Order Compactifications

In this section a totally bounded uniformity associated with a linearly ordered set will be constructed and some propreties of the associated compactification derived. The idea is quite simple and undoubtedly known, although I have no specific reference.

Let X be assumed to be a set linearly ordered by $<$. Standard variations on the order, that is, $>, \leq, \geq$ will have the expected meanings. Given $a, b \in X$, sets of the forms $[a,b], [a,b), (a,b], (a,b)$, $(-\infty,a], [a,\infty), (-\infty,a)$, and (a,∞) will have the usual meanings and, if non-empty, will be called intervals. The last four types will also be called rays, possibliy left or right, and the last two types open rays. The order topology for X is the topology with a subbasis consisting of X and all open rays. $\tau(<)$ will denote the order topology for X. Intervals of the form $(a,b), (a,\infty)$ or $(-\infty,b)$ are always open sets. The other types of interval may or may not be in $\tau(<)$, depending on the order. An open interval means an interval which is in $\tau(<)$.

In the terminology used here, all intervals must have at least one endpoint in X, with rays having one endpoint and other intervals two. Note that the endpoints of an interval need not be distinct ($[x,x] = \{x\}$ for all x) and a non-empty intersection of two intervals is also an interval. Moreover, the endpoints of an interval need not be unique. For example, if X has a smallest element a_0 and a largest b_0, then $X = [a_0,b_0] = (-\infty,b_0] = [a_0,\infty)$. An I-set I is defined by the property that, if $x, y \in I$ and $x < z < y$ with $z \in X$, then $z \in I$. Every interval is an I-set but the converse fails as the examples of $\emptyset$ and $X = (-\infty,\infty)$ show. Even for the non-empty, bounded case, without order-completeness, an I-set need not have endpoints. For example, in $\mathbf{Q}$ with the order inherited from the reals, the set $\mathbf{Q} \cap (-\pi,\pi)$ is a bounded I-set but has no endpoints in $\mathbf{Q}$.

The following terminology may differ from standard usage in that the collections of sets need not cover X.

Definition R21.1 Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of X. $\mathcal{A}$ refines $\mathcal{B}$ provided every $A \in \mathcal{A}$ is a subset of some $B \in \mathcal{B}$ and $\cup\{A : A \in \mathcal{A}\} = \cup\{B : B \in \mathcal{B}\}$.

Definition R21.2 Let $\mathcal{A}$ be a collection of subsets of X. $R(\mathcal{A})$ is defined to be $\cup\{A \times A : A \in \mathcal{A}\}$.

Lemma R21.3 Let $\mathcal{A}$ be a pairwise disjoint collection of subsets of X.
Then $R(\mathcal{A}) \circ R(\mathcal{A}) = R(\mathcal{A})$.

Proof: This is an easy consequence of the definitions.

Lemma R21.4 Let $(X,<)$ be a set with a linear order. Let $\mathcal{I}$ be a finite set of intervals. Then there is $\mathcal{I}_1 \subseteq \mathcal{I}$ such that $\cup\mathcal{I}_1 = \cup\mathcal{I}$, no element of $\mathcal{I}_1$ contains any other element of $\mathcal{I}_1$, and $R(\mathcal{I}_1) \subseteq R(\mathcal{I})$.

Proof: Let $\mathcal{I}_1$ be the set of $I \in \mathcal{I}$ not contained in any other element of $\mathcal{I}$. Clearly $\mathcal{I}_1 \subseteq \mathcal{I}$, no element of $\mathcal{I}_1$ contains any other element of $\mathcal{I}_1$, and by definition $R(\mathcal{I}_1) \subseteq R(\mathcal{I})$. Let $x \in I$ for some I in $\mathcal{I}$. Either $I \in \mathcal{I}_1$ or there is a finite increasing chain of supersets of I in $\mathcal{I}$, the largest of which must be in $\mathcal{I}_1$. Thus $x \in \cup\mathcal{I}_1$.

Lemma R21.5 Let $(X,<)$ be a set with a linear order. Let $\mathcal{I}$ be a finite set of intervals with no element of $\mathcal{I}$ containing any other. If X has a smallest element a_0, at most one member of $\mathcal{I}$ contains a_0. Similarly, if X has a largest element b_0, at most one member of $\mathcal{I}$ contains b_0. Finally, $\mathcal{I}$ contains at most one left ray and at most one right ray.

Proof: Suppose X contains a smallest element a_0, which is in both of I, J from $\mathcal{I}$, with $I \neq J$. Whether I, J are rays or have right endpoints, it easily follows that either $I \subseteq J$ or $J \subseteq I$, contradicting the assumption about $\mathcal{I}$. The assertions about a largest element and rays follow in much the same way.

The proofs of the next few lemmas involve a tedious multiplicity of cases, but I have been unable to find more elegant arguments.

Lemma R21.6 Let $(X, <)$ be a set with a linear order. Let I_1 and I_2 be open intervals in X, with neither of them rays. Assume $a \leq b$ are the endpoints of I_1 and $c \leq d$ are the endpoints of I_2. Assume neither is a subset of the other. Then there is a finite collection of open intervals $\mathcal{J}$ such that $\mathcal{J}$ refines $\{I_1, I_2\}$ and $R(\mathcal{J}) \circ R(\mathcal{J}) \subseteq R(\{I_1, I_2\})$.

Proof: By relabeling the intervals if necessary, assume $a \leq c$. If $I_1 \cap I_2 = \emptyset$, then $\mathcal{J} = \{I_1, I_2\}$ has the required properties and so also assume that $I_1 \cap I_2 \neq \emptyset$. For any $x \in I_1 \cap I_2$, $a \leq x \leq b$ and $c \leq x \leq d$ so that $c \leq b$. As case A, assume $I_1 \cap I_2 = \{x\}$ so that $\{x\}$ and $[x, \infty)$ are both open. Subcase Ai: If $a < c$ and $b < d$, let $J_1 = I_1 \cap (-\infty, x)$, $J_2 = I_2 \cap (x, \infty)$, and $J_3 = \{x\}$. Let $\mathcal{J}$ be the non-empty elements of $\{J_1, J_2, J_3\}$. $\mathcal{J}$ consists of open intervals, each of which is contained in either I_1 or I_2. By R21.3 $R(\mathcal{J}) \circ R(\mathcal{J}) = R(\mathcal{J}) \subseteq R(\{I_1, I_2\})$. To see that $\mathcal{J}$ refines $\{I_1, I_2\}$, let $t \in I_1 \cup I_2$. If $t < x$, then $t \notin I_2$, for otherwise $a < c \leq t < x \leq b$ would imply t is a second point in $I_1 \cap I_2$, a contradiction. Thus $t \in I_1$, which implies $t \in J_1$. Similarly, $t > x$ implies $t \in J_2$. If $t = x$, clearly $t \in J_3$. Thus $I_1 \cup I_2 \subseteq \cup\mathcal{J}$, as needed. Subcase Aii: Assume $a < c$ and $d \leq b$. Since $I_2 \not\subseteq I_1$, it must be that $b = d$ and $d \in I_2 - I_1$. In this subcase, $\{d\} = (x, \infty) \cap I_2$, i.e., $\{d\}$ is open. Then $\mathcal{J} = \{I_1, \{d\}\}$ has the required properties. Subcase Aiii: Assume $a = c$ and $b < d$. This is similar to the preceding subcase: $\{a\}$ is open and $\mathcal{J} = \{I_2, \{a\}\}$ works. Subcase Aiv: Assume $a = c$ and $b > d$. This is again similar: $\{c\}$ is open and $\mathcal{J} = \{I_1, \{c\}\}$ works. Subcase Av: Assume $a = c$ and $b = d$. In this subcase, $(a, b) \subseteq I_1 \subseteq [a, b]$ and similarly for I_2. Since neither is a subset of the other, one must be $[a, b)$ and the other $(a, b]$. Here $\mathcal{J} = \{[a, x), [x, b]\}$ works. As case B, suppose $|I_1 \cap I_2| \geq 2$, and let x, y be in $I_1 \cap I_2$ with $x < y$. Subcase Bi: Assume $(x, y) = \emptyset$ and $a < c$ or $b < d$. The former says both $(-\infty, x] = (-\infty, y)$ and $[y, \infty) = (x, \infty)$ are open. The latter implies $b \leq d$ since, if $b > d$, $a < c$ would yield $I_2 \subseteq I_1$, a contradiction. Let $J_1 = I_1 \cap (-\infty, x]$ and $J_2 = I_2 \cap [y, \infty)$, which are disjoint open intervals so that $R(\{J_1, J_2\}) \circ R(\{J_1, J_2\}) = R(\{J_1, J_2\})$. Let $t \in I_1$. If $t \leq x$, then $t \in J_1$. If $t > x$, since $(x, y) = \emptyset$, $t \geq y$. Since $a \leq c \leq x < t \leq b \leq d$, if $b < d$, then $t \in J_2$. If $b = d$, $a < c$ and since $I_2 \not\subseteq I_1$, $d \in I_2 - I_1$ so that again $t \in J_2$. Thus $I_1 \subseteq J_1 \cup J_2$. Similarly, $I_2 \subseteq J_1 \cup J_2$ so that $\{J_1, J_2\}$ refines $\{I_1, I_2\}$. Subcase Bii: Assume $(x, y) = \emptyset$ and $a = c$ and $b \geq d$. If $b = d$, $(a, b) \subseteq I_1 \cap I_2 \subseteq [a, b]$ Since neither is a subset of the other, one must be $[a, b)$ and the other $(a, b]$. Then $\mathcal{J} = \{[a, b) \cap (-\infty, x], (a, b] \cap [y, \infty)\}$ works. If $b > d$, since $I_2 \not\subseteq I_1$, $c \in I_2 - I_1$. Then $\mathcal{J} = \{I_2 \cap (-\infty, x], I_1 \cap [y, \infty)\}$ has the required properties. Subcase Biii: Assume $x < z < y$ for some $z \in X$ and $a < c$ or $b < d$. Let $J_1 = I_1 \cap (-\infty, z)$, $J_2 = (x, y)$, and $J_3 = I_2 \cap (z, \infty)$. Let $t \in I_1$. If $t < z$, then $t \in J_1$. If $z \leq t < y$, then $t \in J_2$. If $t \geq y$ and $b < d$, then $t \in J_3$. If $b \geq d$, since $a < c$ and $I_2 \not\subseteq I_1$, $b = d$ and $d \in I_2 - I_1$ so that again $t \in J_3$. After a similar argument for $t \in I_2$, we have $I_1 \cup I_2 \subseteq J_1 \cup J_2 \cup J_3$ and so $\mathcal{J} = \{J_1, J_2, J_3\}$ refines $\{I_1, I_2\}$. Next let (p, q) and (q, r) be in $R(\mathcal{J})$. If both pairs are in the same $J_k \times J_k$, clearly $(p, r) \in R(\{I_1, I_2\})$. Thus assume $(p,q) \in J_k \times J_k$ and $(q,r) \in J_m \times J_m$ with $k \neq m$. Since $J_1 \cap J_3 = \emptyset$, one of k, m must be 2. Since $J_2 \subseteq I_1 \cap I_2$, $J_1 \subseteq I_1$, and $J_3 \subseteq I_2$, it follows easily that (p,r) must be in $R(\{I_1, I_2\})$. Thus $R(\mathcal{J}) \circ R(\mathcal{J}) \subseteq R(\{I_1, I_2\})$. Subcase Biv: Assume $x < z < y$ for some $z \in X$ and $a = c$ and $b \geq d$. If $b = d$, as in subcase Bii, one of I_1, I_2 must be $[a,b)$ and the other $(a,b]$. Let $J_1 = [a,b) \cap (-\infty, z)$, $J_2 = (x,y)$, and $J_3 = (a,b] \cap (z,\infty)$. Each of J_1, J_2, J_3 is an open interval and clearly $\mathcal{J} = \{J_1, J_2, J_3\}$ refines $\{I_1, I_2\}$. As in subcase Biii, $R(\mathcal{J}) \circ R(\mathcal{J}) \subseteq R(\{I_1, I_2\})$ as required. If $b > d$, since $I_2 \not\subseteq I_1$, $a = c \in I_2 - I_1$. Let $J_1 = I_2 \cap (-\infty, z)$, $J_2 = (x,y)$, and $J_3 = I_1 \cap (z, \infty)$. Each of J_1, J_2, J_3 is an open interval and for $\mathcal{J} = \{J_1, J_2, J_3\}$, as in subcase Biii, $R(\mathcal{J}) \circ R(\mathcal{J}) \subseteq R(\{I_1, I_2\})$. Let $t \in I_1$. If $t > z$, then $t \in J_3$. If $x < t \leq z$, then $t \in J_2$. If $t \leq x$, $c = a \leq t \leq x < z < d$ and, since $c \in I_2$, $t \in I_2$ and so in J_1. Similarly, $I_2 \subseteq J_1 \cup J_2 \cup J_3$ and so $\mathcal{J}$ refines $\{I_1, I_2\}$.

Lemma R21.7 Let $(X, <)$ be a set with a linear order. Let I_1 and I_2 be open intervals in X. Assume I_1 is a left ray with endpoint b and I_2 is a non-ray with endpoints c and d. Assume neither is a subset of the other. Then there is a finite collection of open intervals $\mathcal{J}$ such that $R(\mathcal{J}) \circ R(\mathcal{J}) \subseteq R(\{I_1, I_2\})$ and $\mathcal{J}$ refines $\{I_1, I_2\}$.

