Special Cases of Extensions

In [11] some general facts about compactifications corresponding to extensions of a totally bounded, separated uniformity by a uniformity generated by equivalence relations were presented. Here cases involving the remnant rings, finite point compactifications, Stone-Čech compactifications, and zero-dimensional compactifications of discrete spaces are considered. Throughout this section the generalized ordering of compactifications described in [7] is used.

The Remnant Rings

In the introduction to the fourth subsection of [11] it was suggested that the uniformity corresponding to the remnant ring $\mathbf{R}_k$ may be an extension of a totally bounded separated uniformity for $\mathbb{N}$ properly contained in it. The first goal here is to verify that conjecture.

Let's begin by recalling some facts and notation from [8] and [10]. Let $k \in \mathbb{N}$ with $k \geq 2$. The map f_k from $\mathbf{Z}$ into $\mathbf{R}_k$ is defined by letting $f_k(z)$ be the ultrafilter corresponding to $\{x_n\}$, where x_n is the element in $\{1, 2, \ldots, k^n\}$ such that $x_n \equiv z \bmod k^n$. As in R16.15, $(\mathbf{R}_k, f_k)$ is a T_2 compactification of $(\mathbf{Z}, \tau_k)$, where τ_k is the topology with clopen basis $\mathcal{B}_k$ consising of all equivalence classes mod k^n over all $n \in \mathbb{N}$.

Let $\mathcal{V}_k$ be the uniformity for $\mathbf{Z}$ corresponding to $[(\mathbf{R}_k, f_k)]$, let $\tilde{\mathcal{V}}_k$ be the subspace uniformity on $\mathbb{N}$ induced by $\mathcal{V}_k$, and let $\tilde{\tau}_k$ be the subspace topology on $\mathbb{N}$ from τ_k. By R21.42 $(\mathbf{R}_k, \tilde{f}_k)$ is a T_2 compactification of $(\mathbb{N}, \tilde{\tau}_k)$, where the map $\tilde{f}_k$ is the restriction of f_k to $\mathbb{N}$. In addition, $\tilde{\mathcal{V}}_k$ is the corresponding uniformity.

Notation for ordered spaces is summarized at the start of [10]. Let $<_k$ be the linear order generating the topology of $\mathbf{R}_k$ described in [9] and let $\prec_k$ be the induced order on $\mathbb{N}$, i.e., $m \prec_k n$ if and only if $\tilde{f}_k(m) <_k \tilde{f}_k(n)$. This order is radically different from the usual order on $\mathbb{N}$ and so all $\prec_k$-intervals will be provided with a subscript of k. By R21.45 $\mathcal{U}(\prec_k)$ is a proper subset of $\tilde{\mathcal{V}}_k$.

The conjecture mentioned above can now be expressed as a positive answer to the following: Is $\tilde{\mathcal{V}}_k$ the extension of $\mathcal{U}(\prec_k)$ by the countable chain $\{[n, \infty)_k : n \in \mathbb{N}\}$? Or, in the notation of [11], does $\mathcal{U}(\prec_k)_e(\{[n, \infty)_k : n \in \mathbb{N}\})$ equal $\tilde{\mathcal{V}}_k$?

In verifying this equality it will be convenient to use the following simplified notation: For each $n \in \mathbb{N}$, $E([n, \infty)_k)$, which by definition is $[n, \infty)_k \times [n, \infty)_k \cup (-\infty, n)_k \times (-\infty, n)_k$, will be denoted $E(n)$.

Lemma R23.1.1 $\mathcal{U}(\prec_k)_e(\{[n, \infty)_k : n \in \mathbb{N}\}) \subseteq \tilde{\mathcal{V}}_k$.

