Extensions and Compactification

In several sections of this website (e.g., [3] and [5]), uniformities generated by equivalence relations (see Levine [1]) have been used to generate new compactifications. In this section, the relation between a compactification and larger compactifications generated with such uniformities will be examined.

Preliminary Facts

As in definition R5.2.1, given an equivalence relation E on set X, $\mathcal{U}_E$ will denote the uniformity for X with $\{E\}$ as a basis. For $A \subseteq X$, $E(A) = A \times A \cup (X - A) \times (X - A)$. Clearly $\mathcal{U}_{E(A)}$ is totally bounded.

Definition R22.1.1 Let $(X,\mathcal{U})$ be a uniform space and let $\mathcal{A}$ be a non-empty family of subsets of X. $\mathcal{U}_e(\mathcal{A})$ is defined to be $\mathcal{U} \vee (\vee\{\mathcal{U}_{E(A)} : A \in \mathcal{A}\}$.

Lemma R22.1.2 Let $(X,\mathcal{U})$ be a uniform space and let $\mathcal{A}$ be a non-empty family of subsets of X. $\mathcal{U}_e(\mathcal{A}) = \vee\{\mathcal{U}_e(\{A\}) : A \in \mathcal{A}\}$.

Proof: This is easily verified by using the definition of a supremum.

Lemma R22.1.3 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A}$ be a non-empty family of subsets of X. Then $\mathcal{U}_e(\mathcal{A})$ is separated and totally bounded.

Proof: This is immediate from P2.13.

Lemma R22.1.4 Let $(X,\mathcal{U})$ be a uniform space and let $\mathcal{A}$ be a non-empty family of subsets of X. Let $S : D \to X$ be a net. Then S is $\mathcal{U}_e(\mathcal{A})$-Cauchy if and only if S is $\mathcal{U}$-Cauchy and, for every $A \in \mathcal{A}$, S is eventually in A or S is eventually in $X - A$.

Proof: First note that, since $E(A) = A \times A \cup (X - A) \times (X - A)$, clearly S is eventually in A or S is eventually in $X - A$ if and only if $S \times S$ is eventually in $E(A)$. Now assume S is $\mathcal{U}_e(\mathcal{A})$-Cauchy. Since $\mathcal{U} \subseteq \mathcal{U}_e(\mathcal{A})$, clearly S must be $\mathcal{U}$-Cauchy. Let A be in $\mathcal{A}$. Then $E(A) \in \mathcal{U}_e(\mathcal{A})$ and so there is $d \in D$ such that $m, n \geq d$ implies $(S(m), S(n)) \in E(A)$, i.e, $S \times S$ is eventually in $E(A)$. Conversely, let $U \in \mathcal{U}$ and $A_1, A_2, \ldots, A_k \in \mathcal{A}$. It is sufficient to show that $S \times S$ is eventually in $U \cap (\cap_{i=1}^k E(A_i))$, a typical basic entourage in $\mathcal{U}_e(\mathcal{A})$. By hypothesis there is $d_0 \in D$ such that $m, n \geq d_0$ implies $(S(m), S(n)) \in U$ and for each $i \leq k$ there is $d_i \in D$ such that $m, n \geq d_i$ implies $(S(m), S(n)) \in E(A_i)$. By the directed set property, there $d \in D$ such that $d \geq d_0$ and $d \geq d_i$ for all $i \leq k$. It is easy to check that $m, n \geq d$ implies $(S(m), S(n)) \in U \cap (\cap_{i=1}^k E(A_i))$.

Lemma R22.1.5 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A}$ be a non-empty family of subsets of X. Suppose $\mathcal{U}_e(\mathcal{A})$ corresponds to the compactification class $[(Z, g)]$ and $\mathcal{U}$ corresponds to the compactification class $[(Y, f)]$.
Then $[(Y, f)] \leq [(Z, g)]$.

Proof: This is immediate from R13.1.2 since $\mathcal{U} \subseteq \mathcal{U}_e(\mathcal{A})$.

The Compactification of an Extension

The following is a general observation.

Proposition R22.2.1 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A}$ be a non-empty family of subsets of X. The compactification class of $\mathcal{U}_e(\mathcal{A})$ is the supremum of the compactification classes of $\mathcal{U}_e(\{A\})$ over all $A \in \mathcal{A}$.

Proof: This follows easily from R22.1.2 and R13.1.2.

Since the compactification class of $\mathcal{U}_e(\{A\})$ can be described as a disjoint union by R15.1.19, the class of $\mathcal{U}_e(\mathcal{A})$ is the supremum of those disjoint unions. When $\mathcal{A}$ is finite, the next few results provide a simpler description of that supremum.

Definition R22.2.2 Let X be a non-empty set and let $\mathcal{A}$ be a non-empty collection of subsets of X. Let $C(\mathcal{A}) = \Pi\{\{A, X - A\} : A \in \mathcal{A}\}$. For each $p \in C(\mathcal{A})$, let $B_p = \cap\{p(A) : A \in \mathcal{A}\}$.

Of course, B_p may be empty, but the following does hold.

Lemma R22.2.3 Let X be a non-empty set and let $\mathcal{A}$ be a non-empty collection of subsets of X. The non-empty elements of $\{B_p : p \in C(\mathcal{A})\}$ form a partition of X.

Proof: Let p, q be in $C(\mathcal{A})$ with $p \neq q$. There is $A \in \mathcal{A}$ such that $p(A) \neq q(A)$, i.e., one is A and the other is $X - A$. Thus one of B_p, B_q is contained in A and the other in $X - A$ so that $B_p \cap B_q = \emptyset$. Now let $x \in X$ and define $q \in C(\mathcal{A})$ by $q(A) = A$ if $x \in A$ and $q(A) = X - A$ if $x \notin A$. Clearly $x \in q(A)$ for all $A \in \mathcal{A}$, i.e., $x \in B_q$.

Lemma R22.2.4 Let X be a non-empty set and let $\mathcal{A}$ be a non-empty collection of subsets of X. Let E be the equivalence relation on X determined by non-empty elements of $\{B_p : p \in C(\mathcal{A})\}$. Then $E = \cap\{E(A) : A \in \mathcal{A}\}$.

Proof: Let $(x, y) \in E$. There is $p \in C(\mathcal{A})$ such that $x, y \in B_p$. For all $A \in \mathcal{A}$, both x and y are in $p(A)$, which is either A or $X - A$, so that $(x, y) \in E(A)$. Conversely, let (x, y) be in the intersection. Define $q \in C(\mathcal{A})$ by $q(A) = A$ if $(x, y) \in A \times A$ and $q(A) = X - A$ if $(x, y) \in X - A \times X - A$. Then x and y are both in B_q so that $(x, y) \in E$.

The next lemma fails without the finiteness of $\mathcal{A}$, although $\vee\{\mathcal{U}_{E(A)} : A \in \mathcal{A}\} \subseteq \mathcal{U}_E$ does hold in general.

Lemma R22.2.5 Let X be a non-empty set and let $\mathcal{A}$ be a non-empty, finite collection of subsets of X. Let E be the equivalence relation on X determined by non-empty elements of $\{B_p : p \in C(\mathcal{A})\}$. Then $\mathcal{U}_E = \vee\{\mathcal{U}_{E(A)} : A \in \mathcal{A}\}$.

Proof: Since $E = \cap\{E(A) : A \in \mathcal{A}\}$ is a finite intersection, E, and so any element of $\mathcal{U}_E$, is in $\vee\{\mathcal{U}_{E(A)} : A \in \mathcal{A}\}$. Conversely, let U be in $\vee\{\mathcal{U}_{E(A)} : A \in \mathcal{A}\}$. There is $\mathcal{F}$, a finite subset of $\mathcal{A}$, such that $\cap\{E(A) : A \in \mathcal{F}\} \subseteq U$. Then $E \subseteq \cap\{E(A) : A \in \mathcal{F}\}$ and so $U \in \mathcal{U}_E$.

In the following proposition $C(\mathcal{A})$ is used as an index set for a disjoint union instead of natural numbers. The description is simplified a bit by implicitly using the fact that the disjoint union of a space T with the empty space is homeomorphic with T.