Proof: If $I_1 \cap I_2 = \emptyset$, then $\mathcal{J} = \{I_1, I_2\}$ has the required properties and so also assume that $I_1 \cap I_2 \neq \emptyset$. Note that, if $x \in I_1 \cap I_2$, then $c \leq x \leq b$. Since $I_2 \not\subseteq I_1$, $c \leq b \leq d$. As a final preliminary observation, if $b = d$, since $(-\infty, b) \subseteq I_1 \subseteq (-\infty, b]$ and $(c,b) \subseteq I_2 \subseteq [c,b]$ and $I_2 \not\subseteq I_1$, $I_1 = (-\infty, b)$ and $b \in I_2$. As a first case, suppose $I_1 \cap I_2 = \{x\}$. Since $\{x\}$ is open, $(-\infty, x]$ is also open. Let $J_1 = I_1 \cap (-\infty, x]$ and $J_2 = I_2 \cap (x, \infty)$. These open intervals are disjoint so that $R(\{J_1, J_2\}) \circ R(\{J_1, J_2\}) = R(\{J_1, J_2\})$. To see that $\mathcal{J} = \{J_1, J_2\}$ refines $\{I_1, I_2\}$, let $t \in I_1 \cup I_2$. If $t \leq x \leq b$, then $t \in I_1$ and so in J_1. If $t > x$, then $t \in I_2$ would imply $t \in J_2$. Suppose $t \notin I_2$ so that $t \in I_1$. If $b < d$, then $c \leq x < t \leq b < d$, contradicting $t \notin I_2$. If $b = d$, as noted above $I_1 = (-\infty, b)$ so that $t < b$. Then $c \leq x < t < b = d$, again contradicting $t \notin I_2$. As a second case assume $|I_1 \cap I_2| \geq 2$ and let $x, y \in I_1 \cap I_2$ with $x < y$. If $(x,y) = \emptyset$, then both $(-\infty, x] = (-\infty, y)$ and $[y, \infty) = (x, \infty)$ are open. Let $J_1 = I_1 \cap (-\infty, x]$ and $J_2 = I_2 \cap [y, \infty)$. These open intervals are disjoint so that $R(\{J_1, J_2\}) \circ R(\{J_1, J_2\}) = R(\{J_1, J_2\})$. To see that $\mathcal{J} = \{J_1, J_2\}$ refines $\{I_1, I_2\}$, let $t \in I_1 \cup I_2$. If $t \leq x < y \leq b$, then $t \in I_1$ and so in J_1. If $t \geq y$, then $t \in I_2$ would imply $t \in J_2$. Suppose $t \notin I_2$ so that $t \in I_1$. If $b < d$, then $c \leq x < y \leq t \leq b < d$, contradicting $t \notin I_2$. If $b = d$, since $I_1 = (-\infty, b)$, $c < y \leq t < b = d$, again contradicting $t \notin I_2$. Finally, if $(x,y) \neq \emptyset$, pick $z \in X$ with $x < z < y$. Let $J_1 = I_1 \cap (-\infty, z)$, $J_2 = (x,y)$, and $J_3 = I_2 \cap (z, \infty)$, all open intervals. Note that $J_2 \subseteq I_1 \cap I_2$. To see that $\mathcal{J} = \{J_1, J_2, J_3\}$ refines $\{I_1, I_2\}$, let $t \in I_1 \cup I_2$. If $t \leq x < z < b$, then $t \in I_1$ and so in J_1. If $x < t < y$, then $t \in J_2$. If $t \geq y$, $t \in I_2$ would imply $t \in J_3$. Suppose $t \notin I_2$ so that $t \in I_1$. If $b < d$, then $c < y \leq t \leq b < d$, contradicting $t \notin I_2$. If $b = d$, since $I_1 = (-\infty, b)$, $c < y \leq t < b = d$, again contradicting $t \notin I_2$. To verify the composition requirement, let (p,q) and (q,r) be in $R(\mathcal{J})$. If both pairs are in the same $J_k \times J_k$, clearly $(p,r) \in R(\{I_1, I_2\})$. Thus assume $(p,q) \in J_k \times J_k$ and $(q,r) \in J_m \times J_m$ with $k \neq m$. Since $J_1 \cap J_3 = \emptyset$, one of k, m must be 2. Since $J_2 \subseteq I_1 \cap I_2$, $J_1 \subseteq I_1$, and $J_3 \subseteq I_2$, it follows easily that (p,r) must be in $R(\{I_1, I_2\})$. Thus $R(\mathcal{J}) \circ R(\mathcal{J}) \subseteq R(\{I_1, I_2\})$ and $\mathcal{J}$ has the required properties.

Lemma R21.8 Let $(X, <)$ be a set with a linear order. Let I_1 and I_2 be open intervals in X. Assume I_1 is a non-ray with endpoints a and b and I_2 is a right ray with endpoint c. Assume neither is a subset of the other. Then there is a finite collection of open intervals

$\mathcal{J}$ such that $R(\mathcal{J}) \circ R(\mathcal{J}) \subseteq R(\{I_1, I_2\})$ and $\mathcal{J}$ refines $\{I_1, I_2\}$.

Proof: If $I_1 \cap I_2 = \emptyset$, then $\mathcal{J} = \{I_1, I_2\}$ has the required properties and so also assume that $I_1 \cap I_2 \neq \emptyset$. For $x \in I_1 \cap I_2$, $a \leq x \leq b$ and $c \leq x$. Thus $c \leq b$. If $c < a$, then $I_1 \subseteq I_2$, contradicting $I_1 \not\subseteq I_2$. Thus $a \leq c \leq b$. As a final preliminary observation, if $a = c$, since $(a, b) \subseteq I_1 \subseteq [a, b]$ and $(c, \infty) \subseteq I_2 \subseteq [c, \infty)$ and $I_1 \not\subseteq I_2$, $I_2 = (c, \infty)$ and $a \in I_1$. As a first case, suppose $I_1 \cap I_2 = \{x\}$. Since $\{x\}$ is open, $[x, \infty)$ is also open. Let $J_1 = I_1 \cap (-\infty, x)$ and $J_2 = I_2 \cap [x, \infty)$. These open intervals are disjoint so that $R(\{J_1, J_2\}) \circ R(\{J_1, J_2\}) = R(\{J_1, J_2\})$. To see that $\mathcal{J} = \{J_1, J_2\}$ refines $\{I_1, I_2\}$, let $t \in I_1 \cup I_2$. If $t \geq x \geq c$, then $t \in I_2$ and so in J_2. If $t < x$, then $t \in I_1$ would imply $t \in J_1$. Assume $t \notin I_1$ so that $t \in I_2$. If $a < c$, then $a < c \leq t < x \leq b$, contradicting $t \notin I_1$. If $a = c$, as noted above $I_2 = (c, \infty)$ and so $a = c < t < x \leq b$, again contradicting $t \notin I_1$. As a second case assume $|I_1 \cap I_2| \geq 2$ and let $x, y \in I_1 \cap I_2$ with $x < y$. If $(x, y) = \emptyset$, then both $(-\infty, x] = (-\infty, y)$ and $[y, \infty) = (x, \infty)$ are open. Let $J_1 = I_1 \cap (-\infty, x]$ and $J_2 = I_2 \cap [y, \infty)$. These open intervals are disjoint so that $R(\{J_1, J_2\}) \circ R(\{J_1, J_2\}) = R(\{J_1, J_2\})$. To see that $\mathcal{J} = \{J_1, J_2\}$ refines $\{I_1, I_2\}$, let $t \in I_1 \cup I_2$. If $t \geq y > x \geq c$, then $t \in I_2$ and so in J_2. If $t \leq x$, then $t \in I_1$ would imply $t \in J_1$. Suppose $t \notin I_1$ so that $t \in I_2$. If $a < c$ then $a < c \leq t \leq x < y \leq b$, contradicting $t \notin I_1$. If $a = c$, since $I_2 = (c, \infty)$, $a = c < t \leq x < y \leq b$, again contradicting $t \notin I_1$. Finally, if $(x, y) \neq \emptyset$, pick $z \in X$ with $x < z < y$. Let $J_1 = I_1 \cap (-\infty, z)$, $J_2 = (x, y)$, and $J_3 = I_2 \cap (z, \infty)$, all open intervals. Note that $J_2 \subseteq I_1 \cap I_2$. To see that $\mathcal{J} = \{J_1, J_2, J_3\}$ refines $\{I_1, I_2\}$, let $t \in I_1 \cup I_2$. If $t \geq y > z > x \geq c$, then $t \in I_2$ and so in J_3. If $x < t < y$, then $t \in J_2$. If $t \leq x$, then $t \in I_1$ would imply $t \in J_1$. Suppose $t \notin I_1$ so that $t \in I_2$. If $a < c$, $c \leq t < y \leq b$ contradicts $t \notin I_1$. If $a = c$, $I_2 = (c, \infty)$ and so $a = c < t < x < y \leq b$, again contradicting $t \notin I_1$. To verify the composition requirement, let (p, q) and (q, r) be in $R(\mathcal{J})$. If both pairs are in the same $J_k \times J_k$, clearly $(p, r) \in R(\{I_1, I_2\})$. Thus assume $(p, q) \in J_k \times J_k$ and $(q, r) \in J_m \times J_m$ with $k \neq m$. Since $J_1 \cap J_3 = \emptyset$, one of k, m must be 2. Since $J_2 \subseteq I_1 \cap I_2$, $J_1 \subseteq I_1$, and $J_3 \subseteq I_2$, it follows easily that (p, r) must be in $R(\{I_1, I_2\})$. Thus $R(\mathcal{J}) \circ R(\mathcal{J}) \subseteq R(\{I_1, I_2\})$ and $\mathcal{J}$ has the required properties.

Lemma R21.9 Let $(X, <)$ be a set with a linear order. Let I_1 and I_2 be open intervals in X. Assume I_1 is a left ray with endpoint b and I_2 is a right ray with endpoint c. Assume neither is a subset of the other. Then there is a finite collection of open intervals $\mathcal{J}$ such that $R(\mathcal{J}) \circ R(\mathcal{J}) \subseteq R(\{I_1, I_2\})$ and $\mathcal{J}$ refines $\{I_1, I_2\}$.

Proof: If $I_1 \cap I_2 = \emptyset$, then $\mathcal{J} = \{I_1, I_2\}$ has the required properties and so also assume that $I_1 \cap I_2 \neq \emptyset$ and consequently that $c \leq b$. As a first case, suppose $I_1 \cap I_2 = \{x\}$. Since $\{x\}$ is open, $(-\infty, x]$ is also open. Let $J_1 = I_1 \cap (-\infty, x]$ and $J_2 = I_2 \cap (x, \infty)$. These open intervals are disjoint so that $R(\{J_1, J_2\}) \circ R(\{J_1, J_2\}) = R(\{J_1, J_2\})$. To see that $\mathcal{J} = \{J_1, J_2\}$ refines $\{I_1, I_2\}$, let $t \in I_1 \cup I_2$. If $t \leq x \leq b$, then $t \in I_1$ and so in J_1. If $t > x \geq c$, then $t \in I_2$ and so in J_2. As a second case assume $|I_1 \cap I_2| \geq 2$ and let $x, y \in I_1 \cap I_2$ with $x < y$. If $(x, y) = \emptyset$, then both $(-\infty, x] = (-\infty, y)$ and $[y, \infty) = (x, \infty)$ are open. Let $J_1 = I_1 \cap (-\infty, x]$ and $J_2 = I_2 \cap [y, \infty)$. These open intervals are disjoint so that $R(\{J_1, J_2\}) \circ R(\{J_1, J_2\}) = R(\{J_1, J_2\})$. To see that $\mathcal{J} = \{J_1, J_2\}$ refines $\{I_1, I_2\}$, let $t \in I_1 \cup I_2$. If $t \leq x \leq b$, then $t \in I_1$ and so in J_1. If $t \geq y \geq c$, then $t \in I_2$ and so in J_2. If $(x, y) \neq \emptyset$, pick $z \in X$ with $x < z < y$. Let $J_1 = I_1 \cap (-\infty, z)$, $J_2 = (x, y)$, and $J_3 = I_2 \cap (z, \infty)$, all open intervals. Note that $J_2 \subseteq I_1 \cap I_2$. To see that $\mathcal{J} = \{J_1, J_2, J_3\}$

refines $\{I_1, I_2\}$, let $t \in I_1 \cup I_2$. If $t \leq x < z < b$, then $t \in I_1$ and so in J_1. If $x < t < y$, then $t \in J_2$. If $t \geq y$, since $y > c$, $t \in I_2$ and so in J_3. Next let (p, q) and (q, r) be in $R(\mathcal{J})$. If both pairs are in the same $J_k \times J_k$, clearly $(p, r) \in R(\{I_1, I_2\})$. Thus assume $(p, q) \in J_k \times J_k$ and $(q, r) \in J_m \times J_m$ with $k \neq m$. Since $J_1 \cap J_3 = \emptyset$, one of k, m must be 2. Since $J_2 \subseteq I_1 \cap I_2$, $J_1 \subseteq I_1$, and $J_3 \subseteq I_2$, it follows easily that (p, r) must be in $R(\{I_1, I_2\})$. Thus $R(\mathcal{J}) \circ R(\mathcal{J}) \subseteq R(\{I_1, I_2\})$ and $\mathcal{J}$ has the required properties.

Lemma R21.10 Let $(X, <)$ be a set with a linear order. Let I_1 and I_2 be open intervals in X. Then there is a finite collection of open intervals $\mathcal{J}$ such that $\mathcal{J}$ refines $\{I_1, I_2\}$ and $R(\mathcal{J}) \circ R(\mathcal{J}) \subseteq R(\{I_1, I_2\})$.

Proof: If either is a subset of the other, this is trivial. The non-trivial cases are covered in R21.6 through R21.9.

Lemma R21.11 Let I_1 and I_2 be intervals in X with $I_1 \not\subseteq I_2$ and $I_2 \not\subseteq I_1$. If a is a left endpoint of both I_1 and I_2, then a is in one but not the other. If b is a right endpoint of both I_1 and I_2, then b is in one but not the other.

Proof: Assume a is a left endpoint of both I_1 and I_2. First suppose neither contains a. Clearly neither can be a right ray and so let c, d be right endpoints of I_1, I_2 respectively. Then $(a, c) \subseteq I_1 \subseteq (a, c]$ and $(a, d) \subseteq I_2 \subseteq (a, d]$. If $c < d$, then $I_1 \subseteq I_2$, a contradiction. If $c > d$, then $I_2 \subseteq I_1$, a contradiction. If $c = d$ and both contain $c = d$ or both do not contain $c = d$, then $I_1 = I_2$, a contradiction. If $c = d$ is in one but not the other, $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$, contradiction. Thus it cannot be that neither contains a. In much the same way, the case of both containing a cannot hold. The argument for the claim about right endpoints is similar.