Proof: By R21.45 $\mathcal{U}(\prec_k) \subseteq \tilde{\mathcal{V}}_k$ and in R22.1.1 $\mathcal{U}(\prec_k)_e(\{[n, \infty)_k : n \in \mathbb{N}\})$ is defined as $\mathcal{U}(\prec_k) \vee (\vee\{\mathcal{U}_{E(n)} : n \in \mathbb{N}\})$. Thus it is sufficient to show that $\mathcal{U}_{E(n)} \subseteq \tilde{\mathcal{V}}_k$, i.e., $E(n) \in \tilde{\mathcal{V}}_k$, for all $n \in \mathbb{N}$. For $n = 1$, $f_k(1)$ is the $<_k$-smallest element of $\mathbf{R}_k$ by R19.1.8 and so 1 is the $\prec_k$-smallest in $\mathbb{N}$. Thus $E(1) = \mathbb{N} \times \mathbb{N}$, which is in $\tilde{\mathcal{V}}_k$. Now let $n \in \mathbb{N}$ with $n \geq 2$. By R19.1.20 $f_k(n)$ is the larger of a consecutive pair in $\mathbf{R}_k$ with the smaller being of the form $f_k(j)$, where j is a negative integer. Then $\mathcal{I} = \{(-\infty, f_k(j)], [f_k(n), \infty)\}$ is an open cover of $\mathbf{R}_k$ and so $U = R(\mathcal{I})$ is in $\mathcal{U}(<_k)$, the unique uniformity for $\mathbf{R}_k$, so that $(f_k \times f_k)^{-1}[U \cap f_k[\mathbb{N}] \times f_k[\mathbb{N}]] \in \tilde{\mathcal{V}}_k$. That $(f_k \times f_k)^{-1}[U \cap f_k[\mathbb{N}] \times f_k[\mathbb{N}]] \subseteq E(n)$ is easily checked and the conclusion follows.

To verify the converse, more detailed information about finite covers by intervals open in $\mathbf{R}_k$ is needed, including the following general fact about ordered spaces.

Lemma R23.1.2 Let $(X, <)$ be a linearly ordered space. Let $a, b \in X$ with $a < b$. Then $[a, b) \in \tau(<)$ if and only if a is the smallest element of X or a is the larger of a consecutive pair in X. Also $(a, b] \in \tau(<)$ if and only if b is the largest element of X or b is the smaller of a consecutive pair.

Proof: If a is the smallest element, the $[a, b) = (-\infty, b)$, which is in $\tau(<)$. If a is the larger of a consecutive pair, let c be the smaller, i.e., $c < a$ and $(c, a) = \emptyset$. Then $[a, b) = (c, b)$, which is in $\tau(<)$. Conversely assume $[a, b) \in \tau(<)$ and that the conclusion is false, i.e., a is not the smallest in X and a is not the larger of a consecutive pair. Then there is a finite intersection of rays from the subbasis containing a and contained in $[a, b)$. Since the intersection of two left (right) rays is a left (right) ray, the expected case is an intersection of one of each: there exist $c, d \in X$ such that $a \in (-\infty, d) \cap (c, \infty) \subseteq [a, b)$. Since $c < a$ and a is not the larger of a consecutive pair, there is x such that $c < x < a < d$ so that $x \in [a, b)$, a contradiction. The case of one right ray, i.e., $a \in (c, \infty) \subseteq [a, b)$, proceeds in the same way. In the final case (one left ray), $a \in (-\infty, d) \subseteq [a, b)$. Here, since a is not the smallest, there is $x < a < d$ so that $x \in [a, b)$, a contradiction. The proof of the second claim is similar.

In [9] the consecutive pairs of $\mathbf{R}_k$ were described complelely and so the last lemma has an application there. As in [10], the word 'interval' includes rays and assumes endpoints. An interval I has a left-closed endpoint x means I has the form $[x, \infty)$ or $[x, y)$ or $[x, y]$. The terms 'right-closed endpoint,''left-open endpoint,' and 'right-open endpoint' have the analogous meanings.

Lemma R23.1.3 Let I be an interval open in $\mathbf{R}_k$. If I has a left-closed endpoint z, then there is $n \in \mathbb{N}$ with $z = f_k(n)$. If I has a right-closed endpoint w, then there is $j \in \mathbf{Z}$ with $j \leq 0$ and $w = f_k(j)$.

Proof: Apply R23.1.2: Let z be a left-closed endpoint of I. If z is the smallest element of $\mathbf{R}_k$, by R19.1.8 $z = f_k(1)$. If z is the larger of a consecutive pair in $\mathbf{R}_k$, by R19.1.19 there is some $n \in \mathbb{N}$ with $n \geq 2$ such that $z = f_k(n)$. The second claim follows similarly since by R19.1.8 $f_k(0)$ is the largest element of $\mathbf{R}_k$ and by R19.1.19 the smaller of a consecutive pair in $\mathbf{R}_k$ must be the f_k-image of some negative integer.