Proposition R22.2.6 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A}$ be a finite, non-empty family of subsets of X. Assume $\mathcal{U}$ corresponds to the compactification class $[(Y, f)]$ and for $S \subseteq Y$ let $\overline{S}$ denote the closure of S in Y. Let E be the equivalence relation on X determined by non-empty elements of $\{B_p : p \in C(\mathcal{A})\}$. Let $g : X \to \coprod\{\overline{f[B_p]} : p \in C(\mathcal{A})\}$ by $g(x) = (f(x), p)$ where $x \in B_p$. Then

i)$(\coprod\{\overline{f[B_p]} : p \in C(\mathcal{A})\}, g)$ is a T_2 compactification of $(X, \tau(\mathcal{U}_e(\mathcal{A})))$.

ii)$[(\coprod\{\overline{f[B_p]} : p \in C(\mathcal{A})\}, g)]$ is the compactifition class corresponding to $\mathcal{U}_e(\mathcal{A})$.

Proof: Since $\mathcal{A}$ is finite, $C(\mathcal{A})$ is finite and so E has finitely many equivalence classes. By R15.1.19 $(\coprod\{\overline{f[B_p]} : p \in C(\mathcal{A})\}, g)$ is a T_2 compactification of $(X, \tau(\mathcal{U} \vee \mathcal{U}_E))$ and the compactification class corresponding to $\mathcal{U} \vee \mathcal{U}_E$ is $[(\coprod\{\overline{f[B_p]} : p \in C(\mathcal{A})\}, g)]$. By R22.2.5 and definition R22.1.1, the desired conclusions follow.

Lemma R22.2.7 Let X be a set and let $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ be a family of subsets of X with $A_i \subseteq A_{i+1}$ for all i between 1 and $n-1$. If $B_p \neq \emptyset$ for some $p \in C(\mathcal{A})$, then B_p is in $\{A_1, X - A_n\} \cup \{A_{i+1} - A_i : 1 \leq i \leq n-1\}$.

Proof: By induction on n. For $n = 1$, the only possible values of B_p are A_1 and $X - A_1$. Asume the conclusion holds for any chain of n subsets. Given a chain of $n+1$ subsets and $B_p \neq \emptyset$, $\cap_{i=1}^{n} p(A_i) \neq \emptyset$ and so by the induction hypothesis must be an element of $\{A_1, X - A_n\} \cup \{A_{i+1} - A_i : 1 \leq i \leq n-1\}$. If $p(A_{n+1}) = A_{n+1}$, the non-empty B_p must be in $\{A_1\} \cup \{A_{i+1} - A_i : 1 \leq i \leq n\}$. If $p(A_{n+1}) = X - A_{n+1}$, only $(X - A_{n+1}) \cap (X - A_n) = X - A_{n+1}$ could be non-empty. Thus the conclusion holds.

Corollary R22.2.8 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A}$ be a chain of n subsets of X. Assume $\mathcal{U}$ corresponds to the compactification class $[(Y, f)]$ and for $S \subseteq Y$ let $\overline{S}$ denote the closure of S in Y. Let E be the equivalence relation on X determined by non-empty elements of $\{A_1, X - A_n\} \cup \{A_{i+1} - A_i : 1 \leq i \leq n-1\}$. Let $Z = \overline{f[A_1]} \coprod \overline{f[X - A_n]} \coprod (\coprod\{\overline{f[A_{i+1} - A_i]} : i : 1 \leq i \leq n-1\})$. Let $g : X \to Z$ by $g(x) = (f(x), 1)$ if $x \in A_1$, $g(x) = (f(x), i+1)$ if $x \in A_{i+1} - A_i$ for some $1 \leq i \leq n-1$ and $g(x) = (f(x), n+1)$ if $x \in X - A_n$. Then (Z, g) is a T_2 compactification of $(X, \tau(\mathcal{U}_e(\mathcal{A})))$ and $[(Z, g)]$ is the compactifition class corresponding to $\mathcal{U}_e(\mathcal{A})$.

Proof: Because of R22.2.7, if one does some insignificant re-indexing of the disjoint union, this is immediate from R22.2.6.

Describing the Smaller Compactification

Now the opposite problem will be considered: given the compactification class of $\mathcal{U}_e(\mathcal{A})$ with $\mathcal{A}$ finite, describe the class of $\mathcal{U}$. The following continues to use $C(\mathcal{A})$ and the sets B_p as defined in R22.2.2.

Proposition R22.3.1 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A}$ be a finite, non-empty family of subsets of X. Suppose $\mathcal{U}_e(\mathcal{A})$ corresponds to the compactification class $[(Z, g)]$ and $\mathcal{U}$ corresponds to the compactification class $[(Y, f)]$. Let $\phi : Z \to Y$ be the continuous surjection such that $\phi \circ g = f$. For $S \subseteq Y$ let $\overline{S}$ denote the closure of S in Y and let c denote the closure operator in Z. Then

i) For $p \in C(\mathcal{A})$, $\phi|_{c(g[B_p])}$ is a homeomorhism from $c(g[B_p])$ onto $\overline{f[B_p]}$.

ii) For $p \neq q$ in $C(\mathcal{A})$, $c(g[B_p])$ and $c(g[B_q])$ are disjoint.

iii) Z is the disjoint union of $\{c(g[B_p]) : p \in C(\mathcal{A})\}$.

Proof: By R22.2.6, (Z, g) is equivalent to $(\coprod\{\overline{f[B_p]} : p \in C(\mathcal{A})\}, h)$, where $h : X \to \coprod\{\overline{f[B_p]} : p \in C(\mathcal{A})\}$ by $h(x) = (f(x), p)$ if $x \in B_p$. By the definition of equivalence of compactifications, there is a homeomorphism $\psi : \coprod\{\overline{f[B_p]} : p \in C(\mathcal{A})\} \to Z$ such that $\psi \circ h = g$. For i), by the continuity of ϕ any restriction is continuous and, for any $p \in C(\mathcal{A})$, $\phi[c(g[B_p]) \subseteq \overline{\phi \circ g[B_p]} = \overline{f[B_p]}$. In addition $f[B_p] = \phi \circ g[B_p]$ is a subset of the closed $\phi[c(g[B_p])]$ and so $\phi|_{c(g[B_p])}$ is onto $\overline{f[B_p]}$. By compactness of the domain and the T_2 property of the image, it is sufficient to verify that $\phi|_{c(g[B_p])}$ is one-to-one. Let $s, t \in c(g[B_p])$ with $\phi(s) = \phi(t)$. There exist nets $S : D \to B_p$ and $T : E \to B_p$ such that $g \circ S$ converges to s and $g \circ T$ converges to t. Since ϕ is continuous, $\phi \circ g \circ S$ and $\phi \circ g \circ T$ both converge to the common value $\phi(s) = \phi(t)$. Since $\phi \circ g = f$, that says $f \circ S$ and $f \circ T$ converge to $\phi(s) = \phi(t)$. Since $\psi \circ h = g$, $\psi \circ h \circ S$ converges to s and $\psi \circ h \circ S$ converges to t. Since S, T map into B_p, for all $d \in D$, $h \circ S(d) = (f \circ S(d), p)$ and for all $e \in E$, $h \circ T(e) = (f \circ T(e), p)$. Thus in the disjoint union $h \circ S$ and $h \circ T$

both converge to $(\phi(s), p) = (\phi(t), p)$. Since $\psi^{-1} \circ g = h$, both $\psi^{-1} \circ g \circ S$ and $\psi^{-1} \circ g \circ T$ converge to $(\phi(s), p) = (\phi(t), p)$. By continuity of ψ^{-1}, $\psi^{-1} \circ g \circ S$ converges to $\psi^{-1}(s)$ and $\psi^{-1} \circ g \circ T$ converges to $\psi^{-1}(t)$, i.e., $\psi^{-1}(s) = \psi^{-1}(t)$. Since ψ^{-1} is one-to-one, s=t. For ii), let $p \in C(\mathcal{A})$ and note that, since ψ^{-1} is a homeomorphism, $\psi^{-1}[c(g[B_p])]$ is the closure (in the disjoint union) of $\psi^{-1} \circ g[B_p] = h[B_p]$, which is $\overline{f[B_p]} \times \{p\}$. For $p \neq q$, $\overline{f[B_p]} \times \{p\} \cap \overline{f[B_q]} \times \{q\} = \emptyset$ and so ii) follows. Lastly, since $X = \cup\{B_p : p \in C(\mathcal{A})\}$ and the finite union $\cup\{g[B_p] : p \in C(\mathcal{A})\} = g[X]$ is dense in Z, $\cup\{c(g[B_p]) : p \in C(\mathcal{A})\} = Z$. Since this is a finite union of pairwise disjoint closed sets, Z is the disjoint union as claimed in iii).