Corollary R21.12 Let $\mathcal{I}$ be a set of intervals of X such that $I, J \in \mathcal{I}$ implies $I \not\subseteq J$ and $J \not\subseteq I$. Then at most two members of $\mathcal{I}$ have a common left endpoint. Likewise, at most two members of $\mathcal{I}$ have a common right endpoint.

Proof: Immediate fron R21.11.

Lemma R21.13 Let $\mathcal{I}$ be a finite set of open intervals in X. Then there is a finite collection of open intervals $\mathcal{J}$ such that $R(\mathcal{J}) \circ R(\mathcal{J}) \subseteq R(\mathcal{I})$ and $\mathcal{J}$ refines $\mathcal{I}$.

Proof: By R21.4, we can assume no element of $\mathcal{I}$ contains any other element of $\mathcal{I}$, which implies by R21.5 that $\mathcal{I}$ contains at most one left ray and at most one right ray. Now proceed by induction on $|\mathcal{I}|$: When $|\mathcal{I}| = 1$ the statement is trivial and the case $|\mathcal{I}| = 2$ is R21.10. Assume the statement is true for any collection of cardinality n, where $n \geq 2$. Let $\mathcal{I} = \{I_1, I_2, \ldots, I_n, I_{n+1}\}$ be a set of $n + 1$ open intervals. If $I_k \cap \cup\{I_j : j \neq k\} = \emptyset$ for any k, apply the induction hypothesis to obtain $\mathcal{J}_1$ which satisfies the conclusion for $\mathcal{I} - \{I_k\}$ and let $\mathcal{J} = \mathcal{J}_1 \cup \{I_k\}$. In this case $R(\mathcal{J}) \circ R(\mathcal{J}) = R(\mathcal{J}_1) \circ R(\mathcal{J}_1) \cup I_k \times I_k$ is easily verified so the conclusion holds. Thus make another assumption: $I_k \cap \cup\{I_j : j \neq k\} \neq \emptyset$ for all k. Since $n + 1 > 2$, $\{a : a \text{ is a left endpoint of some } I_j\}$ is non-empty and finite. Let a_0 be the largest left endpoint. Re-subscript so that I_{n+1} has a_0 as a left endpoint and, if $\mathcal{I}$ contains two intervals with left endpoint a_0, so that I_{n+1} is the one (by R21.11) <u>not</u> containing a_0. Note that, if there is a right ray in $\mathcal{I}$, it must be I_{n+1}. Otherwise, for right ray I_k with left endpoint a, since $a \leq a_0$ and $a_0 \notin I_{n+1}$, $I_{n+1} \subseteq I_k$, a contradiction. Next there are at least two intervals in $\mathcal{I}$ other than I_{n+1}, at least one of which has right endpoint. Let b_0 be the largest right endpoint of the remaining intervals. If necessary, again relabel the remaining intervals so that I_n has right endpoint b_0 and, if there are two such, so that I_n is the one containing b_0. Note that $a_0 \le b_0$ since otherwise $(\cup_{k=1}^n I_k) \cap I_{n+1} = \emptyset$. By the induction hypothesis there is a finite family of open intervals, $\mathcal{A}$, such that $\mathcal{A}$ refines $\{I_1, I_2, \ldots, I_n\}$ and $R(\mathcal{A}) \circ R(\mathcal{A}) \subseteq R(\{I_1, I_2, \ldots, I_n\})$. By R21.10 there is a finite set of open intervals, $\mathcal{B}$, such that $\mathcal{B}$ refines $\{I_n, I_{n+1}\}$ and $R(\mathcal{B}) \circ R(\mathcal{B}) \subseteq R(\{I_n, I_{n+1}\})$. As a first case, assume $a_0 = b_0$. Since $I_n \cap I_{n+1} \neq \emptyset$, $I_n \cap I_{n+1} = \{b_0\}$, which is open so that $(-\infty, b_0]$ is also open. Let $\mathcal{J}$ be the set of non-empty elements from $\{A \cap (-\infty, b_0] : A \in \mathcal{A}\} \cup \{B \cap (b_0, \infty) : B \in \mathcal{B}\}$. Clearly each element of the finite set $\mathcal{J}$ is an open interval contained in some I_j. Since $(-\infty, b_0]$ and (b_0, ∞) are disjoint, $R(\mathcal{J}) \circ R(\mathcal{J}) \subseteq R(\mathcal{A}) \circ R(\mathcal{A}) \cup R(\mathcal{B}) \circ R(\mathcal{B})$ so that $R(\mathcal{J}) \circ R(\mathcal{J}) \subseteq R(\mathcal{I})$. To see that $\mathcal{J}$ refines $\mathcal{I}$, let $t \in \cup_{j=1}^{n+1} I_j$. If $t > b_0 = a_0$, t must be in I_{n+1} and so in $B \cap (b_0, \infty)$ for some $B \in \mathcal{B}$. If $t \le b_0$, then $t \in \cup_{j=1}^n I_j$ and so in $A \cap (-\infty, b_0]$ for some $A \in \mathcal{A}$. As a second case, assume $a_0 < b_0$ and $(a_0, b_0) = \emptyset$. Then $[b_0, \infty) = (a_0, \infty)$ and $(-\infty, a_0] = (-\infty, b_0)$ are open. Let $\mathcal{J}$ be the set of non-empty elements from the collection

$$\{A \cap (-\infty, a_0] : A \in \mathcal{A}\} \cup \{B \cap [b_0, \infty) : B \in \mathcal{B}\} \cup \{I_{n+1} \cap (-\infty, a_0]\}.$$

Clearly each element of the finite set $\mathcal{J}$ is an open interval contained in some I_j. Since $(-\infty, a_0]$ and $[b_0, \infty)$ are disjoint and $I_{n+1} \cap (-\infty, a_0]$ is either $\emptyset$ or $\{a_0\}$, $R(\mathcal{J}) \circ R(\mathcal{J})$ is contained in $R(\mathcal{A}) \circ R(\mathcal{A}) \cup R(\mathcal{B}) \circ R(\mathcal{B})$ so that $R(\mathcal{J}) \circ R(\mathcal{J}) \subseteq R(\mathcal{I})$. To see that $\mathcal{J}$ refines $\mathcal{I}$, let $t \in \cup_{j=1}^{n+1} I_j$. If $t < a_0$, then $t \in \cup_{j=1}^n I_j$ and so there is $A \in \mathcal{A}$ such that $t \in A \cap (-\infty, a_0]$. If $t = a_0$ and $I_{n+1} \cap (-\infty, a_0] = \emptyset$, again $t \in \cup_{j=1}^n I_j$ and $t \in A \cap (-\infty, a_0]$ for some $A \in \mathcal{A}$. If $t = a_0$ and $I_{n+1} \cap (-\infty, a_0] = \{a_0\}$, then $t \in I_{n+1} \cap (-\infty, a_0]$. If $t \ge b_0$, then t must be in $I_n \cup I_{n+1}$ and there is $B \in \mathcal{B}$ such that $t \in B \cap [b_0, \infty)$. As a final case, assume $a_0 < b_0$ and $(a_0, b_0) \neq \emptyset$. Since I_n is either a left ray or has a left endpoint less than or equal a_0 by the choice of a_0, $(a_0, b_0) \subseteq I_n$. Since $I_{n+1} \not\subseteq I_n$ and $a_0 \notin I_{n+1}$ if a_0 is also the left endpoint of I_n, $(a_0, b_0) \subseteq I_{n+1}$ as well. Pick z with $a_0 < z < b_0$ and let $\mathcal{J}$ be the set of non-empty elements of $\{A \cap (-\infty, z) : A \in \mathcal{A}\} \cup \{B \cap (z, \infty)\} \cup \{A \cap B \cap (a_0, b_0) : A \in \mathcal{A}, B \in \mathcal{B}\}$. Clearly each element of the finite set $\mathcal{J}$ is an open interval contained in some I_j. Since $(-\infty, z)$ and (z, ∞) are disjoint, $R(\mathcal{J}) \circ R(\mathcal{J}) \subseteq R(\mathcal{A}) \circ R(\mathcal{A}) \cup R(\mathcal{B}) \circ R(\mathcal{B})$ so that $R(\mathcal{J}) \circ R(\mathcal{J}) \subseteq R(\mathcal{I})$. To see that $\mathcal{J}$ refines $\mathcal{I}$, let $t \in \cup_{j=1}^{n+1} I_j$. If $t < a_0$, $t \notin I_{n+1}$ and so t must be in $\cup_{j=1}^n I_j$ and there is $A \in \mathcal{A}$ with $t \in A \cap (-\infty, z)$. If $t > b_0$, $t \notin \cup_{j=1}^n I_j$ and so t must be in I_{n+1} and there is $B \in \mathcal{B}$ with $t \in B \cap (z, \infty)$. If $a_0 < t < b_0$, since $(a_0, b_0) \subseteq I_n \cap I_{n+1}$, $t \in A \cap B \cap (a_0, b_0)$ for some $A \in \mathcal{A}, B \in \mathcal{B}$. If $t = a_0$ and $a_0 \in I_{n+1}$, by labeling I_{n+1} is the only interval in $\mathcal{I}$ with left endpoint a_0. By the choice of a_0, I_n is either a left ray or has left endpoint smaller than a_0 so that $t = a_0 \in I_n$ and there is $A \in \mathcal{A}$ with $t \in A \cap (-\infty, z)$. If $t = a_0$ and $a_0 \notin I_{n+1}$, then t must be in $\cup_{j=1}^n I_j$ and there is $A \in \mathcal{A}$ with $t \in A \cap (-\infty, z)$. If $t = b_0$ and $b_0 \notin I_n$, by labeling I_n is the only interval in $\{I_1, \ldots, I_n\}$ with right endpoint b_0. Since I_{n+1} is the only possible right ray in $\mathcal{I}$, it must be that $t = b_0 \notin \cup_{j=1}^n I_j$, i.e., t must be in I_{n+1} and there is $B \in \mathcal{B}$ with $t \in B \cap (z, \infty)$. If $t = b_0$ and $b_0 \in I_n$, then there is $B \in \mathcal{B}$ with $t \in B \cap (z, \infty)$.

The following definition and proposition are almost definitely known. Banaschewski [1] may present them in some form or may indicate a source.

Definition R21.14 Let $(X, <)$ be a set with a linear order. $\mathcal{U}(<)$ is defined to be the union of $\{X \times X\}$ and the set of $U \in \mathcal{P}(X \times X)$ such that U is a superset of some $R(\mathcal{I})$, where $\mathcal{I}$ is a finite collection of open intervals covering X.

Proposition R21.15 Let $(X,<)$ be a set with a linear order. Then

i) $\mathcal{U}(<)$ is a uniformity for X.

ii) $\tau(\mathcal{U}(<)) = \tau(<)$.

iii) $\mathcal{U}(<)$ is separated and totally bounded.

Proof: For i): If $\mathcal{I}$ is a cover of X, clearly the diagonal of X is contained in $R(\mathcal{I})$ and $R(\mathcal{I})$ is symmetric. Thus diagonal and symmetry requirements (P2.1i and P2.1iv) in the definition of a uniformity hold for $\mathcal{U}(<)$. Obviously the superset requirement P2.1ii also holds. If $\mathcal{I}_1$ and $\mathcal{I}_2$ are two finite collections of open intervals covering X, let $\mathcal{I}$ be the set of non-empty elements from $\{I \cap J : I \in \mathcal{I}_1, J \in \mathcal{I}_2\}$. $\mathcal{I}$ is a finite collection of open intervals covering X, and it is straightforward to check that $R(\mathcal{I}) \subseteq R(\mathcal{I}_1) \cap R(\mathcal{I}_2)$. The intersection requirement P2.1iii follows easily. Finally, the triangle inequality requirement P2.1v is immediate from R21.13. For ii): First note that if $\mathcal{I}$ is a finite collection of open intervals covering X, then, for any $x \in X$, $R(\mathcal{I})[x] = \cup\{I \in \mathcal{I} : x \in I\}$, which is open in $\tau(<)$. It follows easily that $\tau(\mathcal{U}(<)) \subseteq \tau(<)$. Next let $x \in G \in \tau(<)$. There an open interval $I \in \tau(<)$ such that $x \in I \subseteq G$. Let $\mathcal{I} = \{I, (-\infty, x), (x, \infty)\}$, which is a finite set of open intervals covering X. Then $R(\mathcal{I}) \in \mathcal{U}(<)$ and $R(\mathcal{I})[x] = I \subseteq G$. Thus $G \in \tau(\mathcal{U}(<))$ and so $\tau(<) \subseteq \tau(\mathcal{U}(<))$. For iii): Since $\tau(<)$, i.e., $\tau(\mathcal{U}(<))$, is T_2, $\mathcal{U}(<)$ must be separated. If $\mathcal{I}$ is a finite collection of open intervals covering X, pick $x_I \in I$ for each $I \in \mathcal{I}$ and let $F = \{x_I : I \in \mathcal{I}\}$. F is finite and clearly $R(\mathcal{I})[F] = \cup\{I : I \in \mathcal{I}\} = X$. It follows easily that $\mathcal{U}(<)$ is totally bounded.

Corollary R21.16 Let $(X,<)$ be a set with a linear order. $\mathcal{U}(<)$ is complete if and only if $(X, \tau(<))$ is compact.

Proof: This is immediate from R21.15 and P2.7.

The last proposition and P2.3 show that $\tau(<)$ must be $T_{3\frac{1}{2}}$. A better result shows that $\tau(<)$ is actually completely normal.

Proposition R21.17 Let $(X,<)$ be a set with a linear order. Then $(X, \tau(<))$ is T_5.

Proof: See Steen and Seebach [3; pp. 66-67].

Definition R21.18 Let $(X,<)$ be a set with a linear order. $\mathcal{U}_M(<)$ is defined to be $\{U \subseteq X \times X : R(\mathcal{C}) \subseteq U$, where $\mathcal{C}$ is a finite $\tau(<)$-open cover of $X\}$.

Corollary R21.19 Let $(X,<)$ be a set with a linear order. Then $\mathcal{U}_M(<)$ is a totally bounded uniformity generating $\tau(<)$. Moreover, the T_2 compactification associated with $\mathcal{U}_M(<)$ is the Stone-Čech compactification of $(X, \tau(<))$.