Lemma R23.1.4 Let $\mathcal{I}$ be a cover of $\mathbf{R}_k$ by non-empty intervals in $\tau(<_k)$ and let $I \in \mathcal{I}$. Then there exists $\mathcal{I}_1$, a set containing at most three non-empty intervals in $\tau(<_k)$, such that $I \subseteq \cup\{J : J \in \mathcal{I}_1\}$, each element of $\mathcal{I}_1$ is contained in some element of $\mathcal{I}$, and all the endpoints of intervals in $\mathcal{I}_1$ are in $f_k[\mathbf{Z}\,]$.

Proof: By R19.1.8 $\mathbf{R}_k = [f_k(1), f_k(0)]$ and so assume without loss of generality that each element of $\mathcal{I}$ has two endpoints in $\mathbf{R}_k$. Also each element in $\mathcal{I}$ has two different endpoints, since it is non-empty and f_k is one-to-one. Let I have endpoints $z <_k w$. If both z, w are in $f_k[\mathbf{Z}\,]$, let $\mathcal{I}_1 = \{I\}$. The other possibilities are covered in the following cases. Case I: Suppose z is not in $f_k[\mathbf{Z}\,]$ and $w \in f_k[\mathbf{Z}\,]$. By R23.1.3 z must be a left-open endpoint of I. Let J with endpoints $a <_k b$ be an interval in $\mathcal{I}$ such that $z \in J$. By R23.1.3 $a <_k z <_k b$ and by density of $f_k[\mathbb{N}\,]$ (R12.6.9) there exist $m, n \in \mathbb{N}$ such that $a <_k f_k(m) <_k z <_k f_k(n) <_k \min\{w, b\}$. Let I_1 be the open interval $(f_k(m), f_k(n))$, which is contained in J. Note that $I_2 = [f_k(n), w] \cap I$ is an interval containing $f_k(n)$ with endpoints $f_k(n)$ and w. By R19.1.15 and R23.1.2 I_2 is in $\tau(<_k)$. Clearly $\mathcal{I}_1 = \{I_1, I_2\}$ has the required properties. Case II: Suppose z is in $f_k[\mathbf{Z}\,]$ and w is not in $f_k[\mathbf{Z}\,]$. By

R23.1.3 w must be a right-open endpoint of I. Let K with endpoints $c <_k d$ be an interval in $\mathcal{I}$ such that $w \in K$. By R23.1.3 $c <_k w <_k d$ and by density of $f_k[\mathbb{N}\,]$ there exist $p, q, r \in \mathbb{N}$ such that $\max\{c, z\} <_k f_k(p) <_k f_k(q) <_k w <_k f_k(r) <_k d$. Let I_1 be the open interval $(f_k(p), f_k(r))$, which is contained in K. Note that $I_2 = [z, f_k(q)) \cap I$ is an interval containing $f_k(p)$ with endpoints z and $f_k(q)$. Also I_2 is in $\tau(<_k)$. Clearly $\mathcal{I}_1 = \{I_1, I_2\}$ has the required properties. Case III: Suppose both z and w are not in $f_k[\mathbf{Z}\,]$. By R23.1.3 z must be a left-open endpoint of I. Let J with endpoints $a <_k b$ be an interval in $\mathcal{I}$ such that $z \in J$. By R23.1.3 $a <_k z <_k b$ and by density of $f_k[\mathbb{N}\,]$ there exist $m, n \in \mathbb{N}$ such that $a <_k f_k(m) <_k z <_k f_k(n) <_k \min\{w, b\}$. Let I_1 be the open interval $(f_k(m), f_k(n))$, which is contained in J. Note that $[f_k(n), w)$ is an interval in $\tau(<_k)$ by R19.1.15 and R23.1.2. By R23.1.3 w must be a right-open endpoint and so the technique of the second case applies to $[f_k(n), w)$: Let K with endpoints $c <_k d$ be an interval in $\mathcal{I}$ such that $w \in K$. Pick $p, q, r \in \mathbb{N}$ such that $\max\{f_k(n), c\} <_k f_k(p) <_k f_k(q) <_k w <_k f_k(r) <_k d$. Let I_2 be the open interval $(f_k(p), f_k(r))$ and let $I_3 = [f_k(n), f_k(q))$, which is in $\tau(<_k)$ by R23.1.3. Clearly $\mathcal{I}_1 = \{I_1, I_2, I_3\}$ has the required properties.

