$.

The rest of this subsection examines the relationship between the topology on $\mathbf{R}_\infty$, which will be denoted τ, and $\tau(\leq_\infty)$, the order topology generated by $\leq_\infty$.

Recall that $\mathbf{R}_\infty$ is a subspace of $\mathbb{N}_\infty$, the compactification generated by $\mathcal{Z}_\infty = \cup\{\mathcal{Z}_k : k \in \mathbb{N}\}$. It will again be necessary to make temporary use of the following cumbersome notation: For $W \in \mathcal{Z}_k \subseteq \mathcal{Z}_\infty$, ${}^\infty W^\omega = \{\mathcal{F} \in \omega(\mathcal{Z}_\infty) : W \in \mathcal{F}\}$ and ${}^k W^\omega = \{\mathcal{F} \in \omega(\mathcal{Z}_k) : W \in \mathcal{F}\}$. As in R17.3.6, for $k \in \mathbb{N}$ with $k \geq 2$, let $T(k) = \{p : p \text{ is prime and } p|k\}$

Lemma R19.2.8 $\{{}^\infty Z^\omega : Z \in \mathcal{Z}_\infty\}$ is a clopen basis for $\mathbb{N}_\infty$.

Proof: As noted in P3.6, this set is a closed basis and so the collection of complements is an open basis. By R9.1.7 and definition R10.1.6, $\mathcal{Z}_\infty$ is closed under complementation. It is routine to check that the complement of the closed ${}^\infty(\mathbb{N} - Z)^\omega$ is ${}^\infty Z^\omega$. The conclusion follows easily.

Lemma R19.2.9 Let $\mathcal{F}$ be in $\mathbf{R}_\infty$, and let $Z = C_1^4(6)$. Let $\mathcal{F} \cap \mathcal{Z}_2$ correspond to $\{x_n\}$ and $\mathcal{F} \cap \mathcal{Z}_3$ correspond to $\{y_n\}$. Then $Z \in \mathcal{F}$ if and only if $x_1 = 2$ and $y_1 = 1$.

Proof: Let $\mathcal{F} \cap \mathcal{Z}_6$ correspond to $\{a_n\}$. By R12.5.17i for $i = 2, 3$ $\mathcal{Z}_i \subseteq \mathcal{Z}_6$ and so $(\mathcal{F} \cap \mathcal{Z}_6) \cap \mathcal{Z}_i = \mathcal{F} \cap \mathcal{Z}_i$. Thus $x_1 \equiv a_1 \mod 2$ and $y_1 \equiv a_1 \mod 3$ by R17.2.3. As an element of $\mathcal{Z}(E_1(6))$, Z is associated with $\{1, 2, 3, 5, 6\}$. With those preliminaries, assume $Z \in \mathcal{F}$. By definition of a_1, $a_1 = 4$. Since $x_1 \in \{1, 2\}$ and $y_1 \in \{1, 2, 3\}$, $x_1 = 2$ and $y_1 = 1$

by the congruences. Conversely assume $x_1 = 2$ and $y_1 = 1$. Since $a_1 \in \{1,2,3,4,5,6\}$, $y_1 = 1$ and the second congruence imply $a_1 \in \{1,4\}$. The first congruence now yields $a_1 = 4$. By definition of a_1, $Z \in \mathcal{F}$.

Proposition R19.2.10 τ is not a subset of $\tau(\leq_\infty)$.