Proof: This is immediate from R6.3.4 and R1.8.

Clearly $\mathcal{U}(<) \subseteq \mathcal{U}_M(<)$. A question of interest is whether they are equal. Certainly equality holds if $(X, \tau(<))$ is compact, but the following example shows that $\mathcal{U}(<)$ may be a proper subset.

Example R21.20 The reals with the usual order will be used. Let $O_1 = (-\infty, 1)$, $O_2 = \cup\{(n-1, n+1) : n$ is an even positive integer$\}$, and $O_3 = \cup\{(n-1, n+1) : n$ is an odd positive integer$\}$. It is easily checked that $\{O_1, O_2, O_3\}$ is an open cover of $\mathbb{R}$ so that $R(\{O_1, O_2, O_3\})$ is in $\mathcal{U}_M(<)$. Note that O_2 contains no odd integers and O_3 contains no evens. Let $\mathcal{I}$ be a finite set of open intervals covering $\mathbb{R}$. Clearly $\mathcal{I}$ must contain at least one right ray, say (a, ∞). Pick a positive integer n such that $n > a$. The ordered pair $(n, n+1)$ is in $(a, \infty) \times (a, \infty)$ and so in $R(\mathcal{I})$. Clearly that pair is not in $O_1 \times O_1$. Since one of $n, n+1$ is even and the other odd, that pair is not in $O_2 \times O_2 \cup O_3 \times O_3$ and so not in $R(\{O_1, O_2, O_3\})$. Thus $R(\mathcal{I}) \not\subseteq R(\{O_1, O_2, O_3\})$ and so $R(\{O_1, O_2, O_3\})$ is not in $\mathcal{U}(<)$.

The next definition is familiar, although the terminology may be non-standard.

Definition R21.22 Let $(X, <)$ be a set with a linear order. X is order-complete provided every non-empty subset of X which is bounded above in X has a least upper bound in X.

Note the importance of the phrase 'in X.' By this terminology, for $X = [0, 1)$ with the usual ordering, X is order-complete. X itself has no upper bounds in X, even though it has upper bounds in $\mathbb{R}$.

A standard argument shows that, if $(X, <)$ is order-complete, then every non-empty subset of X which is bounded below in X has a greatest lower bound in X. The next two theorems are also familiar, with proofs found in many expositions, including Steen and Sternbach [3; pp. 67-68].

Theorem R21.23 Let $(X, <)$ be a set with a linear order, where $X \neq \emptyset$. Then $(X, \tau(<))$ is compact if and only if X has a largest element, X has a smallest element, and X is order-complete.

Corollary R21.24 Let $(X, <)$ be a set with a linear order. If $\mathcal{U}(<)$ is complete, then X is order-complete.

Proof: This is immediate from R21.16 and R21.23.

The converse of R21.24 is false, as the example of $\mathbb{R}$ with the usual order shows.

Theorem R21.25 Let $(X, <)$ be a set with a linear order. Then $(X, \tau(<))$ is connected if and only if X has no consecutive points and X is order-complete.

Definition R21.26 Let $(X, <)$ be a set with a linear order and let $A \subseteq X$. $<^A$ is the linear order on A inherited from $(X, <)$, i.e., for $a_1, a_2 \in A$, $a_1 <^A a_2$ if and only if $a_1 < a_2$.

To avoid nuisance cases, $|A| \geq 2$ will often be assumed. $<^A$ may also be referred to as the restriction of $<$ to A. Some additional notation and terminology will also be used. For $A \subseteq X$, intervals and rays in $(A, <^A)$ will be called A-intervals and A-rays and denoted with a subscript. For example, given $a, b \in A$, $[a, b)_A = \{t \in A : a \leq^A t <^A b\}$. This somewhat cumbersome notation can also be used for intervals and rays of X but will normally be avoided except for emphasis or clarity.

Proposition R21.27 Let $(X, <)$ be a set with a linear order and assume X is order complete. Let $A \subseteq X$ with $|A| \geq 2$. If A is an I-set in X, then $(A, <^A)$ is also order-complete.

Proof: Let $B \subseteq A$ be non-empty with an upper bound in A. By the order-completeness of X there is $b_0 \in X$, the least upper bound. Let $b \in B$ and let $a \in A$ be an upper bound of B. Then $b \leq b_0 \leq a$ so that, since A is an I-set in X, $b_0 \in A$, i.e., B has a least upper bound in A as required.

The following example illustrates a complication with subspaces. In $A = [0, 1) \cup \{2\}$ as a subset of $\mathbb{R}$ with the usual ordering, $\{2\}$ is open in the subspace topology but not $\tau(<^A)$-open, which also holds if the superset is taken to be $X = [0, 1] \cup \{2\}$ with the induced order from $\mathbb{R}$. This example from Munkres [2; p.90] demonstrates the second part of the following and shows that density alone does not guarantee that $\tau(<^A)$ is the subspace topology.

Proposition R21.28 Let $(X, <)$ be a set with a linear order and let $A \subseteq X$ with $|A| \geq 2$. Let τ_A be the subspace topology on A from $(X, \tau(<))$. Then

i) $\tau(<^A) \subseteq \tau_A$.

ii) $\tau(<^A)$ may be a proper subset of τ_A.

iii) If A is an I-set in X, then $\tau(<^A) = \tau_A$.

Proof: For $a \in A$, it is easy to check that $(-\infty, a)_A = (-\infty, a) \cap A$ and $(a, \infty)_A = (a, \infty) \cap A$ from which i) follows. For iii), because of i), it is sufficient to show that the subbasic sets, $(x, \infty) \cap A$ and $(-\infty, x) \cap A$, are in $\tau(<^A)$ for every $x \in X$. Let $x \in X$. If $x \notin A$, since A is an I-set in X, either $A \subseteq (-\infty, x)$ or $A \subseteq (x, \infty)$ so that $(-\infty, x) \cap A$ is either $\emptyset$ or A. If $x \in A$, then $(-\infty, x)_A = (-\infty, x) \cap A$. In either case $(-\infty, x) \cap A$ is in $\tau(<^A)$ as needed. Similarly, $(x, \infty) \cap A$ is in $\tau(<^A)$.

Corollary R21.29 Let $(X, <)$ be a set with a linear order and and assume $(X, \tau(<))$ is connected. Then every I-set of X is connected.

Proof: Let A be an I-set of X. If $|A| \leq 1$, the result is trivial. Otherwise, by R21.28iii there is no ambiguity about the topology on A. By R21.25 X is order-complete and has no consecutive points. Because A is an I-set of X, a pair of consecutive points of $(A, <^A)$ would also be consecutive points in X, i.e., there is no such pair. By R21.26 it is also order-complete. By R21.25 again A is connected.

Proposition R21.30 Let $(X, <)$ be a set with a linear order and let $A \subseteq X$ with $|A| \geq 2$ and A dense in X. Let τ_A be the subspace topology on A from $(X, \tau(<))$. If A contains all consecutive pairs of X, then $\tau(<^A) = \tau_A$.

Proof: Let $x \in X$. If $x \in A$, $(-\infty, x) \cap A = (-\infty, x)_A$. If $x \notin A$, by the hypothesis for consecutive pairs, $t < x$ implies $(t, x) \neq \emptyset$. This and the density of A easily yield $(-\infty, x) \cap A = \cup\{(-\infty, a)_A : a \in A, a < x\}$ so that $(-\infty, x) \cap A \in \tau(<^A)$. Similarly, $(x, \infty) \cap A \in \tau(<^A)$. Since $\tau(<^A)$ contains a subbasis for τ_A, $\tau_A \subseteq \tau(<^A)$ and by R21.28i the conclusion follows.

In this context, A has two uniformities of interest, $\mathcal{U}(<^A)$ and the subspace uniformity from $\mathcal{U}(<)$, which will be denoted $\mathcal{U}^A(<)$. The next few items focus on relations between these two uniformities. The next lemma shows that $\mathcal{U}^A(<)$ need not equal $\mathcal{U}(<^A)$.

Lemma R21.31 Let $(X, <)$ be a set with a linear order and let $A \subseteq X$. Let τ_A be the subspace topology on A from $(X, \tau(<))$. If $\tau(<^A) \neq \tau_A$, then $\mathcal{U}(<^A) \neq \mathcal{U}^A(<)$.

Proof: By R21.15ii $\tau(\mathcal{U}(<)) = \tau(<)$ so that $\tau(\mathcal{U}^A(<))$ is the subspace topology from $\tau(<)$, i.e., τ_A. Since by R21.15ii $\tau(\mathcal{U}(<^A)) = \tau(<^A)$ and the topologies on A are distinct, the uniformities must be different.

Lemma R21.32 Let $(X, <)$ be a set with a linear order and let $A \subseteq X$. Let $a \in A$. If $(-\infty, a]_A \in \tau(<_A)$, then either $(-\infty, a] \in \tau(<)$ or there is $x \in X$ such that $a < x$ and $(a, x) \cap A = \emptyset$.

Proof: Assume $(-\infty, a]_A \in \tau(<_A)$ and $(-\infty, a] \notin \tau(<)$. By R21.28i there is $x \in X$ such that $a \in (-\infty, x) \cap A \subseteq (-\infty, a]_A$. Clearly $a < x$ and $(a, x) \cap A = \emptyset$.

Lemma R21.33 Let $(X, <)$ be a set with a linear order and let $A \subseteq X$. Let $a \in A$. If $[a, \infty)_A \in \tau(<_A)$, then either $[a, \infty) \in \tau(<)$ or there is $y \in X$ such that $y < a$ and $(y, a) \cap A = \emptyset$.

Proof: Similar to R21.32.

Lemma R21.34 Let $(X, <)$ be a set with a linear order and let $A \subseteq X$. Let I be a non-empty A-interval in $\tau(<_A)$. Then there is an X-interval $J(I) \in \tau(<)$ such that $J(I) \cap A = I$. Moreover, if I is a left A-ray with endpoint $a \in A$, then $(-\infty, a) \subseteq J(I)$, and similarly for right A-rays. Lastly, if I has endpoints $a <_A b$, then $(a, b) \subseteq J(I)$.

Proof: This proceeds by cases. If $I = (a, b)_A$ for some $a, b \in A$, let $J(I) = (a, b)$. It is easy to check that $(a, b) \cap A = (a, b)_A$, i.e., $J(I) \cap A = I$ as required. If $I = (a, \infty)_A$, let $J(I) = (a, \infty)$ and, if $I = (-\infty, b)_A$, make the analogous choice. In either case $J(I) \cap A = I$ follows as before. If $I = [a, b)_A$, note that $[a, \infty)_A = [a, b)_A \cup (a, \infty)_A$ is in $\tau(<_A)$. Apply R21.33. If $[a, \infty)$ is in $\tau(<)$, let $J(I) = [a, b)$. It is easy to check that $J(I) \in \tau(<)$ and $J(I) \cap A = I$. Otherwise pick $y \in X$ with $y < a$ and $(y, a) \cap A = \emptyset$ and let $J(I) = (y, b)$, which meets the requirements. The other endpoint-included cases can be handled similarly. In all cases, the subset assertions are clear.

Note that despite the function-like notation in the previous lemma, $J(I)$ need not be unique.

Lemma R21.35 Let $(X, <)$ be a set with a linear order and let $A \subseteq X$. Let $\mathcal{I}$ be a finite collection of $\tau(<_A)$-open A-intervals, which cover A. Then there is $\mathcal{J}$, a finite collection of $\tau(<)$-open X-intervals, which cover X, such that $R(\mathcal{J}) \cap A \times A \subseteq R(\mathcal{I})$.

Proof: By R21.4 there is $\mathcal{I}_1 \subseteq \mathcal{I}$ such that $\cup \mathcal{I}_1$ covers A, no element of $\mathcal{I}_1$ contains any other element of $\mathcal{I}_1$, and $R(\mathcal{I}_1) \subseteq R(\mathcal{I})$. A $\mathcal{J}$ which satisfies the requirements of the conclusion for $\mathcal{I}_1$ also works for the original $\mathcal{I}$. Furthermore, if $|\mathcal{I}_1| = 1$, then $\mathcal{J} = \{(-\infty, \infty)\}$ works. Thus assume $|\mathcal{I}_1| \geq 2$. As a last simplifying assumption, assume $\mathcal{I}_1 = \{I_1, I_2, \ldots, I_n\}$ is labeled using R21.5 and R21.11 as follows: I_1 is the left ray, if there is one, or else I_1 is the element with the smallest left endpoint, say a_1, and with $a_1 \in I_1$. For $j \geq 2$, assume a_j is the left endpoint of I_j, $a_2 \leq a_3 \leq \cdots \leq a_n$, and if $a_j = a_{j+1}$, then I_j is the one containing a_j. Note that, if a_1 exists, $a_1 \leq a_2$.

Next note that if $\mathcal{I}_1$ contains a right ray, it must be I_n. For, if I_j is a right ray and $j < n$, either $a_j = a_{j+1}$ so that $a_j \in I_j$ or $a_j < a_{j+1}$. In either case $I_{j+1} \subseteq I_j$, contradicting a simplifing assumption. Now let b_j be the right endpoint of I_j for $1 \leq j < n$ and, if I_n is not a right ray, let b_n be the right endpoint of I_n. As a final preliminary observation, note that $b_j \leq b_{j+1}$ whenever both are defined. For, if $b_{j+1} < b_j$, either $a_j = a_{j+1}$ so that $a_j \in I_j$ or $a_j < a_{j+1}$. In either case $I_{j+1} \subseteq I_j$, again contradicting a simplifing assumption.