Lemma R23.1.5 Let $\mathcal{I}$ be a finite cover of $\mathbf{R}_k$ by non-empty intervals in $\tau(<_k)$. Then there exists $\mathcal{J}$, a finite cover of $\mathbf{R}_k$ by non-empty intervals in $\tau(<_k)$, such that $\mathcal{J}$ refines $\mathcal{I}$ and all the endpoints of intervals in $\mathcal{J}$ are in $f_k[\mathbf{Z}\,]$.

Proof: For each I_j in $\mathcal{I}$, let $\mathcal{I}_j$ be the collection of intervals in $\tau(<_k)$ guaranteed by R23.1.4 and let $\mathcal{J} = \cup\{\mathcal{I}_j : I_j \in \mathcal{I}\}$. It follows easily from R23.1.4 that $\mathcal{J}$ has the required properties.

Lemma R23.1.6 Let $\mathcal{I}$ be a cover of $\mathbf{R}_k$ by non-empty intervals in $\tau(<_k)$ with all the endpoints of intervals in $\mathcal{I}$ in $f_k[\mathbf{Z}\,]$. Let $I \in \mathcal{I}$. Then either I can be described as an interval with both endpoints in $\{0\} \cup f_k[\mathbb{N}\,]$ or there exists $\mathcal{I}_1$, a collection containing two non-empty intervals in $\tau(<_k)$, such that $I \subseteq \cup\{J : J \in \mathcal{I}_1\}$, each element of $\mathcal{I}_1$ is contained in some element of $\mathcal{I}$, and all endpoints of intervals in $\mathcal{I}_1$ are in $\{0\} \cup f_k[\mathbb{N}\,]$.

Proof: By R19.1.8 $\mathbf{R}_k = [f_k(1), f_k(0)]$ and so assume without loss of generality that each element of $\mathcal{I}$ has two endpoints in $\mathbf{R}_k$. Also each element in $\mathcal{I}$ has two different endpoints, since it is non-empty and f_k is one-to-one. Let I have endpoints $z <_k w$. If z and w are both in $\{0\} \cup f_k[\mathbb{N}\,]$, the conclusion holds. Otherwise let $z = f_k(s)$ and $w = f_k(t)$. If $s < 0$, by R23.1.3 and the fact that f_k is one-to-one, z must be a left-open endpoint of I and by R19.1.19 there is $n \in \mathbb{N}$ such that $z <_k f_k(n)$ is a consecutive pair. Then $I = [f_k(n), w] \cap I$, i.e., I can be redescribed with a left-endpoint in $\{0\} \cup f_k[\mathbb{N}]$. Thus assume $s \geq 0$. The remaining cases assume $t < 0$ with $s \geq 0$. Here $w = f_k(t)$ may be right-open or right-closed for I. By R19.1.19 there is $p \in \mathbb{N}$ such that $w <_k f_k(p)$ is a consecutive pair. If w is right-closed for I, then either $I = [f_k(s), f_k(p))$ or $I = (f_k(s), f_k(p))$, i.e., I can be redescribed with both endpoints in $\{0\} \cup f_k[\mathbb{N}]$. If w is right-open for I, there is J in $\mathcal{I}$ with endpoints $u <_k v$ such that $w \in J$. By the density of $f_k[\mathbb{N}\,]$ there is $q \in \mathbb{N}$ such that $\max\{f_k(s), u\} <_k f_k(q) <_k w$. Note that $f_k(p) \leq_k v$ and so $I_1 = [f_k(q), f_k(p)) \subseteq J$ Let $I_2 = [f_k(s), f_k(q)) \cap I$. Then $\mathcal{I}_1 = \{I_1, I_2\}$ has the required properties.