Proof: Let $Z = C_1^4(6)$ and let $\mathcal{F}_p = f_p(1)$ for each prime $p \geq 3$. Let $x_1 = 2$, $x_{n+1} = x_n + 2^n$ for n odd, and $x_{n+1} = x_n$ for n even. By R10.2.6 there is $\mathcal{F}_2$ in $\mathbf{R}_2$ such that $\mathcal{F}_2$ corresponds to $\{x_n\}$. Note that by R19.1.10 $\mathcal{F}_2$ is not part of a consecutive pair in $\mathbf{R}_2$. By R17.3.16 there is $\mathcal{F}$ in $\mathbf{R}_\infty$ such that $\mathcal{F} \cap \mathcal{Z}_p = \mathcal{F}_p$ for each prime p and by R19.2.7 $\mathcal{F}$ is not part of a consecutive pair in $\mathbf{R}_\infty$. By R19.2.9 $\mathcal{F}$ is in ${}^\infty Z^\omega$. Suppose that $\mathcal{F}$ is in the $\tau(\infty)$-interior of ${}^\infty Z^\omega$. Then, since $\mathcal{F}$ is not the larger of a consecutive pair, there is $\mathcal{G}$ in $\mathbf{R}_\infty$ such that $\mathcal{G} <_\infty \mathcal{F}$ and, whenever $\mathcal{G} <_\infty \mathcal{I} <_\infty \mathcal{F}$, $\mathcal{I} \in {}^\infty Z^\omega$. Since $\mathcal{G} <_\infty \mathcal{F}$ and $\mathcal{G} \cap \mathcal{Z}_p \geq_p f_p(1) = \mathcal{F} \cap \mathcal{Z}_p$ for each prime $p \geq 3$, we must have $\mathcal{G} \cap \mathcal{Z}_2 <_2 \mathcal{F} \cap \mathcal{Z}_2$. By R19.1.13 there is $\mathcal{I}_2$ in $\mathbf{R}_2$ such that $\mathcal{G} \cap \mathcal{Z}_2 <_2 \mathcal{I}_2 <_2 \mathcal{F} \cap \mathcal{Z}_2$. Let $\mathcal{I}_p = f_p(0)$ for each prime $p \geq 3$. Again by R17.3.16 there is $\mathcal{I}$ in $\mathbf{R}_\infty$ such that $\mathcal{I} \cap \mathcal{Z}_p = \mathcal{I}_p$ for each prime p. Clearly $\mathcal{G} <_\infty \mathcal{I} <_\infty \mathcal{F}$ but by R19.2.9 $Z \notin \mathcal{I}$, a contradiction. Thus ${}^\infty Z^\omega$ is not $\tau(\infty)$-open and the conclusion follows from R19.2.8.

Corollary R19.2.11 $\tau(\leq_\infty)$ is not a subset of τ.

Proof: Deny the conclusion. Since τ is the topology of a compact T_2 space, it is minimal T_2. But every order topology, in particular $\tau(\leq_\infty)$, is T_2. Thus $\tau(\leq_\infty) = \tau$, which contradicts R19.2.10.

Lastly we have a positive result relating τ and $\tau(\leq_\infty)$. Note that in the following $q = r$ is possible but not assumed.

Proposition R19.2.12 Let $Z \in \mathcal{Z}_k$ for $k \geq 2$ and let $\mathcal{F}$ be in $\mathbf{R}_\infty$ with $Z \in \mathcal{F}$. Assume there exist primes q, r greater than k such that $\mathcal{F} \cap \mathcal{Z}_q \neq f_q(1)$ and $\mathcal{F} \cap \mathcal{Z}_r \neq f_r(0)$. Then $\mathcal{F}$ is in the $\tau(\leq_\infty)$-interior of ${}^\infty Z^\omega$.

Proof: Let $\mathcal{G}_q = f_q(1)$ and $\mathcal{H}_r = f_r(0)$. Let $\mathcal{G}_p = \mathcal{F} \cap \mathcal{Z}_p$ for $p \neq q$ and $\mathcal{H}_p = \mathcal{F} \cap \mathcal{Z}_p$ for $p \neq r$. By R17.3.16 there are $\mathcal{G}, \mathcal{H}$ in $\mathbf{R}_\infty$ such that $\mathcal{G} \cap \mathcal{Z}_p = \mathcal{G}_p$ and $\mathcal{H} \cap \mathcal{Z}_p = \mathcal{H}_p$ for each prime p. By R19.1.8 and the choice of $\mathcal{G}_q$ and $\mathcal{H}_r$, it is clear that $\mathcal{G} <_\infty \mathcal{F} <_\infty \mathcal{H}$. Let $\mathcal{I}$ be in $\mathbf{R}_\infty$ such that $\mathcal{G} <_\infty \mathcal{I} <_\infty \mathcal{H}$. For any prime $p \leq k$, $\mathcal{I} \cap \mathcal{Z}_p \neq \mathcal{G} \cap \mathcal{Z}_p$ would lead to a contradiction of $\mathcal{I} <_\infty \mathcal{H}$ since $\mathcal{G}_p = \mathcal{H}_p$. Thus, for any prime p in $T(k)$, $\mathcal{I} \cap \mathcal{Z}_p = \mathcal{F} \cap \mathcal{Z}_p$ and so by R17.3.7 $\mathcal{I} \cap \mathcal{Z}_k = \mathcal{F} \cap \mathcal{Z}_k$. Then $Z \in \mathcal{I}$. To summarize, $\mathcal{F} \in (\mathcal{G}, \mathcal{H}) \subseteq {}^\infty Z^\omega$, which yields the conclusion.