Definition R22.3.2 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A}$ be a non-empty collection of subsets of X. Suppose $\mathcal{U}_e(\mathcal{A})$ corresponds to the compactification class $[(Z, g)]$. Define a relation $R(\mathcal{A})$ on Z by $aR(\mathcal{A})b$ provided there exists a $\mathcal{U}$-Cauchy net $S : D \to X$ such that for some $p, q \in C(\mathcal{A})$, $g \circ S$ has a subnet in $g[B_p]$ and a subnet in $g[B_q]$, one of which converges in Z to a and the other to b.

Lemma R22.3.3 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A}$ be a non-empty collection of subsets of X. Suppose $\mathcal{U}_e(\mathcal{A})$ corresponds to the compactification class $[(Z, g)]$ and $\mathcal{U}$ corresponds to the compactification class $[(Y, f)]$. Let $\phi : Z \to Y$ be the continuous surjection such that $\phi \circ g = f$. Suppose $aR(\mathcal{A})b$.
Then $\phi(a) = \phi(b)$.

Proof: Let $S : D \to X$ be a $\mathcal{U}$-Cauchy net such that for some $p, q \in C(\mathcal{A})$, $g \circ S$ has a subnet in $g[B_p]$ and a subnet in $g[B_q]$, one of which converges in Z to a and the other to b. Since S is $\mathcal{U}$-Cauchy and f is uniformly continuous from $(X,\mathcal{U})$ to Y with its unique uniformity, $f \circ S$ is Cauchy in Y, which is complete by the compactness of Y. Let t be the limit in Y of $f \circ S$. Let $T : E \to D$ have the subnet property and be such that $g \circ S \circ T$ converges to a. By continuity $\phi \circ g \circ S \circ T = f \circ S \circ T$ converges to $\phi(a)$. Since every subnet of $f \circ S$ converges to t and limits are unique in a T_2 space, $\phi(a) = t$. Similarly $\phi(b) = t$ and the conclusion follows.

To derive the converse of R22.3.3 in the case when $\mathcal{A}$ is finite, some technical facts about directed sets and nets will be needed. The first lemma constructs a directed set that is somewhat similar to the directed set used for the canonical net associated with a filter. Recall that the ordering of a directed set need not be antisymmetric.

Lemma R22.3.4 Let $(D, \geq_D)$ and $(E, \geq_E)$ be directed sets. Let D^* be the union of $\{(d, (d, e)) : (d, e) \in D \times E\}$ and $\{(e, (d, e)) : (d, e) \in D \times E\}$. Define $\geq^*$ on D^* by $(x, (d, e)) \geq^* (y, (f, g))$ if and only if $d \geq_D f$ and $e \geq_E g$. Then $(D^*, \geq^*)$ is a directed set.

Proof: The reflexivity and transitivity of $\geq^*$ follows easily since $\geq_D$ and $\geq_E$ are reflexive and transitive. For the directed set property, let$(x, (d, e))$ and $(y, (f, g))$ be in D^*. Pick $m \in D$ with $m \geq_D d$ and $m \geq_D f$. Pick $n \in E$ with $n \geq_E e$ and $n \geq_E g$. By the definition of $\geq^*$, $(m, (m, n)) \geq^* (x, (d, e))$ and $(m, (m, n)) \geq^* (y, (f, g))$ as required.

Lemma R22.3.5 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space. Suppose $\mathcal{U}$ corresponds to the compactification class $[(Y, f)]$. Assume there are $\mathcal{U}$-Cauchy nets $T_1 : D \to X$ and $T_2 : E \to X$ with $D \cap E = \emptyset$. Assume there is $y \in Y$ such that both $f \circ T_1$ and $f \circ T_2$ converge to y. Let D^* be the directed set described in R22.3.4 and define $T : D^* \to X$ by $T(x, (d, e)) = T_1(d)$ if $x = d$ and $T(x, (d, e)) = T_2(e)$ if $x = e$. Then T is $\mathcal{U}$-Cauchy.

Proof: Let $U \in \mathcal{U}$ with $U = U^{-1}$. There exist $d_0 \in D$ and $e_0 \in E$ such that $p, q \in D$ with $p \geq_D d_0, q \geq_D d_0$ implies $(T_1(p), T_1(q)) \in U$ and $r, s \in E$ with $r \geq_E e_0, s \geq_E e_0$ implies $(T_2(r), T_2(s)) \in U$. Since f is a uniform embedding, there is V in the unique uniformity for Y with $(f \times f)[U] = (f[X] \times f[X]) \cap V$. Pick $W = W^{-1}$ in the unique uniformity for Y with $W \circ W \subseteq V$. By convergence there exist $d_1 \in D$ and $e_1 \in E$ such that $d \geq_D d_1$ implies $(f(T_1(d)), y) \in W$ and $e \geq_E e_1$ implies $(f(T_2(e)), y) \in W$. Then $d \geq_D d_1$ and $e \geq_E e_1$ implies $(f(T_1(d)), f(T_2(e))) \in V$ and so $(T_1(d), T_2(e)) \in U$. Pick $d_2 \in D$ larger than both d_0, d_1 and $e_2 \in E$ larger than both e_0, e_1. It is easy to check by cases that, if $(x, (p, r))$ and $(z, (q, s))$ are both larger in D^* than $(d_2, (d_2, e_2))$, then $(T(x, (p, r)), T(z, (q, s))) \in U$. Thus T is $\mathcal{U}$-Cauchy.

At this point it is be helpful to observe that directed sets can be treated as disjoint by a technique similar to the construction of a disjoint union. Let D and E be directed sets and let $D_1 = D \times \{1\}$ and $E_2 = E \times \{2\}$. Then $D_1 \cap E_2 = \emptyset$. D_1 becomes a directed set with the order transferred from D: $(p, 1) \geq (q, 1)$ if and only if $p \geq q$. The order on E_2 is analogous. If $(X, \mathcal{U})$ is a uniform space and $S : D \to X$ is $\mathcal{U}$-Cauchy, the associated net $S_1 : D_1 \to X$ by $S_1(d, 1) = S(d)$ is clearly also $\mathcal{U}$-Cauchy, and if S is convergent, S_1 has the same limit.

Lemma R22.3.6 Let $(X, \mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A}$ be a non-empty family of subsets of X. Suppose $\mathcal{U}$ corresponds to the compactification class $[(Y, f)]$ and $p, q \in C(\mathcal{A})$. Let $T_1 : D \to B_p$ and $T_2 : E \to B_q$ be $\mathcal{U}$-Cauchy nets. Assume there is $y \in Y$ such that both $f \circ T_1$ and $f \circ T_2$ converge to y. Then there is a $\mathcal{U}$-Cauchy net in X with a subnet in B_p and another in B_q such that the images of these two subnets under f both converge to y.

Proof: By the observation preceeding this lemma one can assume without loss of generality that $D \cap E = \emptyset$. Let D^* be constructed as in R22.3.4 and let T be the $\mathcal{U}$-Cauchy net described in R22.3.5. Note that $D \times E$ is a directed set with ordering defined by $(m, r) \geq (n, s)$ if and only if $m \geq_D n$ and $r \geq_E s$. Define $u : D \times E \to D^*$ by $u(d, e) = (d, (d, e))$. For any $(x, (m, r)) \in D^*$, if $(d, e) \geq (m, r)$, $u(d, e) \geq^* u(m, r)$. Thus u is a finalizing map, i.e., it has the subnet property, i.e., $T \circ u$ is a subnet of T. In addition, $T \circ u(d, e) = T_1(d)$, which is in B_p for all (d, e), and it is easly checked that $f \circ T \circ u$ converges to y. Similarly, $v : D \times E \to D^*$ by $v(d, e) = (e, (d, e))$ has the subnet property, the subnet $T \circ v$ is always in B_q, and $f \circ T \circ v$ converges to y.