Now for $1 \leq j \leq n$ use R21.34 to pick $J(I_j)$ and let $\mathcal{J}_1$ be the set of $J(I_j)$ so chosen. $\mathcal{J}_1$ is a finite set of $\tau(<)$-open X-intervals such that $R(\mathcal{J}_1) \cap A \times A \subseteq R(\mathcal{I}_1)$, but $\mathcal{J}_1$ may not cover X. The objective is to enlarge $\mathcal{J}_1$ by adding a finite number of $\tau(<)$-open intervals each disjoint from A in such a way that the enlarged collection does cover X. First, if a_1 is defined, it must be the smallest element of A and so add $(-\infty, a_1)$ if it is non-empty. Likewise, if b_n is defined, it must be the largest element of A and so add (b_n, ∞) if it is non-empty. If $I_j \cap I_{j+1} = \emptyset$, then $b_j \leq a_{j+1}$ and note that (b_j, a_{j+1}) must be disjoint from A as follows: Suppose $x \in (b_j, a_{j+1}) \cap A$. Then $x \in I_k$ for some k. If $k \leq j$, then $x \leq b_k \leq b_j$, which contradicts $x > b_j$. If $k \geq j+1$, then $a_{j+1} \leq a_k \leq x$, which contradicts $x < a_{j+1}$. Add (b_j, a_{j+1}) if it is non-empty.

Finally, let $\mathcal{J}$ be $\mathcal{J}_1$ together with the additions described in the last paragraph. Clearly, $\mathcal{J}$ is a finite collection of $\tau(<)$-open X-intervals. Since each added interval is disjoint from A, $R(\mathcal{J}) \cap A \times A = R(\mathcal{J}_1) \cap A \times A \subseteq R(\mathcal{I}_1)$. To verify that $\mathcal{J}$ covers X, let $x \in X$. If $x \in A$, then x is in some I_k, which equals $J(I_k) \cap A$. If x is not in A, note that it cannot be an endpoint of any I_k, since all such endpoints are in A. If $x < b_1$, then x is in either $J(I_1)$ or the left ray added if a_1 is defined. If $x > a_n$, then x is in $J(I_n)$ or the right ray added because b_n is defined. Thus assume $b_1 < x < a_n$. Let j =min$\{k : b_k < x\}$ and note $j < n$. If $x < a_{j+1}$, then $I_j \cap I_{j+1} = \emptyset$ and $x \in (b_j, a_{j+1})$, which is one of the additions. If $x > a_{j+1}$, then $j+1 < n$ and $b_j < x < b_{j+1}$. By R21.34 $(a_{j+1}, b_{j+1}) \subseteq J(I_{j+1})$ so that $x \in J(I_{j+1})$.

Proposition R21.36 Let $(X, <)$ be a set with a linear order and let $A \subseteq X$. Then $\mathcal{U}(<^A) \subseteq \mathcal{U}^A(<)$.

Proof: By the definition of $\mathcal{U}(<)$ and $\mathcal{U}(<^A)$ and of the subspace uniformity, this conclusion easily follows from R21.35.

Lemma R21.37 Let $(X, <)$ be a set with a linear order and let A be an interval in X. For any other X-interval, I, $I \cap A$ is either $\emptyset$, A, or an A-interval.

Proof: Assume $I \cap A$ is not $\emptyset$. As a sample case, assume A has endpoints $a \leq b$ and I has endpoints $c \leq d$. Let x =max$\{a, c\}$ and y =min$\{b, d\}$. $I \cap A$ is an X-interval with endpoints x, y, which may or may not be included in $I \cap A$ depending on I and A. If x, y are both in A, clearly $I \cap A$ is an A-interval. If $x \notin A$ and $y \in A$, the set $I \cap A$ can be described as a left A-ray with endpoint y. If $y \notin A$ and $x \in A$, the set $I \cap A$ can be described as a right A-ray with endpoint x. If neither of x, y is in A, $I \cap A = A$. The other cases are similar.

Corollary R21.38 Let $(X, <)$ be a set with a linear order and let A be an interval in X. Then $\mathcal{U}(<^A) = \mathcal{U}^A(<)$.

Proof: Given I, a $\tau(<)$-open interval in X, by R21.37, $I \cap A$ is either $\emptyset$, A, or a $\tau(<^A)$-open interval in A. It follows that a finite collection of $\tau(<)$-open intervals which covers X induces a finite collection of $\tau(<_A)$-open intervals, which cover A. With that, it is routine to verify that $\mathcal{U}^A(<) \subseteq \mathcal{U}(<^A)$. The conclusion is now immediate from R21.36.

Next the compactification determined by $\mathcal{U}(<)$ will be identified in the case that $(X, \tau(<))$ is connected and non-compact. Assuming connectedness, one has by R21.25, R21.28iii, R21.27, and R21.23 that $[a, b]$ is compact for any $a \leq b$ in X. It follows easily that, if $(X, \tau(<))$ is connected, it is also locally compact, which is necessary to apply R5.1.1 in the next three propositions.

Proposition R21.39 Let $(X, <)$ be a set with a linear order. Assume $(X, \tau(<))$ is connected, non-compact, and has neither a largest nor a smallest element. Then the compactification determined by $\mathcal{U}(<)$ is a two-point compactification.

Proof: Let $x_0 \in X$, let $G_1 = (-\infty, x_0)$, and let $G_2 = (x_0, \infty)$. G_1 and G_2 are disjoint, $\tau(<)$-open sets with $X - (G_1 \cup G_2) = \{x_0\}$ compact and $G_i \cup \{x_0\}$ non-compact. Because of local compactness R5.1.1 applies, and so this pair determines a two-point compactification, which, as in the proof of R5.1.1, can be constructed as follows: Pick $p_1 \neq p_2$ not in X and let $Y = X \cup \{p_1, p_2\}$. Let σ be the set of all $O \subseteq Y$ such that $O \cap X$ is $\tau(<)$-open and, for $i = 1, 2$, $p_i \in O$ implies $(X - O) \cap G_i$ has compact closure in X. Y with the inclusion map ι is a two-point compactification of X.

Now extend $<$ to Y by declaring p_1 the smallest element and p_2 the largest element. More precisely, define $<^*$ on Y by $x <^* y$ if and only if $x = p_1 \neq y$ or $x \neq p_2 = y$ or $x, y \in X$ and $x < y$. It can be easily checked that $<^*$ is a linear order on Y and $<^*$ restricted to X is $<$. Denote $<^*$-rays with an $*$ superscript and observe that $(-\infty, p_1)^* = \emptyset$, for $y \in X$ $(-\infty, y)^* = \{p_1\} \cup (-\infty, y)$, and $(-\infty, p_2)^* = \{p_1\} \cup X$. When the ray contains p_1, $[X - (-\infty, y)^*] \cap (-\infty, x_0)$is contained in $[y, x_0]$ for $y \in X$ and is $\emptyset$ for $y = p_2$. It follows from the definition of σ that $(-\infty, y)^* \in \sigma$ and similarly $(y, \infty)^* \in \sigma$. Thus $\tau(<^*) \subseteq \sigma$. Since (Y, σ) is compact and T_2, σ is minimal T_2. Since $\tau(<^*)$ is also T_2, $\tau(<^*) = \sigma$. Since $\tau(\mathcal{U}(<_*)) = \tau(<_*)$, $\mathcal{U}(<^*)$ is the unique uniformity for σ, and, as shown in R1.4, (Y, ι) is the compactification determined by the subspace uniformity on X from $\mathcal{U}(<^*)$. Since $X = (p_1, p_2)^*$ and $<^*$ restricted to X is $<$, by R21.38 that subspace uniformity is $\mathcal{U}(<)$.

One can easily see that, under the hypothesis of R21.39, using the same argument as R5.1.8, all two-point compactifications of $(X, \tau(<))$ are equivalent and, using the argument of R5.1.7, $(X, \tau(<))$ has no n-point compactification for $n \geq 3$.

Proposition R21.40 Let $(X, <)$ be a set with a linear order. Assume $(X, \tau(<))$ is connected and has a largest but no smallest element. Then the compactification determined by $\mathcal{U}(<)$ is the one-point compactification.

Proof: Similar to R21.39: Extend $<$ by adding the point at infinity as the smallest element.

Proposition R21.41 Let $(X, <)$ be a set with a linear order. Assume $(X, \tau(<))$ is connected and has a smallest but no largest element. Then the compactification determined by $\mathcal{U}(<)$ is the one-point compactification.

Proof: Similar to R21.39: Extend $<$ by adding the point at infinity as the largest element.

This section will be concluded with some results related to the remnant rings. The first lemma is a general fact which will be used implicitly.

Lemma R21.42 Let (X, τ) be a $T_{3\frac{1}{2}}$ space, let (Y, f) be a T_2 compactification of (X, τ), and let $\mathcal{U}$ be the separated totally bounded uniformity for X corresponding to (Y, f). Let A be a dense subset of X, let τ_A denote the subspace topology on A, and let $\mathcal{U}^A$ denote the subspace uniformity induced on A by $\mathcal{U}$. Then

i) $\tau_A = \tau(\mathcal{U}^A)$.

ii)$(Y, f|_A)$ is a T_2 compactification of (A, τ_A).

iii) $\mathcal{U}^A$ is the uniformity corresponding to $(Y, f|_A)$.

Proof: The first part is a standard fact, easy to verify. It is routine to check that $f[A]$ is dense in Y and that $f|_A$ is a uniform embedding, from which the second and third parts follow.

Let $k \in \mathbb{N}$ with $k \geq 2$. The following facts and notation will be used in the rest of this section. For $n \in \mathbb{N}$ and $z \in \mathbf{Z}$, $D^z_n(k)$ is the $\mathbf{Z}$-equivalence class of z mod k^n. Let $\mathcal{B}_k$ be the set of $D^z_n(k)$ over all n, z. By R16.7 and R16.9 $\mathcal{B}_k$ is a clopen basis for a topology on $\mathbf{Z}$. In R16.15 it is shown that $(\mathbf{R}_k ,f_k)$ is a T_2 compactification of $(\mathbf{Z} ,\tau_k)$, where τ_k is the topology with basis $\mathcal{B}_k$. As in R16.24 $\mathcal{V}_k$ denotes the uniformity for $\mathbf{Z}$ corresponding to $(\mathbf{R}_k ,f_k)$.

In $\mathbb{N}$ the subspace topology from τ_k will be denoted by $\tilde{\tau}_k$, the subspace uniformity from $\mathcal{V}_k$ by $\tilde{\mathcal{V}}_k$, and the restriction of f_k to $\mathbb{N}$ by $\tilde{f}_k$. By R12.6.9 $f_k[\mathbb{N}]$ is dense in $\mathbf{R}_k$ and so R21.42 applies: $(\mathbf{R}_k ,\tilde{f}_k)$ is a T_2 compactification of $(\mathbb{N} ,\tilde{\tau}_k)$ with $\tilde{\mathcal{V}}_k$ the corresponding uniformity.

Finally $<_k$ denotes the linear order generating the topology on $\mathbf{R}_k$ as in R19.1.7. Recall that the consecutive pairs of $<_k$ were completely described in R19.1.15, R19.1.17, and R19.1.19. Those facts will be utilized in what follows.

Proposition R21.43 Let $(X, <)$ be a set with a linear order with consecutive points $x_0 < x_1$. Let A be dense in X and assume at least one of x_0, x_1 is not in A. Then $\mathcal{U}(<^A)$ is a proper subset of $\mathcal{U}^A(<)$.

Proof: Since x_0, x_1 are consecutive, $(x_0, x_1) = \emptyset$, $(-\infty, x_0] = (-\infty, x_1)$ is in $\tau(<)$, and $[x_1, \infty) = (x_0, \infty)$ is in $\tau(<)$. Thus $\mathcal{I} = \{(-\infty, x_0], [x_1, \infty)\}$ is a finite cover of X by $\tau(<)$-open intervals. Suppose $x_0 \notin A$ and there is $\mathcal{J}$, a finite set of $\tau(<_A)$-open A-intervals, such that $R(\mathcal{J}) \subseteq R(\mathcal{I}) \cap A \times A$. For any $J \in \mathcal{J}$, since $J \times J \subseteq R(\mathcal{I})$ and there are only two elements of $\mathcal{I}$, either $J \subseteq (-\infty, x_0]$ or $J \subseteq [x_1, \infty)$. Since A is dense in X, $A \cap (-\infty, x_0] \neq \emptyset$ and so, since $\mathcal{J}$ covers A, there is at least one element of $\mathcal{J}$ contained in $(-\infty, x_0]$. Each such element has a right endpoint and so, since $\mathcal{J}$ is finite, let $b \in A$ be the largest such right end point. Since $x_0 \notin A$, $b < x_0$. Since $(b, x_0]$ is non-empty and in $\tau(<)$, by density there is $a \in A$ such that $b < a \leq x_0$. By the definition of b, $a \notin \cup\mathcal{J}$, a contradiction. The argument is similar but uses a smallest left endpoint, if $x_0 \in A$ so that $x_1 \notin A$.

The last proposition can be applied to the remnant rings, as follows.

Corollary R21.44 For any natural number $k \geq 2$, the uniformity determined by $<_k$ restricted to the subset $f_k[\mathbb{N}]$ is a proper subset of the subspace uniformity.

Proof: Fix $k \geq 2$. $\mathcal{U}(<_k)$ must be the unique uniformity for the compact space $\mathbf{R}_k$. For any $j \in \mathbb{N}$ with $j \geq 2$, by R19.1.21 $f_k(j)$ is the larger of a consecutive pair with the smaller being the image of a negative integer under f_k. By R21.43 the conclusion is immediate.

The last result transfers to $\mathbb{N}$. First, more notation: For $m, n \in \mathbb{N}$, $m \prec_k n$ if and only if $f_k(m) <_k f_k(n)$.

Corollary R21.45 For any natural number $k \geq 2$, $\mathcal{U}(\prec_k)$ is a proper subset of $\tilde{\mathcal{V}}_k$.

Proof: The one-to-one map $\tilde{f}_k$ induces two weak uniformities on $\mathbb{N}$, $\mathcal{U}(\prec_k)$ from the uniformity determined by $<_k$ restricted to the subset $f_k[\mathbb{N}]$ and $\tilde{\mathcal{V}}_k$ from the subspace uniformity for $f_k[\mathbb{N}]$. By R21.44, since $f_k[\mathbb{N}]$ is the range of $\tilde{f}_k$, the first is a proper subset of the second.

The final results will be used to describe relationship of $\tau(\mathcal{U}(\prec_k))$ to $\tau(\tilde{\mathcal{V}}_k)$, i.e., $\tilde{\tau}_k$. The first lemma contrasts with R21.30.