Lemma R23.1.7 Let $\mathcal{I}$ be a finite cover of $\mathbf{R}_k$ by non-empty intervals in $\tau(<_k)$ with all the endpoints of intervals in $\mathcal{I}$ in $f_k[\mathbf{Z}\,]$. Then there exists $\mathcal{J}$, a finite cover of $\mathbf{R}_k$ by non-empty intervals in $\tau(<_k)$, such that $\mathcal{J}$ refines $\mathcal{I}$ and all the endpoints of intervals in $\mathcal{J}$ are in $\{0\} \cup f_k[\mathbb{N}\,]$.

Proof: This follows immediately from R23.1.5 and R23.1.6.

Lemma R23.1.8 Let $\mathcal{I}$ be a finite cover of $\mathbf{R}_k$ by non-empty intervals in $\tau(<_k)$ with all endpoints in $\{0\} \cup f_k[\mathbb{N}]$. Then $(f_k \times f_k)^{-1}[R(\mathcal{I}) \cap (f_k[\mathbb{N}] \times f_k[\mathbb{N}])]$ is in the uniformity $\mathcal{U}(\prec_k)_e(\{[n,\infty)_k : n \in \mathbb{N}\})$.

Proof: By R19.1.8 $\mathbf{R}_k = [f_k(1), f_k(0)]$ and so assume without loss of generality that each element of $\mathcal{I}$ has two endpoints in $\mathbf{R}_k$. By R23.1.2, R19.1.8, and R19.1.19 the following holds: For $I \in \mathcal{I}$, since $n \geq 2$ implies $f_k(n)$ is the larger of a consecutive pair, I may have a left-closed endpoint but the right endpoint of I can only be right-closed if it is $f_k(0)$. Since each I is non-empty, this implies each I has two distinct endpoints. Construct a collection of $\prec_k$-intervals $\tilde{\mathcal{I}}$ as follows: Let $I \in \mathcal{I}$ and let $f_k(p) <_k f_k(q)$ be the endpoints of I. Then $p \geq 1$ and $q \geq 0$. If $f_k(p)$ is left-closed for I and $p > 1$, by R19.1.15 $f_k(p)$ is the larger of a consecutive pair. By R19.1.19 the smaller of the pair must be $f_k(-j)$ for some $j \in \mathbb{N}$. Let $J(I) \in \mathcal{I}$ with $f_k(-j)$ in $J(I)$. Let v be the left endpoint of $J(I)$. Since f_k is one-to-one, $v <_k f_k(-j)$. By density pick $x(I) \in \mathbb{N}$ such that $v <_k f_k(x(I)) <_k f_k(-j)$. Note that $(f_k(x(I)), f_k(-j)] \subseteq J(I)$. Now proceed by cases. Case I: $q = 0$. In this case whether $f_k(q)$ right-open or right-closed for I is irrelevant in $\mathbb{N}$. If $f_k(p)$ is left-open for I, let $\tilde{I} = (p,\infty)_k$. If $f_k(p)$ is left-closed for I and $p > 1$, let $\tilde{I} = (x(I),\infty)_k$. If $f_k(p)$ is left-closed for I and $p = 1$, let $\tilde{I} = [1,\infty)_k$, which of course is $\mathbb{N}$. Case II: $q > 0$. If $f_k(p)$ is left-open for I, let $\tilde{I} = (p,q)_k$. If $f_k(p)$ is left-closed for I and $p > 1$, let $\tilde{I} = (x(I),q)_k$. If $f_k(p)$ is left-closed for I and $p = 1$, let $\tilde{I} = [1,q)_k$. Let $\tilde{\mathcal{I}} = \{\tilde{I} : I \in \mathcal{I}\}$. It is easy to check that $\tilde{\mathcal{I}}$ is a finite cover of $\mathbb{N}$ by $\prec_k$-intervals in $\tau(\prec_k)$ and so $R(\tilde{\mathcal{I}})$ is in $\mathcal{U}(\prec_k)$ by definition R21.14. Let $U = R(\tilde{\mathcal{I}}) \cap (\cap\{E(p) : p \in L\})$, where L is the set $\{p \in \mathbb{N} : f_k(p)$ is a left-closed endpoint of some $I \in \mathcal{I}\}$. Note that U is in $\mathcal{U}(\prec_k)_e(\{[n,\infty)_k : n \in \mathbb{N}\})$. Let $V = (f_k \times f_k)^{-1}[R(\mathcal{I}) \cap (f_k[\mathbb{N}] \times f_k[\mathbb{N}])]$. To see that U is a subset of V, let the ordered pair (m,n) be in U. Since the pair is in $R(\tilde{\mathcal{I}})$, there must be $I \in \mathcal{I}$ such that the pair is in $\tilde{I} \times \tilde{I}$. By symmetry assume $m \preceq_k n$. Let $f_k(p)$ be the left endpoint of I. If $f_k(p)$ is left-open for I or left-closed with $p = 1$, by the construction of $\tilde{I}$ the ordered pair $(f_k(m), f_k(n))$ is in $I \times I \subseteq R(\mathcal{I})$ so that the pair (m,n) is in V. If $f_k(p)$ is left-closed for I and $p \geq 2$, $\tilde{I}$ has left endpoint $x(I)$. Since the ordered pair (m,n) is in $E(p)$, either both are in $[p,\infty)_k$ or both are in $(-\infty,p)_k$. By the construction of $\tilde{I}$, in the first case the ordered pair $(f_k(m), f_k(n))$ is in $I \times I \subseteq R(\mathcal{I})$ and in the second case the ordered pair $(f_k(m), f_k(n))$ is in $J(I) \times J(I) \subseteq R(\mathcal{I})$. Thus the pair (m,n) is in V and the conclusion holds.