Finally the promised converse of R22.3.3 can be presented.

Proposition R22.3.7 Let $(X, \mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A}$ be a finite, non-empty family of subsets of X. Suppose $\mathcal{U}_e(A)$ corresponds to the compactification class $[(Z, g)]$ and $\mathcal{U}$ corresponds to the compactification class $[(Y, f)]$. Let $\phi : Z \to Y$ be the continuous surjection such that $\phi \circ g = f$. Suppose $\phi(a) = \phi(b)$ for some $a, b \in Z$. Then $aR(A)b$.

Proof: Let c denote the closure in Z. By R22.3.1iii there exist $p, q \in C(\mathcal{A})$ such that $a \in c(g[B_p])$ and $b \in c(g[B_q])$. Then there exist nets $S_1 : D \to B_p$ and $S_2 : E \to B_q$ such that $g \circ S_1$ converges to a and $g \circ S_2$ to b. Since g is a unimorphism onto $g[X]$, S_1 and S_2 are both $\mathcal{U}_e(\mathcal{A})$-Cauchy and so $\mathcal{U}$-Cauchy. $f \circ S_1 = \phi \circ g \circ S_1$ converges to $\phi(a)$ and $f \circ S_2 = \phi \circ g \circ S_2$ to $\phi(b)$ by continuity. Since $\phi(a) = \phi(b)$, R22.3.6 applies: there a $\mathcal{U}$-Cauchy net T in X with two subnets, say T_1 in B_p and T_2 in B_q, such that $f \circ T_1$ and

$f \circ T_2$ both converge to $\phi(a) = \phi(b)$. By R22.1.4 both T_1 and T_2 are $\mathcal{U}_e(A)$-Cauchy and so $g \circ T_1$ and $g \circ T_2$ are Cauchy and convergent in the compact Z. $g \circ T_1$ is in $g[B_p]$ and so converges to a point of $c(g[B_p])$. Since $f \circ T_1 = \phi \circ g \circ T_1$ converges to $\phi(a)$ and $\phi|_{c(g[B_p])}$ is a homeomorphism by R22.3.1i, $g \circ T_1$ converges to a. Simlarly $g \circ T_2$ converges to b. By definition R22.3.2 $aR(A)b$.

Corollary R22.3.8 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A}$ be a finite, non-empty collection of subsets of X. Suppose $\mathcal{U}_e(A)$ corresponds to the compactification class $[(Z, g)]$. Then $R(\mathcal{A})$ is an equivalence relation on Z.

Proof: By R22.3.3 and R22.3.7 $aR(A)b$ if and only if $\phi(a) = \phi(b)$, i.e., $R(\mathcal{A})$ is the equivalence relation on Z induced by the map ϕ.

The next lemma is so obvious that it perhaps should be left implicit. Note that nothing need be assumed about the map s, not even continuity.

Lemma R22.3.9 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let (Y, f) be a T_2 compactification of (X, τ). Suppose W is a topological space and $s : X \to W$ is a map. Assume $\psi : Y \to W$ is a homeomorphism such that $\psi \circ f = s$. Then (W, s) is a T_2 compactification of (X, τ) and (W, s) is equivalent to (Y, f).

Proof: As a homeomorphic image of Y, W is compact and T_2. Since ψ and f are continuous and one-to-one, so is s. Since $f[X]$ is dense in Y, $\psi[f[X]] = s[X]$ is dense in W. For $O \in \tau$, there is G open in Y such that $f[O] = G \cap f[X]$. $\psi[G]$ is open in W and $s[O] = \psi \circ f[O] = \psi[G] \cap \psi[f[X]] = \psi[G] \cap s[X]$. Thus (W, s) is a T_2 compactification of (X, τ). By the definition of equivalence and the hypotheses for ψ, (W, s) is equivalent to (Y, f).

The next proposition provides an answer to the basic question of this subsection: how can the compactification corresponding to $\mathcal{U}$ be recovered from the compactification corresponding to $\mathcal{U}_e(\mathcal{A})$ when $\mathcal{A}$ is finite?

Proposition R22.3.10 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A}$ be a finite, non-empty collection of subsets of X. Suppose $\mathcal{U}_e(\mathcal{A})$ corresponds to the compactification class $[(Z, g)]$. Let π denote the canonical projection from $Z \to Z/R(\mathcal{A})$. Then $[(Z/R(\mathcal{A}), \pi \circ g)]$ is the T_2 compactification class corresponding to $\mathcal{U}$.

Proof: Suppose $\mathcal{U}$ corresponds to the compactification class $[(Y, f)]$. Let $\phi : Z \to Y$ be the continuous surjection such that $\phi \circ g = f$ and let $R_\phi = \{(a, b) : \phi(a) = \phi(b)\}$. By general theory R_ϕ is an equivalence relation and, since ϕ must be a closed map, the compact, T_2 space Y is homeomorphic to the quotient space Z/R_ϕ via the map ψ, which maps y to the equivalence class $\phi^{-1}[\{y\}]$. By R22.3.3 and R22.3.7 $R(A) = R_\phi$ and so it is easy to check that $\psi \circ f = \pi \circ g$. R22.3.9 applies and so $[(Z/R(A), \pi \circ g)] = [(Y, f)]$, i.e., $[(Z/R(A), \pi \circ g)]$ is the T_2 compactification class corresponding to $\mathcal{U}$.

Increasing Sequences

In R21.45 a totally bounded separated uniformity for $\mathbb{N}$ properly contained in the uniformity corresponding to the remnant ring $\mathbf{R}_k$ was identified. R21.49 suggests that the larger uniformity may be an extension of the smaller by a countable chain, although that connection was not demonstrated there. This observation leads to the focus of this subsection, the extension of a uniformity by an increasing sequence, a special case of a countable chain.

In this subsection $(X,\mathcal{U})$ will denote a totally bounded separated uniform space and $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$ will denote a family of subsets of X with $A_i \subseteq A_{i+1}$ for $i \in \mathbb{N}$. For notational convenience $A_0 = \emptyset$ will be added to the sequence. Clearly $\mathcal{U}_e(\mathcal{A}) = \mathcal{U}_e(\mathcal{A} \cup \{A_0\})$ so that results obtained for the augmented sequence also apply to $\mathcal{A}$. Because $E(S) = E(X - S)$, a decreasing sequence $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$ can be routinely converted into an increasing sequence $\{X - B_i : i \in \mathbb{N}\}$ with $\mathcal{U}_e(\mathcal{B}) = \mathcal{U}_e(\{X - B_i : i \in \mathbb{N}\})$, so that results for the increasing case determine results for the decreasing case.

The notation introduced in R22.2.2 will continue to be used here. The next two lemmas show that, in the case of an increasing sequence, the non-empty sets B_p can be simply described.

Lemma R22.4.1 Let $n \in \mathbb{N}$ and $p \in C(\{A_1, \ldots, A_n\})$. If $B_p \neq \emptyset$, then either $B_p = X - A_n$ or there is i with $1 \leq i \leq n$ such that $B_p = A_i - A_{i-1}$.

Proof: By induction on n: For $n = 1$ the elements of $C(\{A_1\})$ are p, q where $p(A_1) = A_1$ and $q(A_1) = X - A_1$. $B_p = A_1 = A_1 - A_0$ and $B_q = X - A_1$. Assume the lemma holds for n and $p \in \mathcal{C}(\{A_1, A_2, \ldots, A_{n+1}\})$ with $B_p \neq \emptyset$. As a first case assume $p(A_{n+1}) = X - A_{n+1}$. For $1 \leq i \leq n$, since $A_i \subseteq A_{n+1}$, $p(A_i) = A_i$ would imply $B_p = \emptyset$. Thus $p(A_i) = X - A_i \supseteq X - A_{n+1}$ and so $B_p = X - A_{n+1}$. In the other case $p(A_{n+1}) = A_{n+1}$. Define $q \in C(\{A_1, \ldots, A_n\})$ by $q(A_i) = p(A_i)$ and apply the induction hypothesis. Since $B_p = A_{n+1} \cap (\cap_{i=1}^{n} q(A_i))$, if $\cap_{i=1}^{n} q(A_i) = X - A_n$, then $B_p = A_{n+1} - A_n$. If $\cap_{i=1}^{n} q(A_i) = A_j - A_{j-1}$ for some $1 \leq j \leq n$, since $A_j \subseteq A_{n+1}$, $B_p = A_j - A_{j-1}$ also.