Lemma R21.46 Let $(X, <)$ be a set with a linear order. Let x_0, x_1 be a consecutive pair in X with $x_0 < x_1$. Suppose x_0 is not the larger of some other consective pair. Assume A is a dense subset of X with $x_0 \notin A$ but $x_1 \in A$. Let τ_A denote the subspace topology on A. Then $[x_1, \infty)_A \in \tau_A$ but $[x_1, \infty)_A \notin \tau(<^A)$ so that $\tau(<^A)$ is a proper subset of τ_A.

Proof: First note that $[x_1, \infty) = (x_0, \infty) \in \tau(<)$ since $x_0 < x_1$ and $(x_0, x_1) = \emptyset$. Thus we have $[x_1, \infty) \cap A = [x_1, \infty)_A \in \tau_A$. Moreover $(-\infty, x_0] = (-\infty, x_1) \in \tau(<)$ and, since A is dense, $(-\infty, x_0] \cap A \neq \emptyset$ so that $[x_1, \infty)_A \neq A$. Now suppose $[x_1, \infty)_A \in \tau(<^A)$. Then there is $a \in A$ with $x_1 \in (a, x_1]_A \subseteq [x_1, \infty)_A$. $a <^A x_1$ means $a < x_1$ so that $a \leq x_0$. Since $x_0 \notin A$, $a \neq x_0$. Since a, x_0 is not a consecutive pair in X and A is dense, there is $a_1 \in A$ with $a < a_1 < x_0$. But $a <^A a_1 <^A x_1$ and so $a_1 \in [x_1, \infty)_A$, a contradiction. The second part of the conclusion follows from the first and R21.28i.

Example R21.47 Let $k \geq 2$ be in $\mathbb{N}$. The lemma will be applied with X being the remnant ring $\mathbf{R}_k$, A being $f_k[\mathbb{N}]$, and the ordering being $<_k$ from [10]. Let $x_1 = f_k(2)$, which is in A. By R19.1.21 it is the larger of a consecutive pair, the smaller being $x_0 = f_k(-k)$, which is not in A. By R19.1.19 x_0 is not the larger of some other consecutive pair in X. By R21.46 $\tau(<_k^A)$ is a proper subset of the subspace topology from $\tau(<_k)$, i.e., $\tau(\mathcal{U}(<_k^A))$ is a proper subset of $\tau(\mathcal{U}^A(<_k))$.

Lemma R21.48 Let $(X,<)$ be a set with a linear order. Let A be a dense subset of X such that, for every consecutive pair in X, the larger is in A and the smaller is not in A. Let τ^* be the topology for A with basis $\{A\} \cup \{[x,\infty)_A : x$ is the larger of a consecutive pair in $X\}$. Then $\tau(<^A) \vee \tau^*$ is the subspace topology induced on A by $\tau(<)$.

Proof: First note that X can have no consecutive triple. Assume otherwise, i.e., $a < b < c$ in X with $(a,b) = \emptyset$ and $(b,c) = \emptyset$. Then b must be in A as the larger of consecutive pair a, b and not in A as the smaller of consecutive pair b, c, a contradiction. This observation shows that the hypothesis of R21.46 is satisfied for every consecutive pair in X.

Now let τ_A denote the subspace topology induced on A by $\tau(<)$. By R21.28i and R21.46 $\tau(<^A) \vee \tau^* \subseteq \tau_A$. Now let $x \in X$. If $x \in A$, then $(-\infty,x) \cap A = (-\infty,x)_A$ and $(x,\infty) \cap A = (x,\infty)_A$ so that both are in $\tau(<^A)$. If $x \notin A$, since x is not the larger of a consecutive pair, by density $(-\infty,x) \cap A = \cup\{(-\infty,a)_A : a \in A, a < x\}$, which is in $\tau(<^A)$. If x is not the smaller of a consecutive pair, by density $(x,\infty) \cap A = \cup\{(a,\infty) \cap A : a \in A, x < a\}$, which is in $\tau(<^A)$. If x is the smaller of a consecutive pair, let y be the larger of the pair. Then $(x,\infty) = [y,\infty)$ and $(x,\infty) \cap A = [y,\infty)_A$, which is in τ^*. Thus $\tau(<^A) \vee \tau^*$ contains a subbasis for τ_A, i.e., $\tau_A \subseteq \tau(<^A) \vee \tau^*$.

The following proposition again uses the linear order $\prec_k$, which last appeared in R21.45. Since it differs radically from the usual order on the natural numbers, rays will be labelled with the subscript k.

Proposition R21.49 Let $k \geq 2$ be in $\mathbb{N}$. Let τ_k^* be the topology for $\mathbb{N}$ with basis $\{[n,\infty)_k : n \in \mathbb{N}\}$. Then $\tilde{\tau}_k = \tau(\mathcal{U}(\prec_k)) \vee \tau_k^*$.

Proof: R19.1.19 shows that every $<_k$-consecutive pair in $\mathbf{R}_k$ has a smaller element of the form $f_k(-j)$ for some $j \in \mathbb{N}$ and a larger element $f_k(l)$ for some $l \geq 2$ in $\mathbb{N}$. It follows that R21.48 applies, with X being the remnant ring $\mathbf{R}_k$, A being $f_k[\mathbb{N}]$, and the ordering being $<_k$. Continuing with that notation, we have that the objects of interest are all obtained by transference via $\tilde{f}_k$ from $f_k[\mathbb{N}]$: $\tilde{\tau}_k$ is the weak topology on $\mathbb{N}$ induced by $\tilde{f}_k$ and the subspace topology on A. $\mathcal{U}(\prec_k)$ is the weak uniformity on $\mathbb{N}$ induced by $\tilde{f}_k$ and $\mathcal{U}(<_k^A)$ so that $\tau(\mathcal{U}(\prec_k))$ is the weak topology on $\mathbb{N}$ induced by $\tilde{f}_k$ and $\tau(\mathcal{U}(<_k^A))$. $[n,\infty)_k = \tilde{f}_k^{-1}[[f_k(n),\infty)_A]$ for all $n \in \mathbb{N}$. By R19.1.8 $f_k(1)$ is the smallest element of $\mathbf{R}_k$ so that $[1,\infty)_k = \mathbb{N}$ and τ_k^* is the weak topology on $\mathbb{N}$ induced by $\tilde{f}_k$ and the topology with basis $\{A\} \cup \{[x,\infty)_A : x$ is the larger of a consecutive pair in $X\}$.

The conclusion follows immediately by transference from R21.48.

Corollary R21.50 Let $k \geq 2$ be in $\mathbb{N}$. $\tau(\mathcal{U}(\prec_k))$ is a proper subset of $\tilde{\tau}_k$.

Proof: Use R21.46 and transference as in the proof of R21.49 to see that $[2,\infty)_k$ is in $\tilde{\tau}_k$ but not in $\tau(\mathcal{U}(\prec_k))$. Containment is clear from R21.49.

Albert J. Klein 2011
http://susanjkleinart.com/compactification/

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Added Comment 2013

Dr. Scott Williams (http://www.nsm.buffalo.edu/˜sww/mathprof.html) has pointed out that the compactification associated with $\mathcal{U}(<)$ is called the Dedekind Compactification.

Added 2018

In R21.39-R21.41 representations of the compactification class corresponding to $\mathcal{U}(<)$ were described in certain special cases. This note extends that description to the general case. Whether the comment of Dr. Williams applies only to the special cases, or to the general case as well, is unknown to me.

The construction below relies on an idea related to the Dedekind cut method of constructing the reals from the rationals. Most, maybe all, of this is probably known. Proofs are provided since I have no reference. The presentation and notation may be unconventional.

Definition R21.Add.1 Let X be a set with linear order $<$. For $S \subseteq X$, S is a left R-set provided $x \in S$ and $y < x$ imply $y \in S$.

Clearly left rays, $\emptyset$, and X are left R-sets. It is easy to give examples in $\mathbf{Q}$ of non-empty left R-sets which do not have an endpoint, i.e., are not rays.

The collection of all left R-sets, ordered by containment, is linearly ordered, order complete, and has smallest element $\emptyset$ and largest element X. That collection is unsuitable for the purpose here because, for any $x \in X$, $(-\infty, x)$ and $(-\infty, x]$ are a consecutive pair. As a result, it is necessary to proceed by identifying such pairs.

Definition R21.Add.2 Let X be a set with linear order $<$. The set $\mathcal{D}$ consists of all $\{S\}$, where S is a left R-set but not a left ray, and all doubletons of the form $\{(-\infty, x), (-\infty, x]\}$, where $x \in X$. For $D \in \mathcal{D}$, $R(D) = \cup\{S : S \in D\}$.

Clearly, every left R-set is in at least one element of $\mathcal{D}$ and any $R(D)$ must be a left R-set.

Note that $\mathcal{D}$ need not be a partition of the collection of left R-sets: Let a, b be a consecutive pair in X with a smaller. Then $(-\infty, a] = (-\infty, b)$ and so $\{(-\infty, a), (-\infty, a]\}$ and $\{(-\infty, b), (-\infty, b]\}$ are distinct elements of $\mathcal{D}$ which have a non-empty intersection.

Definition R21.Add.3 Let X be a set with linear order $<$. Let D_1, D_2 be in $\mathcal{D}$. $D_1 \leq^* D_2$ provided $R(D_1) \subseteq R(D_2)$.

Lemma R21.Add.4 Let X be a set with linear order $<$. Then $<^*$ is a linear order on $\mathcal{D}$.

Proof: Transitivity holds since containment is transitive. Note that $R(\{S\}) = S$ and $R(\{(-\infty, x), (-\infty, x]\}) = (-\infty, x]$. Let $D_1 <^* D_2$ and $D_2 <^* D_1$. If both are singletons or both are doubletons, clearly $D_1 = D_2$. The case with one singleton and one doubleton cannot occur: If, for example, $D_1 = \{S\}$ and $D_2 = \{(-\infty, x), (-\infty, x]\}$, the assumption implies $S = (-\infty, x]$, a contradiction since S is not a left ray. Thus $<^*$ is anti-symmetric. Finally, let D_1, D_2 be in $\mathcal{D}$ and assume $D_1 \not\leq^* D_2$, i.e., $R(D_1)$ is not a subset of $R(D_2)$. Pick $y \in R(D_1)$ with $y \notin R(D_2)$ and let $x \in R(D_2)$. Since $R(D_2)$ is a left R-set, $y \not\leq x$ and so $x < y$. Since $R(D_1)$ is a left R-set, $x \in R(D_1)$. Thus $R(D_2) \subseteq R(D_1)$, i.e., $D_2 <^* D_1$.

Lemma R21.Add.5 Let X be a set with linear order $<$. If X does not have a smallest element, then $\{\emptyset\}$ is the smallest element of $\mathcal{D}$. If x_0 is the smallest element of X, then $\{(-\infty, x_0), (-\infty, x_0]\}$ is the smallest element of $\mathcal{D}$.

Proof: First assume X does not have a smallest element. Then $\emptyset$ is a left R-set but not a left ray and so $\{\emptyset\} \in \mathcal{D}$. Since $R(\{\emptyset\}) = \emptyset$, $\{\emptyset\}$ is the smallest of $\mathcal{D}$. Now assume X has a smallest element, x_0. Then $\emptyset = (-\infty, x_0)$ and so there is a unique element of $\mathcal{D}$ containing it, $\{(-\infty, x_0), (-\infty, x_0]\}$. Since $R(\{(-\infty, x_0), (-\infty, x_0]\}) = \{x_0\}$ and every $R(D)$ is non-empty in this case, $\{(-\infty, x_0), (-\infty, x_0]\}$ is the smallest element of $\mathcal{D}$.

Lemma R21.Add.6 Let X be a set with linear order $<$. If X does not have a largest element, then $\{X\}$ is the largest element of $\mathcal{D}$. If x_1 is the largest element of X, then $\{(-\infty, x_1), (-\infty, x_1]\}$ is the largest element of $\mathcal{D}$.

Proof: First assume X does not have a largest element. Then the left R-set X is not a left ray and so the element of $\mathcal{D}$ containing X is $\{X\}$. Since $R(\{X\}) = X$, $\{X\}$ is the largest element of $\mathcal{D}$. Now assume x_1 is the largest element of X. Then $X = (-\infty, x_1]$ and so the element of $\mathcal{D}$ containing X is $\{(-\infty, x_1), (-\infty, x_1]\}$. Since $R(\{(-\infty, x_1), (-\infty, x_1]\}) = X$, $\{(-\infty, x_1), (-\infty, x_1]\}$ is the largest element of $\mathcal{D}$.

Lemma R21.Add.7 Let X be a set with linear order $<$. Then $\mathcal{D}$ with $<^*$ is order complete.

Proof: Let $\mathcal{S}$ be a non-empty subset of $\mathcal{D}$. ($\mathcal{S}$ is bounded above since $\mathcal{D}$ has a largest element.) Let $S = \cup\{R(D) : D \in \mathcal{S}\}$. S is a left R-set, being a union of left R-sets. If S is not a left ray, let $D^* = \{S\}$. Here $R(D^*) = S$ and so D^* is an upper bound of $\mathcal{S}$. Let $E \in \mathcal{D}$ be an upper bound of $\mathcal{S}$. Clearly $S \subseteq R(E)$ and so $D^* \leq^* E$. Thus D^* is the least upper bound of $\mathcal{S}$. If S is a left ray, let $x \in X$ be its endpoint. If $S = (-\infty, x]$, let $D^* = \{(-\infty, x), (-\infty, x]\}$. Since $R(D^*) = S$, D^* is an upper bound of $\mathcal{S}$. If E is an upper bound of $\mathcal{S}$, $R(D^*) = S \subseteq R(E)$, i.e., $D^* \leq^* E$. Thus D^* is the least upper bound of $\mathcal{S}$. If $S = (-\infty, x)$, there are two cases. If x is the larger of a consecutive pair with y smaller, then $S = (-\infty, y]$ and $D^* = \{(-\infty, y), (-\infty, y]\}$ is the least upper bound of $\mathcal{S}$ as in the previous case. Thus assume x is not the larger of a consecutive pair. Let $D^* = \{(-\infty, x), (-\infty, x]\}$. Since $S \subseteq R(D^*)$, D^* is an upper bound of $\mathcal{S}$. Let E be an upper bound of $\mathcal{S}$. Clearly $S \subseteq R(E)$. If $E = \{R\}$ where R is a left R-set but not a left ray, S must be a proper subset of R. For $y \in R - S$, $x \leq y$ so that $x \in R$. Thus $R(D^*) \subseteq R(E)$, i.e., $D^* \leq^* E$. If $E = \{(-\infty, y), (-\infty, y]\}$, $x > y$ cannot hold: If so, by the hypothesis in this case there is $t \in X$ with $y < t < x$. Since $t \in S$, there is $D \in \mathcal{S}$ with $t \in R(D) \subseteq R(E) = (-\infty, y]$, a contradiction. Thus $x \leq y$ so that $R(D^*) \subseteq R(E)$. In both cases, $D^* \leq^* E$ and so D^* is the least upper bound of $\mathcal{S}$.