Albert J. Klein 2009
http://www.susanjkleinart.com/compactification/

References

An asterisk indicates a reference not seen by me.

1* Lynn, I.L., Linearly Orderable Spaces, Proc. Amer. Math. Soc. 13(1962), 454-456.

2. Purisch, S., A History of Results on Orderability and Suborderability, in: A Handbook of the History of General Topology, vol. 2, C.E. Aull and R. Lowen, editors, Kulwer Academic Publishers, 1998.

3. This Website, P3: Normal Bases

4. This Website, R5: Finite-Point Compactifications

5. This Website, R9: Directed Sets of Normal Bases
6. This Website, R10: Some Metric Compactifications of $\mathbb{N}$
7. This Website, R12: Extension of Arithmetic Operations
8. This Website, R16: The Remnant Rings as Compactifications
9. This Website, R17: Algebraic Structure of the Remnant Rings

Added 2019

This note contains two propositiosns related to the orderings on the remnant rings. The first provides a simpler way of determining how two elements in $\mathbf{R}_k$ are related. In effect, the order of two elements of $\mathbf{R}_k$ is determined by the dictionary order of their associated sequences. The second is a minor but interesting fact about $<_k$ and $<_{k^n}$.

Lemma R19.Add.1 Let $k \in \mathbb{N}$ with $k \geq 2$. Let $\mathcal{F} \neq \mathcal{G}$ be in $\mathbf{R}_k$. Assume $\mathcal{F}$ is associated with the sequence $\{x_n\}$ and $\mathcal{G}$ is associated with $\{y_n\}$. Then $\{n \in \mathbb{N} : x_n \neq y_n\}$ is non-empty.

Proof: This is immediate from R10.2.4.

Proposition R19.Add.2 Let $k \in \mathbb{N}$ with $k \geq 2$. Let $\mathcal{F} \neq \mathcal{G}$ be in $\mathbf{R}_k$. Assume $\mathcal{F}$ is associated with the sequence $\{x_n\}$ and $\mathcal{G}$ is associated with $\{y_n\}$. Let M be the smallest element of $\{n \in \mathbb{N} : x_n \neq y_n\}$. Then $\mathcal{F} <_k \mathcal{G}$ if and only if $x_M < y_M$, where $<$ is the usual order for the natural numbers.

Proof: First suppose $x_M < y_M$. As an initial case, assume $M = 1$. Since $x_1 < y_1$, $x_1 r_1 y_1$ so that $x_n r_n y_n$ for all n by R19.1.3. By definition $\mathcal{F} <_k \mathcal{G}$. Now assume $M > 1$. As usual, $x_M = x_{M-1} + c(x_{M-1})k^{M-1}$ and $y_M = y_{M-1} + c(y_{M-1})k^{M-1}$. Since $x_M < y_M$ and $x_{M-1} = y_{M-1}$, it follows that $c(x_{M-1}) < c(y_{M-1})$ so that $x_M r_M y_M$ by definition. Using R19.1.3 again, $x_n r_n y_n$ for all n and so by definition $\mathcal{F} <_k \mathcal{G}$. Conversely, suppose $\mathcal{F} <_k \mathcal{G}$. If $y_M < x_M$, the first half would show that $\mathcal{G} <_k \mathcal{F}$, a contradiction since $<_k$ is antisymmetric. Thus $x_M < y_M$.