Proposition R23.1.9 $\mathcal{U}(\prec_k)_e(\{[n,\infty)_k : n \in \mathbb{N}\}) = \tilde{\mathcal{V}}_k$.

Proof: Let $V \in \tilde{\mathcal{V}}_k$. Then there exists $W \in \mathcal{U}(<_k)$, the unique uniformity for $\mathbf{R}_k$, such that $V = (f_k \times f_k)^{-1}[W \cap (f_k[\mathbb{N}] \times f_k[\mathbb{N}])]$ and there is $\mathcal{I}$, a finite cover of $\mathbf{R}_k$ by $<_k$-intervals in $\tau(<_k)$ with $R(\mathcal{I}) \subseteq W$. Apply R23.1.5 and then R23.1.7 to obtain $\mathcal{J}$, a finite cover of $\mathbf{R}_k$ by $<_k$-intervals in $\tau(<_k)$ with all endpoints in $\{0\} \cup f_k[\mathbb{N}]$, such that $\mathcal{J}$ refines $\mathcal{I}$ and so $R(\mathcal{J}) \subseteq R(\mathcal{I})$. Thus $(f_k \times f_k)^{-1}[R(\mathcal{J}) \cap (f_k[\mathbb{N}] \times f_k[\mathbb{N}])] \subseteq V$. By R23.1.8 $(f_k \times f_k)^{-1}[R(\mathcal{J}) \cap (f_k[\mathbb{N}] \times f_k[\mathbb{N}])]$ is in $\mathcal{U}(\prec_k)_e(\{[n,\infty)_k : n \in \mathbb{N}\})$ as is the superset V. R23.1.1 shows the reverse containment and so equality holds.

Note that $\{[n,\infty)_k : n \in \mathbb{N}\}$ is a countable chain of subsets of $(\mathbb{N}, \prec_k)$. However, despite the labeling which may suggest otherwise, it is not a decreasing sequence with the natural indexing and so subsection R22.4 is not applicable to the collection of complements. The following proposition shows there is no reindexing which makes the collection monotone.

Lemma R23.1.10 Let $(X, <)$ be a linearly ordered set. Let S be a countable subset of X with $|S| \geq 2$. Assume if $s_1, s_2 \in S$ with $s_1 > s_2$ then there is $t \in S$ such that $s_1 > t > s_2$. Then there is no bijective $f : \mathbb{N} \to S$ such that $f(n) > f(n+1)$ for all n.

Proof: Note that $|S| \geq 2$ and the assumed density property imply S is countably infinite. Deny the conclusion and let $f : \mathbb{N} \to S$ be one-to-one and onto with $f(n) > f(n+1)$ for all n. Clearly $m < n$ implies $f(m) > f(n)$. Since $f(1) > f(2)$, there is $t \in S$ with $f(1) > t > f(2)$. Let $t = f(j)$. Since f is a function, $2 < j$ so that $f(2) > f(j) = t$, a contradiction.