Corollary R21.Add.8 Let X be a set with linear order $<$. Then $(\mathcal{D}, \tau(<^*))$ is compact and T_2.

Proof: This follows from the previous lemmas and R21.23.

Definition R21.Add.9 Let X be a set with linear order $<$. Define $g : X \to \mathcal{D}$ by $g(x) = \{(-\infty, x), (-\infty, x]\}$.

Lemma R21.Add.10 Let X be a set with linear order $<$, and let $x \neq y$ be in X. Then $x < y$ if and only if $g(x) <^* g(y)$.

Proof: By definition $R(g(x)) = (-\infty, x]$ and $R(g(y) = (-\infty, y]$. Then $g(x) <^* g(y)$ if and only if $(-\infty, x]$ is a proper subset of $(-\infty, y]$, which holds if and only if $x < y$.

Corollary R21.Add.11 Let X be a set with linear order $<$. Then g is one-to-one.

Proof: This is immediate from the previous lemma.

Lemma R21.Add.12 Let X be a set with linear order $<$. Let $x \in X$ and let $D \in \mathcal{D} - g[X]$ with $g(x) <^* D$. Then there exists $t \in X$ such that $g(x) <^* g(t) <^* D$.

Proof: Since $R(g(x)) = (-\infty, x]$ is a proper subset of $R(D)$, there is $t \in R(D)$ with $t \notin (-\infty, x]$. Then $x < t$ implies $g(x) <^* g(t)$. Since $R(D)$ is a left R-set, $(-\infty, t] \subseteq R(D)$ and so $g(t) \leq^* D$. Since $D \notin g[X]$, $g(t) <^* D$.

Lemma R21.Add.13 Let X be a set with linear order $<$. Let $x \in X$ and let $D \in \mathcal{D} - g[X]$ with $D <^* g(x)$. Then there exists $t \in X$ such that $D <^* g(t) <^* g(x)$.

Proof: Since D is not in $g[X]$, $D = \{R\}$, where R is a left R-set but not a left ray. By hypothesis, $R(D) = R$ is a proper subset of $R(g(x)) = (-\infty, x]$, and $R \neq (-\infty, x)$. Thus there is $t \in (-\infty, x)$ with $t \notin R(D)$. Then $R(D)$ is a proper subset of $(-\infty, t]$ and $t < x$ so that $D <^* g(t) <^* g(x)$.

Lemma R21.Add.14 Let X be a set with linear order $<$. Let $D, E \in \mathcal{D} - g[X]$ with $D <^* E$. Then there exists $t \in X$ such that $D <^* g(t) <^* E$.

Proof: Since $R(D)$ is a proper subset of $R(E)$, there is $t \in X$ with $t \in R(E) - R(D)$. Since $R(D)$ is a left R-set, any $x \in R(D)$ must be less than t and so $R(D)$ is a proper subset of $(-\infty, t]$. Since $R(E)$ is a left R-set, $(-\infty, t] \subseteq R(E)$. Thus $D <^* g(t) \leq^* E$. Since $E \notin g[X]$, $g(t) <^* E$.

Corollary R21.Add.15 Let X be a set with linear order $<$. Then $g[X]$ is dense in $(\mathcal{D}, \tau(<^*))$.

Proof: Let D_0 be the smallest element of $\mathcal{D}$, and D_1 the largest. Possible basic open sets in $\tau(<^*)$ have one of three forms: $[D_0, D)^*$, $(D, E)^*$, or $(D, D_1]^*$, where $D, E \in \mathcal{D}$. The preceding lemmas show in the various cases that every non-empty basic open set has a non-empty intersection with $g[X]$.

Lemma R21.Add.16 Let X be a set with a linear order $<$ and let D, E be in $\mathcal{D}$. Then D, E are a consecutive pair in $\mathcal{D}$ if and only if there is a consecutive pair x, y in X with $D = g(x)$ and $E = g(y)$.

Proof: First assume x, y are a consecutive pair in X with x smaller. By R21.Add.10

$g(x) <^* g(y)$. Suppose there is F in $\mathcal{D}$ with $g(x) <^* F <^* g(y)$. If $F = g(t)$ for some $t \in X$, then by R21.Add.10 $x < t < y$, which contradicts the assumption that x, y are a consecutive pair. If $F \notin g[X]$, by R21.Add.12 there is $t \in X$ with $g(x) <^* g(t) <^* F <^* g(y)$, which yields the same contradiction. Conversely, assume D, E are a consecutive pair in $\mathcal{D}$ with D smaller. Lemmas R21.Add.12 through R21.Add.14 show that both D and E must be in $g[X]$. Let $D = g(x)$ and $E = g(y)$. If $x < t < y$, then $D = g(x) <^* g(t) <^* g(y) = E$, which contradicts the assumption that D and E are consecutive. Thus x and y are consecutive.

Corollary R21.Add.17 Let X be a set with linear order $<$. Then $(\mathcal{D}, \tau(<^*))$ is connected if and only if X has no consecutive pairs.

Proof: By R21.Add.7 $\mathcal{D}$ is order complete and so by R21.25 $(\mathcal{D}, \tau(<^*))$ is connected if and only if $\mathcal{D}$ has no consecutive elements. The conclusion now follows from the prior lemma.

The goal here is to show that, given a linearly ordered space $(X, <)$, $(\mathcal{D}, g)$ is a T_2 compactification in the class corresponding to $\mathcal{U}(<)$. The following general lemmas, which will be needed, involve a tedious multiplicity of cases. At one point, the set A is explicitly assumed to have at least two elements, a harmless assumption since, if A is $\tau(<)$-dense in X, A finite implies $A = X$, in which case the conclusions would be trivial.

Lemma R21.Add.18 Let X be a set with linear order $<$ and let $x \in X$. Then $[x, \infty) \in \tau(<)$ if and only if x is the smallest in X or x is the larger of a consecutive pair. Also $(-\infty, x] \in \tau(<)$ if and only if x is the largest in X or x is the smaller of a consecutive pair in X.

Proof: If x is the smallest in X, $[x, \infty) = X$, which is in $\tau(<)$. If t, x is a consecutive pair with x larger, $[x, \infty) = (t, \infty)$, which is in $\tau(<)$. Now assume $[x, \infty)$ is in $\tau(<)$. Because the rays are a subbasis, there are three cases: First suppose $x \in (-\infty, a) \subseteq [x, \infty)$. For $t \in X$, $t < x < a$ would imply $t \in [x, \infty)$, a contradiction. Thus $t \geq x$, i.e., x is the smallest of X. Secondly, suppose $x \in (b, \infty) \subseteq [x, \infty)$. In this case $b < t < x$ would imply $t \geq x$, a contradiction. Thus b, x are consecutive with x larger. Thirdly, suppose $x \in (-\infty, a) \cap (b, \infty) \subseteq [x, \infty)$. In this case, $b < t < x < a$ would imply $t \geq x$, a contradiction. Thus b, x are consecutive with x larger. The second assertion follows similarly.

Corollary R21.Add.19 Let X be a set with linear order $<$. Let $a, b \in X$ with $a < b$. Then $[a, b) \in \tau(<)$ if and only if a is the smallest element of X or a is the larger of a consecutive pair in X. Also $(a, b] \in \tau(<)$ if and only if b is the largest element of X or b is the smaller of a consecutive pair.

Proof: The first statement follows from the previous lemma and the set equations $[a, b) = [a, \infty) \cap (-\infty, b)$ and $[a, \infty) = [a, b) \cup (a, \infty)$. The second statement follows similarly.

Lemma R21.Add.20 Let X be a set with linear order $<$. Let $A \subseteq X$ be $\tau(<)$-dense in X and assume A contains all consecutive pairs of X. Let $\mathcal{I}$ be a finite cover of X by intervals in $\tau(<)$. Let $I \in \mathcal{I}$ be a ray with $I \neq \emptyset$. Then I can be covered by at most two intervals in $\tau(<)$, each of which is contained in some interval of $\mathcal{I}$ and all of which have all endpoints in A.

Proof: If the endpoint of I is in A, $\{I\}$ works as the required cover. First assume $I = (x, \infty)$ with $x \notin A$. Pick I_1 in $\mathcal{I}$ with $x \in I_1$. If x is the smallest in X, there is $t \in X$

with $[x,t) \subseteq I_1$. Because x is not in A and so not part of a consecutive pair, $(x,t) \neq \emptyset$ and so by density there is $a_1 \in A$ with $x < a_1 < t$. For the same reasons, there is $a_2 \in A$ with $x < a_2 < a_1$. Then $\{(-\infty, a_1), (a_2, \infty)\}$ covers I, $(-\infty, a_1) = [x, a_1) \subseteq I_1$, and $(a_2, \infty) \subseteq I$, i.e., the conclusion holds. Since I is non-empty, x cannot be the largest. In the remaining cases, since x is not part of a consecutive pair, there are $r, s \in X$ with $x \in (r,s) \subseteq I_1$. By density again, there are $a_1, a_2, a_3 \in A$ such that $r < a_1 < x < a_2 < a_3 < s$. Then $\{(a_1, a_3), (a_2, \infty)\}$ covers I, $(a_1, a_3) \subseteq I_1$, and $(a_2, \infty) \subseteq I$, i.e., the conclusion holds. Now assume $I = [x, \infty)$ with $x \notin A$. If x is the smallest, then $I = X$. Pick $a_1, a_2 \in A$ with $a_1 < a_2$. $\{(a_1, \infty), (-\infty, a_2)\}$ is the required cover. The case with x the larger of a consecutive pair cannot occur by hypothesis, since $x \notin A$. The cases when I is a left ray with endpoint not in A are similar.

Lemma R21.Add.21 Let X be a set with linear order $<$. Assume X has a largest element or a smallest element or both. Let $A \subseteq X$ be $\tau(<)$-dense in X and assume A contains all consecutive pairs of X. Let $\mathcal{I}$ be a finite cover of X by intervals in $\tau(<)$. Let $I \in \mathcal{I}$ with $I \neq \emptyset$ have the smallest or largest of X as at least one endpoint. Then I can be covered by at most three intervals in $\tau(<)$, each of which is contained in some interval of $\mathcal{I}$ and all of which have all endpoints in A.

Proof: Let x_0 be the smallest of X and x_1 the largest. (The cases when X has only one or the other are implicit in what follows.) The case with $I = [x_0, x_1] = [x_0, \infty)$ is covered by the previous lemma, as are the cases $I = (x_0, x_1] = (x_0, \infty)$, $I = [x_0, x_1) = (-\infty, x_1)$,$I = [x_0, x) = (-\infty, x)$, $I = [x_0, x] = (-\infty, x]$, $I = (x, x_1] = (x, \infty)$, and $I = [x, x_1] = [x, \infty)$. Also, if both endpoints of I are in A, the cover $\{I\}$ is as required. Now assume $I = (x_0, x_1)$. with $x_0, x_1 \notin A$ so that neither is part of a consecutive pair. There exist $I_0, I_1 \in \mathcal{I}$ with $x_0 \in I_0$ and $x_1 \in I_1$. By density pick $a_1 \in A \cap I_0$ and $a_2 \in (x_0, a_1) \cap A$. Similarly, pick $a_3 \in A \cap I_1$ and $a_4 \in (a_3, x_1) \cap A$. Then $(-\infty, a_1) = [x_0, a_1) \subseteq I_0$, $(a_2, a_4) \subseteq I$, and $(a_3, \infty) = (a_3, x_1] \subseteq I_1$. Also $\{(-\infty, a_1), (a_2, a_4), (a_3, \infty)\}$ covers I and so verifies the conclusion in this case. If $I = (x_0, x_1)$ with $x_0 \notin A$ and $x_1 \in A$, pick I_0, a_1, and a_2 as in the previous case. The set $\{(-\infty, a_1), (a_2, x_1)\}$ works. Likewise, if $I = (x_0, x_1)$ with $x_1 \notin A$ and $x_0 \in A$, pick I_1, a_3, and a_4 as above and use $\{(x_0, a_4), (a_3, \infty)\}$. For the next case assume $I = (x_0, x)$ with $x \neq x_1$ and $x_0, x \notin A$. Pick I_0, a_1, and a_2 as above. There is $I_2 \in \mathcal{I}$ with $x \in I_2$. Since x is not the largest or smallest in X and $x \notin A$, by R21.Add.19 x is not an endpoint of I_2. By density, there are $a_3, a_4, a_5 \in I_2 \cap A$ with $a_3 < a_4 < x < a_5$. Now $(-\infty, a_1) \subseteq I_0$, $(a_2, a_4) \subseteq I$, and $(a_3, a_5) \subseteq I_2$. The set $\{(-\infty, a_1), (a_2, a_4), (a_3, a_5)\}$ covers I and verifies the conclusion in this case. If $I = (x_0, x)$ with $x \neq x_1$, $x_0 \notin A$, and $x \in A$, pick I_0, a_1, a_2 as above and use $\{-\infty, a_1), (a_2, x)\}$. If $I = (x_0, x)$ with $x \neq x_1$, $x \notin A$, and $x_0 \in A$, pick I_2, a_3, a_4, a_5 as above and use $\{(x_0, a_4), (a_3, a_5)\}$. If $I = (x_0, x]$ with $x \neq x_1$ and $x_0 \notin A$, by R21.Add.19 $x \in A$. Pick I_0, a_1, a_2 as above and use $\{(-\infty, a_1), (a_2, x]\}$. Lastly, the various cases associated with (x, x_1) or $[x, x_1)$ with $x \neq x_0$ follow a similar pattern.