Lemma R22.4.2 Let $p \in C(\mathcal{A})$. If $B_p \neq \emptyset$, then either $B_p = X - \cup_{i=1}^{\infty} A_i$ or there is $i \in \mathbb{N}$ such that $B_p = A_i - A_{i-1}$.

Proof: Assume $p(A_n) = X - A_n$ for every $n \in \mathbb{N}$. Then $B_p = \cap_{n=1}^{\infty} (X - A_n) = X - \cup_{i=1}^{\infty} A_i$. In the other case, there exists n such that $p(A_n) = A_n$. For $i > n$, $p(A_i) = X - A_i$ would imply $B_p \subseteq A_n \cap (X - A_i) = \emptyset$. Thus $p(A_i) = A_i \supseteq A_n$ for all $i > n$ and so $B_p = \cap_{i=1}^{n} p(A_i)$. Define $q \in C(\{A_1, \ldots, A_n\})$ by $q(A_i) = p(A_i)$. As in the proof of the previous lemma, since $q(A_n) = A_n$, $B_p = \cap_{i=1}^{n} q(A_i) = A_j - A_{j-1}$ for some $1 \leq j \leq n$.

Given the compactification class corresponding to $\mathcal{U}$, these lemmas allow a simpler description of a compactification in the class corresponding to $\mathcal{U}_e(\{A_1, \ldots, A_n\})$ via R22.2.6. For the problem of subsection 2 in the case of an infinite sequence, I am unable to add anything beyond R22.2.1. What follows will focus on the problem of subsection 3.

For the rest of this subsection, the following assumptions and notation will be used: (Y, f) will denote a compactification in the class corresponding to $\mathcal{U}$. For $n \in \mathbb{N}$, $Z_n = (\coprod_{i=1}^{n} \overline{f[A_i - A_{i-1}]}) \coprod \overline{f[X - A_n]}$ and $g_n : X \to Z_n$ by $g_n(x) = (f(x), j)$, where $j = n + 1$ if $x \in X - A_n$ and $j = i$ if $x \in A_i - A_{i-1}$. By the two previous lemmas and R22.2.6 (with the omission of clearly empty components and the obvious reindexing), (Z_n, g_n) is a compactification in the class corresponding to $\mathcal{U}_e(\{A_1, \ldots, A_n\})$. In addition, for $m \geq k$, $[(Z_m, g_m)] \geq [(Z_k, g_k)]$ and so there exists a unique continuous surjection ϕ_{mk} from Z_m to Z_k such that $\phi_{mk} \circ g_m = g_k$.

Next let $Z = \{p \in \prod_{i=1}^{\infty} Z_i : m \geq k \Rightarrow p(k) = \phi_{mk}(p(m))\}$ and $g : X \to Z$ by $g(x) = y_x$, where $y_x(i) = g_i(x)$. With all initial segments from $\mathbb{N}$ as cofinal set and (Z_n, g_n) as the compactification corresponding to $\{1, 2, \ldots, n\}$, by R13.Add.8, (Z, g) is a compactification in the class corresponding to $\mathcal{U}_e(\mathcal{A})$.

Finally, since $\mathcal{U} \subseteq \mathcal{U}_e(\{A_i : 1 \leq i \leq n\}) \subseteq \mathcal{U}_e(\mathcal{A})$ for each $n \in \mathbb{N}$, let $\psi_n : Z \to Z_n$

and $\phi : Z \to Y$ be the unique continuous surjections such that $\psi_n \circ g = g_n$ and $\phi \circ g = f$. Let $\phi_n : Z_n \to Y$ be the unique continuous surjection such that $\phi_n \circ g_n = f$.

Lemma R22.4.3 For every $n \in \mathbb{N}$, $\phi_n \circ \psi_n = \phi$.

Proof: Fix n. $\phi_n \circ \psi_n \circ g = \phi_n \circ g_n = f$. Since ϕ is the unique such map, it follows immediately that $\phi_n \circ \psi_n = \phi$.

The next three lemmas provide simple descriptions of the maps ψ_n, ϕ_n, and ϕ_{mk}.

Lemma R22.4.4 Let $n \in \mathbb{N}$. ψ_n is the resriction to Z of the natural projection from $\prod_{i=1}^{\infty} Z_i$ onto Z_n.

Proof: Let π_n denote the projection restricted to Z. For $x \in X$, $\pi_n(g(x)) = \pi_n(y_x) = y_x(n) = g_n(x)$, i.e., $\pi_n \circ g = g_n$. Since ψ_n is the unique such map, the conclusion follows.

Lemma R22.4.5 Let $n \in \mathbb{N}$ and let $(y, \alpha) \in Z_n$. Then $\phi_n((y, \alpha)) = y$.

Proof: Define $\rho : Z_n \to Y$ by $\rho((t, \beta)) = t$. Let O be an open set in Y. Then $O = (\cup_{i=1}^{n}(O \cap \overline{f[A_i - A_{i-1}]})) \cup (O \cap \overline{f[X - A_n]})$ since $f[X] = (\cup_{i=1}^{n} f[A_i - A_{i-1}]) \cup f[X - A_n]$ and $f[X]$ is dense. Moreover, as a set, $Z_n = (\cup_{i=1}^{n} \overline{f[A_i - A_{i-1}]} \times \{i\}) \cup \overline{f[X - A_n]} \times \{n+1\}$ and so it is easy to check that $\rho^{-1}[O] = (\cup_{i=1}^{n}(O \cap \overline{f[A_i - A_{i-1}]}) \times \{i\}) \cup (O \cap \overline{f[X - A_n]}) \times \{n+1\}$, which is open in the disjoint union. Thus ρ is continuous. For $x \in X$, $\rho \circ g_n(x) = \rho((f(x), j))$, where j depends on the location of x. Thus $\rho \circ g_n = f$ and, by the density of $f[X]$, ρ is onto. Since ϕ_n is the unique such map, $\phi_n = \rho$.

Lemma R22.4.6 Let $k, m \in \mathbb{N}$ with $k \leq m$. Let (t, α) be in Z_m. Then

i) $\alpha \leq k \Rightarrow \phi_{mk}(t, \alpha) = (t, \alpha)$.

ii) $\alpha > k \Rightarrow \phi_{mk}(t, \alpha) = (t, k+1)$.

Proof: Define $\rho : Z_m \to Z_k$ by $\rho(t, \alpha) = (t, \alpha)$ if $\alpha \leq k$ and $\rho(t, \alpha) = (t, k+1)$ if $\alpha > k$. It is sufficient for both parts to show $\phi_{mk} = \rho$. To see that ρ is continuous, let O be open in Z_k and without loss of generality assume O is contained in one of the components of the disjoint union. Then there is G open in Y such that either $O = (G \cap \overline{f[A_i - A_{i-1}]}) \times \{i\}$ for some $1 \leq i \leq k$ or $O = (G \cap \overline{f[X - A_k]}) \times \{k+1\}$. In the first case, $\rho^{-1}[O] = (G \cap \overline{f[A_i - A_{i-1}]}) \times \{i\}$, which is open in Z_m. In the second case, first note that the chain property implies $X - A_k = (\cup_{i=k+1}^{m}(A_i - A_{i-1})) \cup (X - A_m)$. Since the union is finite, $\overline{f[X - A_k]} = (\cup_{i=k+1}^{m} \overline{f[A_i - A_{i-1}]}) \cup \overline{f[X - A_m]}$. It follows easily that $\rho^{-1}[O] = (\cup_{i=k+1}^{m} G \cap \overline{f[A_i - A_{i-1}]} \times \{i\}) \cup (G \cap \overline{f[X - A_m]} \times \{m+1\}$, which is open in Z_m as needed for continuity. Next let $x \in X$. If $x \in A_k$, then $g_m(x) = (f(x), \alpha)$ for some $\alpha \in \{1, 2, \ldots, k\}$ and $\rho \circ g_m(x) = (f(x), \alpha) = g_k(x)$. If $x \notin A_k$, then $g_m(x) = (f(x), \alpha)$ for some $\alpha \in \{k+1, \ldots, m+1\}$ and $\rho \circ g_m(x) = (f(x), k+1) = g_k(x)$. Thus ρ is continuous, $\rho \circ g_m = g_k$, and ρ is onto since its closed image contains the dense $g_k[X]$. Since ϕ_{mk} is the unique such map, $\phi_{mk} = \rho$.