Proposition R23.1.11 No reindexing of $\{[n, \infty)_k : n \in \mathbb{N}\}$ makes the collection monotone.

Proof: $\mathcal{A} = \{[n, \infty)_k : n \in \mathbb{N}\}$ is a linearly ordered set with containment since $\mathbb{N}$ is linearly ordered by $\prec_k$. $\mathcal{A}$ has the order density property: Let $m, n \in \mathbb{N}$ with $m \prec_k n$ so that $[n, \infty)_k \subset [m, \infty)_k$. Then $f_k(m) <_k f_k(n)$ and by R19.1.19 these are not a consecutive pair. Since $f_k[\mathbb{N}]$ is dense in $\mathbf{R}_k$, there is $j \in \mathbb{N}$ such that $f_k(j)$ is in the non-empty, open $\mathbf{R}_k$-interval $(f_k(m), f_k(n))$. Then $m \prec_k j \prec_k n$ so that $[n, \infty)_k \subset [j, \infty)_k \subset [m, \infty)_k$. Now let $g : \mathbb{N} \to \mathcal{A}$ be a reindexing, i.e., a one-to-one, onto map. By R23.1.10 g cannot be monotone decreasing and g cannot be monotone increasing since $[1, \infty)_k = \mathbb{N}$: if $g(j) = [1, \infty)_k$, then $g(j+1) \subset g(j)$.

In the next proposition notation from R22.2.2 will be used.

Proposition R23.1.12 Let $\mathcal{A} = \{[n, \infty)_k : n \in \mathbb{N}\}$ and let $p \in \mathcal{C}(\mathcal{A})$. Then B_p is either empty or a singleton.

Proof: Let $j \in \mathbb{N}$ and define $q \in \mathcal{C}(\mathcal{A})$ as follows: For $l \in \mathbb{N}$ there are two cases. If $l \preceq_k j$, let $q(l) = [l, \infty)_k$. If $j \prec_k l$, let $q(l) = \mathbb{N} - [l, \infty)_k = [1, l)_k$. Note that $j \in B_q$. Let $t \in \mathbb{N}$ with $t \neq j$. If $t \prec_k j$, then $t \notin q(j)$. If $j \prec_k t$, then $f_k(j) <_k f_k(t)$. By R19.1.19 these are not a consecutive pair. Since $f_k[\mathbb{N}]$ is dense in $\mathbf{R}_k$, there is $l \in \mathbb{N}$ such that $f_k(l)$ is in the non-empty, open $\mathbf{R}_k$-interval $(f_k(j), f_k(t))$. Then $j \prec_k l \prec_k t$ and so $t \notin q(l)$. Thus $B_q = \{j\}$. Since the non-empty B_p form a partition of $\mathbb{N}$ by R22.2.3, the conclusion follows.

Extension by a Countable Chain

The previous subsection shows that the remnant rings are compactifications corresponding to the extension of a certain separated, totally bounded uniformity by a countable chain and that the chain cannot be represented as a monotone sequence, in particular, not as an incresing sequence of the kind studied in R22.4. This subsection will show the results of R22.4 can be partially extended to the chain case by making suitable modifications to the approach in R22.4.

In what follows $(X, \mathcal{U})$ is assumed to be a totally bounded, separated uniform space and $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$ a countably infinite chain of non-empty subsets of X with $A_i \neq A_j$ if $i \neq j$. (Y, f) is assumed to be a T_2 compactification in the class corresponding to $\mathcal{U}$. For each n in $\mathbb{N}$, $\{A_i : 1 \leq i \leq n\}$ is a finite chain, which can be reindexed as an increasing sequence $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_n$, i.e., there is a one-to-one correspondence such that each A_i is some B_j and vice-versa. By R22.2.6 and R22.4.1 the compatification class corresponding to $\mathcal{U}_e(\{A_1,\ldots,A_n\}) = \mathcal{U}_e(\{B_1,\ldots,B_n\})$ is represented by (Z_n, g_n), where (with $B_0 = \emptyset$ and the closure of $S \subseteq Y$ denoted $\overline{S}$) $Z_n = (\coprod_{i=1}^n \overline{f[B_i - B_{i-1}]}) \coprod \overline{f[X - B_n]}$ and $g_n : X \to Z_n$ by $g_n(x) = (f(x), i)$ if $x \in B_i - B_{i-1}$ and $g_n(x) = (f(x), n+1)$ if $x \in X - B_n$.

