Suprema via Quotients of the Stone-Čech Compactification

The existence of suprema of compactifications was shown by Lubben [3] in 1941. In this section that fact is derived, in a relatively straightforward manner, by considering quotients of the Stone-Čech compactification. For an arbitrary space (X, τ) with an equivalence relation E, X/E will denote the set of equivalence classes of E with the identification topology. Other definitions and notation can be found in [4] or in Kelley[2].

Lemma R2.1 Let (X, τ) be an arbitrary topological space and $\{E_\alpha : \alpha \in \Delta\}$ a family of equivalence relations on X. Let $E = \bigcap\{E_\alpha : \alpha \in \Delta\}$. If X/E_α is T_2 for every $\alpha \in \Delta$, then X/E is also T_2.

Proof: The projection maps $\rho_\alpha : X/E \to X/E_\alpha$ given by $\rho_\alpha([x]_E) = [x]_{E_\alpha}$ are continuous for every $\alpha \in \Delta$ since $\pi_\alpha = \rho_\alpha \circ \pi$ is continuous, where π_α and π are the standard projection maps. For $[x]_E \neq [y]_E$ pick any $\alpha \in \Delta$ with $[x]_{E_\alpha} \neq [y]_{E_\alpha}$ and let the latter be separated by disjoint open sets O, G in X/E_α. Then $[x]_E, [y]_E$ are separated in X/E by the disjoint open sets $\rho_\alpha^{-1}[O], \rho_\alpha^{-1}[G]$.

Lemma R2.2 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and $(\beta X, \iota)$ its Stone-Čech compactification. Let (Y, f) be a T_2 compactification of X. Then there is an equivalence relation R on βX and a homeomorphism $h : Y \to \beta X/R$ such that $(\beta X/R, h \circ f)$ is a T_2 compactification equivalent to (Y, f). In addition, $h \circ f = \pi_R \circ \iota$, where π_R is the projection from βX to $\beta X/R$.

Proof: Let $g : \beta X \to Y$ be the unique continuous surjection such that $f = g \circ \iota$. Since Y is compact and T_2, g must be an identification map, and so, letting R be the equivalence relation on βX induced by g, the required homeomorphism is $h : Y \to \beta X/R$ given by $h(y) = g^{-1}\{y\}$. For the second assertion, note that $\pi_R = h \circ g$ so that $\pi_R \circ \iota = h \circ g \circ \iota = h \circ f$.

For the next three results the following framework will be assumed: (X, τ) will be a $T_{3\frac{1}{2}}$ space and $(\beta X, \iota)$ its Stone-Čech compactification. $\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$ will be a non-empty set of T_2 compactifications for X with E_α and h_α the equivalence relation and homeomorphism guaranteed for each α by R2.2. π_α will be the projection from βX onto $\beta X/E_\alpha$. E will denote the equivalence relation $\bigcap\{E_\alpha : \alpha \in \Delta\}$ and π the projection from βX onto $\beta X/E$. Lastly, ρ_α will be the continuous surjection from $\beta X/E$ to $\beta X/E_\alpha$ as in the proof of R2.1.

Proposition R2.3 $(\beta X/E, \pi \circ \iota)$ is a T_2 compactification for X.

Proof: $\beta X/E$ is compact and, by R2.1, T_2 so that it is only necessary to verify that $\pi \circ \iota$ is an embedding with a dense image. Its continuity is clear and, since $\iota[X]$ is dense in βX and π is onto, $\pi \circ \iota[X]$ is dense in $\beta X/E$. Next suppose $\pi \circ \iota(a) = \pi \circ \iota(b)$ and pick any $\alpha \in \Delta$. Since $(\iota(a), \iota(b)) \in E$, the pair is also in E_α so that $\pi_\alpha \circ \iota(a) = \pi_\alpha \circ \iota(b)$. Note that $\pi_\alpha \circ \iota = h_\alpha \circ f_\alpha$, an embedding, so that $a = b$. Thus $\pi \circ \iota$ is one-to-one. To see that the map is a homeomorphism onto its image, let F be closed in X. As noted $\pi_\alpha \circ \iota$ is an embedding for all α so there is H_α closed in $\beta X/E_\alpha$ with $\pi_\alpha \circ \iota[F] = \pi_\alpha \circ \iota[X] \cap H_\alpha$. Also $\pi_\alpha^{-1}[H_\alpha]$ is closed in βX as is $S = \bigcap\{\pi_\alpha^{-1}[H_\alpha] : \alpha \in \Delta\}$. It is easy to check that $\pi \circ \iota[F] = \pi \circ \iota[X] \cap \pi[S]$ and so $\pi \circ \iota$ is an embedding.

Proposition R2.4 $[(\beta X/E, \pi \circ \iota)] \geq [(Y_\alpha, f_\alpha)] \forall \alpha \in \Delta$.

Proof: Clearly $\rho_\alpha \circ (\pi \circ \iota) = \pi_\alpha \circ \iota$ so that ρ_α is the map required to show $[(\beta X/E, \pi \circ \iota)] \geq [(\beta X/E_\alpha, \pi_\alpha \circ \iota)]$. By R2.2 and the equation $\pi_\alpha \circ \iota = h_\alpha \circ f_\alpha$, the latter is $[(Y_\alpha, f_\alpha)]$.