The previous lemma allows a simple description of Z and ϕ.

Corollary R22.4.7 Let $z \in Z$. Then there is $y \in Y$ such that, for every $n \in \mathbb{N}$, $z(n) = (y, \alpha_n)$ for some $\alpha_n \in \{1, 2, \ldots, n+1\}$.

Proof: Let $y \in Y$ be such that $z(1) = (y, j)$ where $j \in \{1, 2\}$ and let $n \in \mathbb{N}$. Since $z(n) \in Z_n$, $z(n) = (w, k)$ for some $k \in \{1, 2, \ldots, n+1\}$. Since $z \in Z$, $(y, j) = z(1) = \phi_{n1}(z(n))$. By R22.4.6 $\phi_{n1}((w, k)) = (w, l)$, with l determined by k. Thus $w = y$ as required.

Corollary R22.4.8 Let $z \in Z$. Suppose $y \in Y$ is the element such that, for every $n \in \mathbb{N}$, $z(n) = (y, \alpha_n)$ for some $\alpha_n \in \{1, 2, \ldots, n+1\}$. Then $\phi(z) = y$.

Proof: Fix $n \in \mathbb{N}$. Using R22.4.3, R22.4.4 and R22.4.5, $\phi(z) = \phi_n \circ \psi_n(z) = \phi_n(z(n)) = \phi_n((y, \alpha_n)) = y$.

If one tries to follow the pattern of the third subsection for the case of an increasing sequence, a complication arises: as the following example shows, Y may contain elements which are not in $\overline{f[B_p]}$ for any $p \in C(\mathcal{A})$.

Example R22.4.9 Let $X = (0, 1)$ and $Y = [0, 1]$ be intervals of reals with the usual topology. Let $f : X \to Y$ be the inclusion map. (Y, f) is a T_2 compactification of X. Define a sequence of subsets of X by letting, for each $i \in \mathbb{N}$, $A_i = (0, 1-1/i)$. Here, with $A_0 = \emptyset$ as usual, $A_1 - A_0 = \emptyset$, $A_2 - A_1 = A_2$ and, for $j \geq 3$, $A_j - A_{j-1} = [1-1/(j-1), 1-1/j)$. Note that by R22.4.2 these are the only non-empty values of B_p since $X - \cup_{n=1}^{\infty} A_n = \emptyset$. Here $1 \notin \overline{f[A_j - A_{j-1}]}$ for all j but $1 \in \cap_{n=1}^{\infty} \overline{f[X - A_n]}$.

To deal with the complication in the current context, it is necessary to enlarge the relation defined in R22.3.2 as follows.

Definition R22.4.10 Let $S : D \to X$ be a net. S is $\mathcal{A}$-compatible provided S is eventually in $X - A_n$ for every n or there exists $j \in \mathbb{N}$ such that S is eventually in $A_j - A_{j-1}$.

Definition R22.4.11 Let $a, b \in Z$. Then $aR_1(\mathcal{A})b$ if and only if there is a $\mathcal{U}$-Cauchy net $S : D \to X$ with two $\mathcal{A}$-compatible subnets, one whose g-image converges to a and the other whose g-image converges to b.

Lemma R22.4.12 $R(\mathcal{A}) \subseteq R_1(\mathcal{A})$.

Proof: Let $(a, b) \in R(\mathcal{A})$. By R22.3.2 there is a $\mathcal{U}$-Cauchy net $S : D \to X$ and $p, q \in C(\mathcal{A})$ such that $g \circ S$ has two subnets, one in $g[B_p]$ converging to a and the other in $g[B_q]$ converging to b. The subnet in $g[B_p]$ determines a subnet of S whose g-image converges to a. By R22.4.2 this subnet is either in $A_j - A_{j-1}$ for some j or it is in $\cap_{n=1}^{\infty}(X - A_n)$, i.e., it is $\mathcal{A}$-compatible. In the same way the subnet in $g[B_q]$ determines an $\mathcal{A}$-compatible subnet of S whose g-image converges to b. Thus $(a, b) \in R_1(\mathcal{A})$.

If the sequence terminates, $\mathcal{A}$ is a finite set. Let m be such that $A_i = A_m$ for all $i \geq m$. Here a net eventually in $X - A_n$ for all n must eventually be in $X - A_m = X - \cup_{i=1}^{\infty} A_i$, which by R22.4.2 is a possible value of B_p. It follows easily that $R(\mathcal{A}) = R_1(\mathcal{A})$. In general equality of the relations does not hold, as will be illustrated after more theoretical development.

Lemma R22.4.13 Let $a, b \in Z$ with $aR_1(\mathcal{A})b$. Then $\phi(a) = \phi(b)$.

Proof: Let $S : D \to X$ be a $\mathcal{U}$-Cauchy net with two $\mathcal{A}$-compatible subnets, one whose g-image converges to a and the other whose g-image converges to b. Since S is $\mathcal{U}$-Cauchy and f is uniformly continuous from $(X, \mathcal{U})$ to Y with its unique uniformity, $f \circ S$ is Cauchy in Y, which is complete by the compactness of Y. Let t be the limit in Y of $f \circ S$. Let $T : E \to D$ have the subnet property and be such that $S \circ T$ is the subnet with $g \circ S \circ T$ converging to a. By continuity $\phi \circ g \circ S \circ T = f \circ S \circ T$ converges to $\phi(a)$. Since every subnet of $f \circ S$ converges to t and limits are unique in a T_2 space, $\phi(a) = t$. Similarly $\phi(b) = t$ and the conclusion follows.

The next definition leads to R22.4.19, which is a modification of R22.3.6 needed for the current context.

Definition R22.4.14 Let $S : D \to X$ be an $\mathcal{A}$-compatible net. The degree of $\mathcal{A}$-compatibility, $\deg_{\mathcal{A}}(S)$, is defined to be j if S is eventually in $A_j - A_{j-1}$ and ∞ if S is eventually in $X - A_n$ for every n.

The following lemma illustrates the previous definition and establishes a key fact.

Lemma R22.4.15 Let $y \in \cap_{n=1}^{\infty} \overline{f[X - A_n]}$. Then there is an $\mathcal{A}$-compatible net $S : D \to X$ such that $\deg_{\mathcal{A}}(S) = \infty$ and $f \circ S$ converges to y.

Proof: Let D_1 be the set of open Y-neighborhoods of y ordered by reverse inclusion, a directed set. For each $j \in \mathbb{N}$ let the net $S_j : D_1 \to X - A_j$ be constructed in the standard way: For $O \in D_1$, $O \cap f[X - A_j] \neq \emptyset$ and so let $S_j(O)$ be any element of $X - A_j$ with $f(S_j(O)) \in O$. As usual $f \circ S_j$ converges to y. Let $D = D_1 \times \mathbb{N}$ with ordering defined by $(O, j) \preceq (G, k)$ if and only if $O \supseteq G$ and $j \leq k$. With this ordering D is a directed set. Define $S : D \to X$ by $S(O, j) = S_j(O)$. It is easy to check that S is an $\mathcal{A}$-compatible net, $\deg_{\mathcal{A}}(S) = \infty$, and $f \circ S$ converges to y.

Lemma R22.4.16 Let $z \in Z$. Then there is an $\mathcal{A}$-compatible net $S : D \to X$ such that $g \circ S$ converges to z.