Note that the proposition does not say that the natural ordering at stage M persists for all larger subscripts. In $\mathbf{R}_2$ for example, $f_2(2)$ is associated with the constant sequence $\{2, 2, 2, \dots\}$, while $f_2(3)$ is associated with the sequence $\{1, 3, 3, \dots\}$.

The question of whether the presentation in this section could be simplified by using R19.Add.2 as the definition of $<_k$ is one that I have not examined.

Before presenting the second proposition, some preliminary comments are needed. In R17.2.12 it was noted that, for $k, n \in \mathbb{N}$ with $k \geq 2$, $\mathcal{Z}_{k^n} = \mathcal{Z}_k$ and so the corresponding Wallman compactifications are identical, i.e., $\mathbb{N}_{k^n} = \mathbb{N}_k$. For emphasis, this means that the underlying sets and topologies as well as the embeddings are identical. Thus $\mathbf{R}_{k^n}$ and $\mathbf{R}_k$ have the same underlying set and topology. In addition, we have the following:

Lemma R19.Add.3 Let $k, n \in \mathbb{N}$ with $k \geq 2$. Then $\mathbf{R}_k$ and $\mathbf{R}_{k^n}$ are identical as rings.

Proof: Since the underlying sets are the same, it is sufficient to show that the binary operations are equal, which was done in R17.2.13.

Recall that the additive identity in $\mathbf{R}_k$ is denoted $\mathcal{O}_k$.

Corollary R19.Add.4 Let $k, n \in \mathbb{N}$ with $k \geq 2$. Then $\mathcal{O}_{k^n} = \mathcal{O}_k$.

Proof: Additive identities are unique.

Notice that, despite all these identities, the associated sequence of an ultrafilter may vary depending on whether it is considered an element of $\mathbf{R}_k$ or $\mathbf{R}_{k^n}$. For instance, $\mathcal{O}_k$ is associated with $\{k, k^2, k^3, \dots\}$ while $\mathcal{O}_{k^n}$ is associated with $\{k^n, k^{2n}, k^{3n}, \dots\}$. It appears that the associated sequence of an element of $\mathbf{R}_{k^n}$ is a subsequence of its sequence as an element of $\mathbf{R}_k$.

Corollary R19.Add.5 Let $k, n \in \mathbb{N}$ with $k \geq 2$. Then $\mathbf{R}_k$ and $\mathbf{R}_{k^n}$ are identical as compactifications.

Proof: Immediate from the comments above and R17.2.14, which shows that the embeddings, f_k and f_{k^n}, are identical.

Proposition R19.Add.6 Let $k, n \in \mathbb{N}$ with $k, n \geq 2$. Then $<_k$ and $<_{k^n}$ are different orders and the identity map from $(\mathbf{R}_k, <_k)$ to $(\mathbf{R}_{k^n}, <_{k^n})$ is neither order-preserving nor order-reversing.

Proof: Let $\{a_n\}$ be associated with $f_k(k)$ and $\{b_n\}$ with $f_k(k+1)$ in $\mathbf{R}_k$. Let $\{x_n\}$ be associated with $f_{k^n}(k)$ and $\{y_n\}$ with $f_{k^n}(k+1)$ in $\mathbf{R}_{k^n}$. Since $a_1 = k$ and $b_1 = 1$, $f_k(k+1) <_k f_k(k)$. The hypothesis implies $k+1 < k^2 \leq k^n$. As a result, $x_1 = k$ and $y_1 = k+1$ so that $f_{k^n}(k) <_{k^n} f_{k^n}(k+1)$. Since $f_k = f_{k^n}$, this shows that $<_k$ and $<_{k^n}$ are different orders and that the identity map is not order-preserving. Next note that the constant sequence $\{1, 1, 1, \dots\}$ is associated with $f_k(1)$ in $\mathbf{R}_k$ and also with $f_{k^n}(1)$ in $\mathbf{R}_{k_n}$. Thus $f_k(1) <_k f_k(k)$ and $f_{k^n}(1) <_{k^n} f_{k^n}(k)$ and so the identity map is not order-reversing.