Proposition R2.5 If $[(Z,g)] \geq [(Y_\alpha, f_\alpha)] \forall \alpha \in \Delta$, then $[(Z,g)] \geq [(\beta X/E, \pi \circ \iota)]$.

Proof: For each $\alpha \in \Delta$, let $g_\alpha : Z \to \beta X/E_\alpha$ be the continous surjection such that $g_\alpha \circ g = \pi_\alpha \circ \iota$. In order to define a map $G : Z \to \beta X/E$ by $G(z) = \bigcap\{g_\alpha(z) : \alpha \in \Delta\}$, it is necessary to show that the intersection is an E-equivalence class. To that end, let $z \in Z$ and let $\{x_\gamma\}$ be a net in X such that $\{g(x_\gamma)\}$ converges to z. For every $\alpha \in \Delta$ $\{g_\alpha(g(x_\gamma))\}$, i.e. $\{\pi_\alpha(\iota(x_\gamma))\}$, converges to $g_\alpha(z)$. Since βX is compact, $\{\iota(x_\gamma)\}$ has a convergent subnet, say $\{\iota(x_{\gamma_\eta})\}$ converging to w. Then, for every $\alpha \in \Delta$, $\{\pi_\alpha(\iota(x_{\gamma_\eta}))\}$, i.e. $\{g_\alpha(g(x_{\gamma_\eta}))\}$, converges to $\pi_\alpha(w)$. Since $\beta X/E_\alpha$ is T_2, $\pi_\alpha(w) = g_\alpha(z)$, i.e. $[w]_{E_\alpha} = g_\alpha(z)$ for every $\alpha \in \Delta$. Thus $w \in \bigcap\{g_\alpha(z) : \alpha \in \Delta\}$. It is now easy to check that the intersection must be $[w]_E$ and so G is defined.

To verify that $G \circ g = \pi \circ \iota$, let $x \in X$ and note that

$$\begin{aligned} G(g(x)) &= \bigcap\{g_\alpha(g(x)) : \alpha \in \Delta\} \\ &= \bigcap\{\pi_\alpha(\iota(x)) : \alpha \in \Delta\} \\ &= \bigcap\{[\iota(x)]_{E_\alpha} : \alpha \in \Delta\}. \end{aligned}$$

This last intersection must be $[\iota(x)]_E$, i.e. $\pi(\iota(x))$.

Next, to verify that G is continuous, let $\{z_\gamma\}$ be a net in Z converging to z. Since $\beta X/E$ is compact, it is sufficient to show that every convergent subnet of $\{G(z_\gamma)\}$ must converge to $G(z)$.Thus assume the subnet $\{G(z_{\gamma_\eta})\}$ converges to $[q]_E$. If $q \notin g_\alpha(z)$ for some $\alpha \in \Delta$, there would be disjoint open sets in $\beta X/E_\alpha$, say V_α and W_α, separating $[q]_\alpha$ and $g_\alpha(z)$. Then $\{G(z_{\gamma_\eta})\}$ would eventually be in $\rho_\alpha^{-1}[V_\alpha]$ and so $\{\rho_\alpha(G(z_{\gamma_\eta}))\}$ would eventually be in V_α. Since $\rho_\alpha \circ G = g_\alpha$, it would follow that $\{g_\alpha(z_\gamma)\}$ is frequently not in W_α, which contradicts the continuity of g_α.

Lastly, since $G \circ g = \pi \circ \iota$, $G \circ g[X]$ is dense in $\beta X/E$. Since $G[Z]$ is closed and contains $G \circ g[X]$, G is onto. By definition, the existence of G shows $[(Z,g)] \geq [(\beta X/E, \pi \circ \iota)]$.

Using a fixed representative from the Stone-Čech class, one can cosider the set $S = \{(\beta X/R, \pi_R \circ \iota) : R$ is an equivalence relation on βX and $(\beta X/R, \pi_R \circ \iota)$ is T_2 compactification of $X\}$. As noted in R2.2, this set contains a representative of each compactification class of X. Equivalence restricted to the set S yields an equivalence relation, and so we can let $\mathcal{COM}(X)$ denote the set of equivalence classes from S. (A question: is equivalence really necessary here, i.e., is it possible for two distinct elements of S to be equivalent?)

Theorem R2.6 [Lubben] Let (X, τ) be a $T_{3\frac{1}{2}}$ space. Then $\mathcal{COM}(X)$ admits a supremum operation.

Proof: This summarizes the previous results.

Corollary R2.7 [Lubben] Let (X, τ) be non-compact, locally compact, and T_2. Then $\mathcal{COM}(X)$ is a complete lattice.

Proof: The one-point compactification pair is known to determine the smallest element of $\mathcal{COM}(X)$. Infima can be defined as suprema of appropriate sets of lower bounds.

Chandler [1], crediting Lubben, shows that if $\mathcal{COM}(X)$ has infima, then X must be locally compact.

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References

An asterisk indicates a reference not seen by me.

1. Chandler, R. E., Hausdorff Compactifications, Marcel Dekker Inc., 1976

2. Kelley, J. R., General Topology, Van Nostrand, 1955.

3* Lubben, R. G., Concerning the decomposition and amalgamation of points, upper semi-continuous collections, and topological extensions, Trans. AMS 49(1941),410-466.

4. This website, P1: Ordering of Compactifications