Proof: By R22.4.7 there is $y \in Y$ such that, for every $n \in \mathbb{N}$, $z(n) = (y, \alpha_n)$ for some $\alpha_n \in \{1, 2, \ldots, n+1\}$. As a first case, suppose $\alpha_n = n+1$ for all n, i.e., $y \in \cap_{n=1}^{\infty} \overline{f[X - A_n]}$. By R22.4.15, there is an $\mathcal{A}$-compatible net $S : D \to X$ such that $\deg_{\mathcal{A}}(S) = \infty$ and $f \circ S$ converges to y. To show that $g \circ S \to z$, it is sufficient to show $\psi_n \circ g \circ S$ converges to $z(n)$ for all n. Fix $n \in \mathbb{N}$. There is $d_0 \in D$ such that $d \geq d_0$ implies $S(d) \in X - A_n$. For $d \geq d_0$, $\psi_n \circ g \circ S(d) = g \circ S(d)(n) = g_n(S(d)) = (f \circ S(d), n+1)$. Thus $\psi_n \circ g \circ S$ is eventually in $\overline{f[X - A_n]} \times \{n+1\}$, a clopen subset of Z_n. Since $f \circ S$ converges to y, it is easy to chsck that $\psi_n \circ g \circ S$ converges to $(y, n+1) = z(n)$. As a second case, assume there is n such that $\alpha_n \neq n+1$. Let m be the smallest of the non-empty set $\{n : \alpha_n \neq n+1\}$. First it is claimed that $\alpha_m = m$. Let $\alpha_m = j$, where $1 \leq j \leq m$. If $m = 1$, clearly $m = j$. If $m > 1$, by the choice of m, $\alpha_{m-1} = m$. Since $z \in Z$, $\phi_{mm-1}((y, j)) = \phi_{mm-1}(z(m)) = z(m-1) = (y, m)$. By R22.4.6, this can only happen when $j = m$, which verifies the claim. Now since $z(m) = (y, m)$ in Z_m, (y, m) must be in $\overline{f[A_m - A_{m-1}]} \times \{m\}$ and so there is a net $S : D \to A_m - A_{m-1}$ such that $f \circ S$ converges to y. Clearly S is $\mathcal{A}$-compatible of degree m. To see that $g \circ S$ converges to z, it is sufficient to show $\psi_n \circ g \circ S$ converges to $z(n)$ for all n. Fix $n \in \mathbb{N}$. If $n < m$, by the choice of m, $z(n) = (y, n+1)$ so that $y \in \overline{f[X - A_n]}$. Since $n \leq m-1$, $A_n \subseteq A_{m-1}$ and so $A_m - A_{m-1} \subseteq X - A_n$. Thus $\psi_n \circ g \circ S(d) = g \circ S(d)(n) = g_n(S(d)) = (f \circ S(d), n+1)$ for all $d \in D$ and so $\psi_n \circ g \circ S$ converges to $(y, n+1)$ as required. If $n \geq m$, since $z \in Z$, $\phi_{nm}((y, \alpha_n)) = \phi_{nm}(z(n)) = z(m) = (y, m)$. By R22.4.6 $\alpha_n > m$ cannot occur and so $\phi_{nm}((y, \alpha_n)) = (y, \alpha_n)$, i.e., $\alpha_n = m$. Here $\psi_n \circ g \circ S(d) = g \circ S(d)(n) = g_n(S(d)) = (f \circ S(d), m)$ for all $d \in D$ and so $\psi_n \circ g \circ S$ converges to $(y, m) = z(n)$ in this case as well.

Lemma R22.4.17 Let $S : D \to X$ be an $\mathcal{A}$-compatible net with $\deg_{\mathcal{A}}(S) = m$. Assume $z \in Z$ with $g \circ S$ converging to z. Let $y \in Y$ be such that, for every $n \in \mathbb{N}$, $z(n) = (y, \delta_n)$, where $\delta_n \in \{1, \ldots, n, n+1\}$. Then $\delta_n = m$ for $n \geq m$ and $\delta_n = n+1$ for $n < m$.

Proof: For all n, $\psi_n \circ g \circ S$ converges to $\psi_n(z) = z(n)$. Pick $d_0 \in D$ such that $d \geq d_0 \Rightarrow S(d) \in A_m - A_{m-1}$. For $n \geq m$ and $d \geq d_0$, $\psi_n \circ g \circ S(d) = g \circ S(d)(n) = g_n(S(d)) = (f \circ S(d), m)$, which is in the clopen subset $\overline{f[A_m - A_{m-1}]} \times \{m\}$ of Z_n. Thus the limit $z(n) = (y, \delta_n)$ must be in $\overline{f[A_m - A_{m-1}]} \times \{m\}$, i.e., $\delta_n = m$. For $n < m$ and $d \geq d_0$, $(f \circ S(d), m)$ is in the clopen subset $\overline{f[X - A_n]} \times \{n+1\}$ of Z_n. Thus the limit

$z(n) = (y, \delta_n)$ must be in $\overline{f[X - A_n]} \times \{n+1\}$, i.e., $\delta_n = n+1$.

Lemma R22.4.18 Let $S : D \to X$ be an $\mathcal{A}$-compatible net with $\deg_{\mathcal{A}}(S) = \infty$. Assume $z \in Z$ with $g \circ S$ converging to z. Let $y \in Y$ be such that, for every $n \in \mathbb{N}$, $z(n) = (y, \delta_n)$, where $\delta_n \in \{1, \ldots, n, n+1\}$. Then $\delta_n = n+1$ for every n.

Proof: Fix $n \in \mathbb{N}$. Then $\psi_n \circ g \circ S$ converges to $\psi_n(z) = z(n)$. There is $d_n \in D$ such that $d \geq d_n \Rightarrow S(d) \in X - A_n$. Since $g \circ S(d) = g_n(S(d)) = (f(S(d)), n+1)$ for $d \geq d_n$, $\psi_n \circ g \circ S$ is eventually in $\overline{f[X - A_n]} \times \{n+1\}$, a clopen subset of Z_n. Thus $z(n) = (y, n+1)$ as claimed.

Lemma R22.4.19 Let $T_1 : D \to X$ and $T_2 : E \to X$ be $\mathcal{A}$-compatible nets and assume $y \in Y$ such that $f \circ T_1$ and $f \circ T_2$ both converge to y. Then there is a $\mathcal{U}$-Cauchy net in X with two $\mathcal{A}$-compatible subnets such that the f-image of each subnet converges to y, the first subnet has the same degree of $\mathcal{A}$-compatibility as T_1, and the second subnet has the same degree of $\mathcal{A}$-compatibility as T_2.

Proof: By the observation preceeding R22.3.6 one can assume without loss of generality that $D \cap E = \emptyset$. Let D^* be constructed as in R22.3.4 and let T be the $\mathcal{U}$-Cauchy net described in R22.3.5. Note that $D \times E$ is a directed set with ordering defined by $(m, r) \geq (n, s)$ if and only if $m \geq_D n$ and $r \geq_E s$. Define $u : D \times E \to D^*$ by $u(d, e) = (d, (d, e))$. For any $(x, (m, r)) \in D^*$, if $(d, e) \geq (m, r)$, $u(d, e) \geq^* (x, (m, r))$. Thus u is a finalizing map, i.e., it has the subnet property, i.e., $T \circ u$ is a subnet of T. In addition, $T \circ u(d, e) = T_1(d)$, which is $\mathcal{A}$-compatible of the same degree as T_1, and it is easly checked that $f \circ T \circ u$ converges to y. Similarly, $v : D \times E \to D^*$ by $v(d, e) = (e, (d, e))$ has the subnet property, the subnet $T \circ v$ is $\mathcal{A}$-compatible of the same degree as T_2, and $f \circ T \circ v$ converges to y.

Lemma R22.4.20 Let $a, b \in Z$ with $\phi(a) = \phi(b)$. Then $aR_1(\mathcal{A})b$.

Proof: Let $y = \phi(a) = \phi(b)$. Apply R22.4.16 to obtain $\mathcal{A}$-compatible nets $T_1 : D_1 \to X$ and $T_2 : D_2 \to X$ such that $g \circ T_1$ converges to a and $g \circ T_2$ converges to b. By R22.4.7 and R22.4.8 $a(n) = (y, \alpha_n)$ and $b(n) = (y, \beta_n)$, where α_n and β_n are determined by the degrees of the nets as in R22.4.17 or R22.4.18. By the continuity of ϕ, $\phi \circ g \circ T_1 = f \circ T_1$ converges to $\phi(a) = y$. In the same way $f \circ T_2$ also converges to y. By R22.4.19 there is a $\mathcal{U}$ Cauchy net $S : D \to X$ with two $\mathcal{A}$-compatible subnets such that the f-images of both converge to y, the first has the same degree of $\mathcal{A}$-compatibility as T_1, and the second has the same degree of $\mathcal{A}$-compatibility as T_2. As subnets of S, both are $\mathcal{U}$ Cauchy and by R22.1.4 both are $\mathcal{U}_e(\mathcal{A})$ Cauchy. Thus the g-image of the first subnet is Cauchy in Z. Let $\overline{a}$ be its limit. Since the first subnet has the same degree of $\mathcal{A}$-compatibility as T_1, by R22.4.17 or R22.4.18 $\overline{a} = a$. In the same way the g-image of the second subnet converges to b. By definition $aR_1(\mathcal{A})b$.