Lemma R21.Add.22 Let X be a set with linear order $<$. Let $A \subseteq X$ be $\tau(<)$-dense in X and assume A contains all consecutive pairs of X. Let $\mathcal{I}$ be a finite cover of X by intervals in $\tau(<)$. Let $I \in \mathcal{I}$ with $I \neq \emptyset$. Then I can be covered by at most three intervals in $\tau(<)$, each of which is contained in some interval of $\mathcal{I}$ and all of which have all endpoints in A.

Proof: By the previous two lemmas, we can assume I has two endpoints, neither of which is the smallest or largest of X. Note that by R21.Add.19 and the hypothesis, since $I \in \tau(<)$, if an endpoint of I is in I, it must also be in A. As before, if both endpoints of I are in A , $\{I\}$ can be taken as the required cover. As a first case, assume $I = (x, y)$ with neither of x, y in A. There exist I_0, I_1 in $\mathcal{I}$ such that $x \in I_0$ and $y \in I_1$. Since x, y are not in A, neither is part of a consecutive pair, nor by assumption the smallest or largest. Thus x is not an endpoint of I_0 and y is not an endpoint of I_1. There exist $r, s, t, u \in X$ such that $x \in (r, s) \subseteq I_0$ and $y \in (t, u) \subseteq I_1$. Since $(r, x) \neq \emptyset$, by density, there exists $a_1 \in (r, x) \cap A$. Similarly, there is $a_3 \in (x, \min\{s, y\}) \cap A$ and $a_2 \in (x, a_3) \cap A$. Note that $r < a_1 < x < a_2 < a_3 < \min\{s, y\}$. In the same way, there exist $a_4 \in (\max\{a_3, t\}, y) \cap A$ and $a_5 \in (y, u) \cap A$ so that $\max\{a_3, t\} < a_4 < y < a_5$. Clearly, $(a_1, a_3) \subseteq I_0$, $(a_2, a_4) \subseteq I$, and $(a_3, a_5) \subseteq I_1$. The set $\{(a_1, a_3), (a_2, a_4), (a_3, a_5)\}$ covers I and so verifies the conclusion in this case. Next suppose $I = (x, y)$ with $x \notin A$ and $y \in A$. Let $x \in I_0 \in \mathcal{I}$. As above, there exist $a_1, a_2, a_3 \in A \cap I_0$ with $a_1 < x < a_2 < a_3 < y$. The set $\{(a_1, a_3), (a_2, y)\}$ has the required properties. Note that the case $I = (x, y]$ with $x \notin A$ is almost identical, since y must be in A. Pick I_0, a_1, a_2, a_3 again and use $\{(a_1, a_3), (a_2, y]\}$. As a next case, let $I = (x, y)$ with $x \in A$ and $y \notin A$. Pick $I_1 \in \mathcal{I}$ and $a_4, a_5, a_6 \in I_1 \cap A$ with $x < a_4 < a_5 < y < a_6$ and $(a_4, a_6) \subseteq I_1$. The set $\{(x, a_5), (a_4, a_6)\}$ works. Lastly, for $I = [x, y)$ with $y \notin A$, pick the same I_1, a_4, a_5, a_6 and use the set $\{[x, a_5), (a_4, a_6)\}$.

Note that the covers constructed in the last three lemmas cannot be expected to be unique.

Lemma R21.Add.23 Let X be a set with linear order $<$. Let $A \subseteq X$ be $\tau(<)$-dense in X and assume A contains all consecutive pairs of X. Let $\mathcal{I}$ be a finite cover of X by intervals in $\tau(<)$. Then there exists $\mathcal{J}$, a finite cover of X by intervals in $\tau(<)$, such that all endpoints of each interval in $\mathcal{J}$ are in A and $\mathcal{J}$ refines $\mathcal{I}$.

Proof: Construct $\mathcal{J}$ by replacing each non-empty $I \in \mathcal{I}$ with the three (or fewer) elements of a cover as guaranteed by the three previous lemmas. Clearly $\mathcal{J}$ is finite. Because of the properties of the covers in the lemmas, $\mathcal{J}$ has the properties listed in the conclusion.

Recall the notation $<^A$ (R21.26) for the linear order inherited by a subset A and $\mathcal{U}^A(<)$ for the subspace uniformity on A from $\mathcal{U}(<)$.

Lemma R21.Add.24 Let X be a set with linear order $<$. Let $A \subseteq X$ be $\tau(<)$-dense in X and assume A contains all consecutive pairs of X. Let J be an interval in $\tau(<)$ with all endpoints of J in A. Then $J \cap A$ is an interval in $\tau(<_A)$.

Proof: As noted in the proofs of R21.28 and R21.30, for $x, y \in A$, $(x, \infty) \cap A = (x, \infty)_A$, $(x, y] \cap A = (x, y]_A$, and so on through the many cases. Thus $J \cap A$ is an A-interval. If no endpoints of J are in J, clearly the A-interval $J \cap A$ is in $\tau(<_A)$. If J is of the form $(x, y]$, since it is in $\tau(<)$, y is either the largest in X or the smaller of a consecutive pair in X. If y is the largest in X, it is the largest in A. If it is the smaller of a consecutive pair in X, by hypothesis it is the smaller of a consecutive pair in A. By R21.Add.19, $J \cap A$ is in $\tau(<_A)$. For the other cases with one or both endpoints in J, $J \cap A$ can be similarly shown to be in $\tau(<_A)$.

Lemma R21.Add.25 Let X be a set with linear order $<$. Let $A \subseteq X$ be $\tau(<)$-dense in X and assume A contains all consecutive pairs of X. Then $\mathcal{U}(<^A) = \mathcal{U}^A(<)$.

Proof: By R21.36 $\mathcal{U}(<^A) \subseteq \mathcal{U}^A(<)$. Conversely, a typical element in $\mathcal{U}^A(<)$ is of the form $U \cap (A \times A)$, where $U \in \mathcal{U}(<)$. There exists $\mathcal{I}$ a finite cover of X by intervals in $\tau(<)$ with $R(\mathcal{I}) \subseteq U$. By R21.Add.23 there is $\mathcal{J}$, a finite cover of X by intervals in $\tau(<)$ such that all endpoints of each interval in $\mathcal{J}$ are in A and $\mathcal{J}$ refines $\mathcal{I}$. Then $R(\mathcal{J})$ is in $\mathcal{U}(<)$ and, since $\mathcal{J}$ refines $\mathcal{I}$, $R(\mathcal{J}) \subseteq R(\mathcal{I})$. For each $J \in \mathcal{J}$, since the endpoints of J are in A, by R21.Add.24 $J \cap A$ is an $<^A$-interval in $\tau(<^A)$. Thus, for $\mathcal{J}_A = \{J \cap A : J \in \mathcal{J}\}$, $R(\mathcal{J}_A)$ is in $\mathcal{U}(<^A)$. Since $R(\mathcal{J}_A) = R(\mathcal{J}) \cap (A \times A) \subseteq R(\mathcal{I}) \cap (A \times A) \subseteq U \cap (A \times A)$, by the superset property, $U \cap (A \times A)$ is in $\mathcal{U}(<^A)$. Thus $\mathcal{U}^A(<) \subseteq \mathcal{U}(<^A)$.

Lemma R21.Add.26 Let X, Y be sets with linear orders $<_X$ and $<_Y$. Let $f : X \to Y$ be a bijection such that $a <_X b$ if and only if $f(a) <_Y f(b)$ for all $a, b \in X$. Then $f : (X,\mathcal{U}(<_X)) \to (Y,\mathcal{U}(<_Y))$ is a unimorphism.

Proof: Clearly f and f^{-1} map intervals to intervals. For example, given $a, b \in X$ and $s, t \in Y$, $f[(a, b]] = (f(a), f(b)]$ and $f^{-1}[(s, t)] = (f^{-1}(s), f^{-1}(t))$. As a result finite covers by intervals in the order topologies map back and forth, and so $(f \times f)[R(\mathcal{I})] = R(\mathcal{J})$, where $\mathcal{J} = \{f[I] : I \in \mathcal{I}\}$. Similarly, $(f^{-1} \times f^{-1})[R(\mathcal{J}_1)] = R(\mathcal{I}_1)$, where $\mathcal{I}_1 = \{f^{-1}[J] : J \in \mathcal{J}_1\}$. Thus both f and f^{-1} are uniformly continuous.

Proposition R21.Add.27 Let X be a set with linear order $<$. Then $(\mathcal{D}, g)$ is a T_2 compactification of $(X, \tau(<))$. It is in the compactification class of $\mathcal{U}(<)$.

Proof: By R21.Add.8 and R21.Add.15 $(\mathcal{D}, \tau(<^*))$ is compact and T_2, and $g[X]$ is $\tau(<^*)$-dense in $\mathcal{D}$. By R21.Add.10 and R21.Add.26, $g : (X,\mathcal{U}(<)) \to (g[X],\mathcal{U}(<^{*,g[X]}))$ is a unimorphism, where $<^{*,g[X]}$ denotes the linear order on $g[X]$ inherited from $<^*$. By R21.Add.16, $g[X]$ contains all consecutive pairs of $\mathcal{D}$. By R21.Add.25 $\mathcal{U}(<^{*,g[X]}) = \mathcal{U}^{g[X]}(<^*)$ and so $(\mathcal{D}, g)$ is a T_2 compactification of $(X, \tau(<))$. By R1.6a it is in the compactification class corresponding to $\mathcal{U}(<)$.

(D, g) may be referred to as the Dedekind compactification of $(X, <)$.

Example R21.Add.28 Let $X = (0, 1) \cup (2, 3)$ with the linear order inherited from the usual order on $\mathbb{R}$. X has exactly three left R-sets which are not left rays: $\emptyset$, X, and $(0, 1)$. Thus $(\mathcal{D}, g)$ is a three-point compactification of $(X, \tau(<))$ and $\mathcal{D}$ has no consecutive pairs. Now let $Y = [0, 1] \cup [2, 3]$ with linear order inherited from the usual order on $\mathbb{R}$. If $i : X \to Y$ is the inclusion map, $a < b$ if and only if $i(a) < i(b)$ and (Y, i) is a four-point compactification of $(X, \tau(<))$. (Y, i) is not equivalent to $(\mathcal{D}, g)$.

As is perhaps suggested by the previous example, the Dedekind compactification is the smallest order-generated compactification of $(X, \tau(<))$, where "order-generated" is defined as in the hypothesis of the next proposition.

Proposition R21.Add.29 Let X be a set with linear order $<$. Let (Y, f) be a T_2 compactification of $(X, \tau(<))$. Assume $\prec$ is a linear order on Y such that $\tau(\prec)$ is the topology of Y and $a < b$ if and only if $f(a) \prec f(b)$. Then $[(\mathcal{D}, g)] \leq [(Y, f)]$.

Proof: Since Y has a unique uniformity, by R21.15 it must be $\mathcal{U}(\prec)$. Let $\mathcal{V}$ be the totally bounded uniformity for $(X, \tau(<))$ that corresponds to the compactification class of (Y, f), i.e., $f : (X, \mathcal{V}) \to (f[X],\mathcal{U}^{f[X]}(\prec))$ is a unimorphism. By hypothesis and R21.Add.26 $f : (X,\mathcal{U}(<)) \to (f[X],\mathcal{U}(\prec^{f[X]}))$ is also a unimorphism. By R21.36 $\mathcal{U}(\prec^{f[X]}) \subseteq \mathcal{U}^{f[X]}(\prec)$. For $U \in \mathcal{U}(<)$, $(f \times f)[U]$ is in $\mathcal{U}(\prec^{f[X]}) \subseteq \mathcal{U}^{f[X]}(\prec)$ and so $(f \times f)^{-1}[(f \times f)[U]] = U$ is in $\mathcal{V}$, i.e., $\mathcal{U}(<) \subseteq \mathcal{V}$. By R1.5 $[(\mathcal{D}, g)] \leq [(Y, f)]$.

Example R21.Add.30 Let $\mathbf{Z}$ have the usual order. There are exactly two left R-sets which are not left rays, $\emptyset$ and $\mathbf{Z}$. In this case the Dedekind compactification is a two-point compactification.

Example R21.Add.30 Let $k \in \mathbb{N}$ with $k \geq 2$. Let $\prec_k$ be the linear order on $\mathbf{Z}$ induced by the order $<_k$ on $\mathbf{R}_k$, i.e., for $m, n \in \mathbf{Z}$, $m \prec_k n$ if and only if $f_k(m) <_k f_k(n)$. Recall that $(\mathbf{R}_k\ , f_k)$ is a T_2 compactification of $(\mathbf{Z}\ , \tau_k)$. Let $\mathcal{V}_k$ be the unique uniformity corresponding to the compactification class of $(\mathbf{R}_k\ , f_k)$. By R19.1.7 $\tau(<_k)$ is the topology of $\mathbf{R}_k$ and so, by R21.15ii, $\mathcal{U}(<_k)$ is the unique uniformity for $\mathbf{R}_k$. By R19.1.19 $f_k[\mathbf{Z}\ ]$ contains all the consecutive pairs of $\mathbf{R}_k$ and so, by R21.Add.25, $\mathcal{U}(<_k^A) = \mathcal{U}^A(<_k)$, where $A = f_k[\mathbf{Z}\ ]$. By R21.26 f_k from $(X, \mathcal{U}(\prec_k))$ to $(A, \mathcal{U}(<_k^A))$ is a unimorphism and so $\mathcal{U}(\prec_k)$ corresponds to the compactification class of $(\mathbf{R}_k\ , f_k)$, i.e., $\mathcal{V}_k = \mathcal{U}(\prec_k)$ and so $\tau_k = \tau(\mathcal{V}_k) = \tau(\prec_k)$. By R21.Add.27 $(\mathcal{D}_k, g_k)$, the Dedekind compactification of $(\mathbf{Z}\ , \prec_k)$, is equivalent to $(\mathbf{R}_k\ , f_k)$.