Lemma R22.4.21 $R_1(\mathcal{A})$ is an equivalence relation.

Proof: Let $R_\phi = \{(a, b) : \phi(a) = \phi(b)\}$, an equivalence relation. By R22.4.13 $R_1(\mathcal{A})$ is a subset of R_ϕ. R22.4.20 says $R_\phi \subseteq R_1(\mathcal{A})$ as well.

The next example, which is a digression, provides an example showing $R(\mathcal{A})$ may be a proper subset of $R_1(\mathcal{A})$, as mentioned after R22.4.12.

Example R22.4.22 Let $X = [0, 1)$ and $Y = [0, 1]$ with each having the usual uniformity, $\mathcal{U}$ for X and $\mathcal{V}$ for Y. Let $f : X \to Y$ be the inclusion map. (Y, f) is in the compactification class corresponding to $\mathcal{U}$. For $i \in \mathbb{N}$ let $A_i = [0, 1 - \frac{1}{n})$. With $A_0 = \emptyset$, $A_1 - A_0 = \emptyset$ and, for $i \geq 2$, $A_i - A_{i-1} = [1 - \frac{1}{i-1}, 1 - \frac{1}{i})$. Let (Z, g) be as described above for these values of X, Y, and $\mathcal{A}$. Let $z \in \prod_{i=1}^{\infty} Z_i$ be defined by $z(n) = (1, n+1)$ for all n. It easily follows from R22.4.6 that $z \in Z$. Let S be a Cauchy net in some non-empty B_p. By R22.4.2 the only non-empty values of B_p are $A_i - A_{i-1}$ for $i \geq 2$, since $X = \cup_{i=1}^{\infty} A_i$. Thus $f \circ S$ converges to some $t < 1$ and $g \circ S$ converges to w where $w(n) = (t, \alpha_n)$ for all n. In particular, no such $g \circ S$ converges to z and so $(z, z) \notin R(\mathcal{A})$, i.e., $R(\mathcal{A})$ is not reflexive. By R22.4.21 $R(\mathcal{A}) \neq R_1(\mathcal{A})$.

The proof of the next proposition is almost identical to the proof of R22.3.10.

Proposition R22.4.23 Let π denote the canonical projection from $Z \to Z/R_1(\mathcal{A})$. Then $[(Z/R_1(\mathcal{A}), \pi \circ g)]$ is the T_2 compactification class corresponding to $\mathcal{U}$.

Proof: Again let $R_\phi = \{(a, b) : \phi(a) = \phi(b)\}$. As in the proof of the previous lemma $R_1(\mathcal{A}) = R_\phi$. By general theory R_ϕ is an equivalence relation and, since ϕ must be a closed map, the compact, T_2 space Y is homeomorphic to the quotient space Z/R_ϕ via the map ψ, which maps y to the R_ϕ-equivalence class $\phi^{-1}[\{y\}]$. It is easy to check that $\psi \circ f = \pi \circ g$. R22.3.9 applies and so $[(Z/R_1(\mathcal{A}), \pi \circ g)] = [(Y, f)]$, i.e., $[(Z/R_1(\mathcal{A}), \pi \circ g)]$ is the T_2 compactification class corresponding to $\mathcal{U}$.

The previous proposition can, unsurprisingly, be modified to apply to an arbitrary representation of the compactification class corresponding to $\mathcal{U}_e(\mathcal{A})$. The rest of this subsection fills in the routine details.

Definition R22.4.24 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A}$ be a countable, non-empty, increasing collection of subsets of X. Suppose $\mathcal{U}_e(\mathcal{A})$ corresponds to the compactification class $[(W, h)]$. Let $a, b \in W$. Then $aR_1^W(\mathcal{A})b$ if and only if there is a $\mathcal{U}$-Cauchy net $S : D \to X$ with two $\mathcal{A}$-compatible subnets, one whose h-image converges to a and the other whose h-image converges to b.

Clearly, the relation $R_1(\mathcal{A})$ defined in R22.4.11 is now denoted $R_1^Z(\mathcal{A})$.

Lemma R22.4.25 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A}$ be a countable, non-empty, increasing collection of subsets of X. Suppose $\mathcal{U}_e(\mathcal{A})$ corresponds to the compactification class $[(W, h)]$. Then $R_1^W(\mathcal{A})$ is an equivalence relation.

Proof: Since (Z, g) is in the compactification class $[(W, h)]$, there is a homeomorphism $\psi : W \to Z$ such that $\psi \circ h = g$. Let $a, b \in W$ with $aR_1^W(\mathcal{A})b$. By definition there is a $\mathcal{U}$-Cauchy net $S : D \to X$ with $\mathcal{A}$-compatible subnets T_1 and T_2 such that $h \circ T_1$ converges to a and $h \circ T_2$ converges to b. By continuity $\psi \circ h \circ T_1 = g \circ T_1$ converges to $\psi(a)$ and $\psi \circ h \circ T_2$ converges to $\psi(b)$. Thus $\psi(a)R_1^Z(\mathcal{A})\psi(b)$. Similarly, for $c, d \in Z$, $cR_1^Z(\mathcal{A})d$ implies $\psi^{-1}(c)R_1^W(\mathcal{A})\psi^{-1}(d)$. Since $R_1^Z(\mathcal{Z})$ is an equivalence relation by R22.4.21, it follows easily that $R_1^W(\mathcal{A})$ is also an equivalence relation.

Proposition R22.4.26 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A}$ be a countable, non-empty, increasing collection of subsets of X. Suppose $\mathcal{U}_e(\mathcal{A})$ corresponds to the compactification class $[(W, h)]$. Let ρ denote the canonical projection from W to $W/R_1^W(\mathcal{A})$. Then $(W/R_1^W(\mathcal{A}), \rho \circ h)$ is a T_2 compactification of $(X, \tau(\mathcal{U}))$ and $[(W/R_1^W(\mathcal{A}), \rho \circ h)]$ is the T_2 compactification class corresponding to $\mathcal{U}$.

Proof: By R22.3.9 and R22.4.23 it is sufficient to find a homeomorhism $\overline{\psi}$ from $W/R_1^W(\mathcal{A})$ to $Z/R_1^Z(\mathcal{A})$ such that $\overline{\psi} \circ \rho \circ h = \pi \circ g$, where π is the canonical projection from Z to $Z/R_1(\mathcal{A})$. Since (Z, g) is in the compactification class $[(W, h)]$, there is a homeomorphism $\psi : W \to Z$ such that $\psi \circ h = g$. Define $\overline{\psi} : W/R_1^W(\mathcal{A}) \to Z/R_1^Z(\mathcal{A})$ by $\overline{\psi}([w]_W) = [\psi(w)]_Z$. As in the proof of R22.4.25 $aR_1^W(\mathcal{A})b$ if and only if $\psi(a)R_1^Z(\mathcal{A})\psi(b)$.

From this fact it follows easily that $\overline{\psi}$ is in fact a function and is one-to-one. Since ψ is onto, $\overline{\psi}$ is also onto. For $w \in W$, $\overline{\psi} \circ \rho(w) = \overline{\psi}([w]_W) = [\psi(w)]_Z = \pi \circ \psi(w)$, i.e., $\overline{\psi} \circ \rho = \pi \circ \psi$, which is continuous. Since ρ is a quotient map, this implies $\overline{\psi}$ is continuous. Since $\psi \circ h = g$, it also implies $\overline{\psi} \circ \rho \circ h = \pi \circ g$. Lastly, since the domain is compact and the image T_2, $\overline{\psi}$ is also closed. Thus $\overline{\psi}$ is the required homeomorphism.

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