The supremum of the Z_n can be represented by applying the addendum to R13. The initial segments of $\mathbb{N}$ would be used as the co-final subset of the finite subsets of $\mathbb{N}$.

For $l \leq m$, $\mathcal{U}_e(\{A_1,\ldots,A_l\}) \subseteq \mathcal{U}_e(\{A_1,\ldots,A_m\})$ and so $[(Z_l, g_l)] \leq [(Z_m, g_m)]$. Let $\phi_{ml} : Z_m \to Z_l$ be the unique continuous surjection such that $\phi_{ml} \circ g_m = g_l$.

Let $Z = \{p \in \prod_{n=1}^\infty Z_n : l \leq m \Rightarrow \phi_{ml}(p(m)) = p(l)\}$ and for $x \in X$ let $y_x(n) = g_n(x)$. By R13.Add.2 each y_x is in Z and so $g : X \to Z$ can be defined by $g(x) = y_x$. By R13.Add.7 and R13.Add.8 (Z, g) is in the compactification class corresponding to $\mathcal{U}_e(\mathcal{A})$.

As above, because of containments among the uniformities, there exist unique continuous surjections $\phi : Z \to Y$ with $\phi \circ g = f$, for every n $\psi_n : Z \to Z_n$ with $\psi_n \circ g = g_n$, and for every n $\phi_n : Z_n \to Y$ with $\phi_n \circ g_n = f$.

Lemma R23.2.1 Let $n \in \mathbb{N}$. Then $\phi = \phi_n \circ \psi_n$.

Proof: Same proof as R22.4.3.

Lemma R23.2.2 Let $n \in \mathbb{N}$. ψ_n is the restriction to Z of the natural projection from $\prod_{i=1}^\infty Z_i$ onto Z_n.

Proof: Same proof as R22.4.4

The proof of the next lemma is almost identical to that of R22.4.5, but minor adjustments are needed because of the complication of ordering shifts at each level.

Lemma R23.2.3 Let $n \in \mathbb{N}$ and let $(y, \alpha) \in Z_n$. Then $\phi_n((y, \alpha)) = y$.

Proof: Define $\rho : Z_n \to Y$ by $\rho((t, \beta)) = t$. Let O be an open set in Y and let $\{B_1,\ldots,B_n\}$ denote the re-indexing of $\{A_1,\ldots,A_n\}$ into a finite increasing chain. With $B_0 = \emptyset$, $O = (\cup_{i=1}^n (O \cap \overline{f[B_i - B_{i-1}]})) \cup (O \cap \overline{f[X - B_n]})$ since $f[X]$ is dense and $f[X] = (\cup_{i=1}^n f[B_i - B_{i-1}]) \cup f[X - B_n]$. Moreover, as a set,

$$Z_n = (\cup_{i=1}^n \overline{f[B_i - B_{i-1}]} \times \{i\}) \cup \overline{f[X - B_n]} \times \{n+1\}$$

and so one easily checks $\rho^{-1}[O] = (\cup_{i=1}^n (O \cap \overline{f[B_i - B_{i-1}]}) \times \{i\}) \cup (O \cap \overline{f[X - B_n]}) \times \{n+1\}$, which is open in the disjoint union. Thus ρ is continuous. For $x \in X$, $\rho \circ g_n(x) = \rho((f(x), j))$, where j depends on the location of x. Thus $\rho \circ g_n = f$ and, by the density of $f[X]$, ρ is onto. Since ϕ_n is the unique such map, $\phi_n = \rho$.

Lemma R23.2.4 Let $l, m \in \mathbb{N}$ with $l \leq m$. Let $(y, \alpha) \in Z_m$. Then $\phi_{ml}((y, \alpha)) = (y, \beta)$ for some $\beta \leq \alpha$.

Proof: Assume $\{A_1,\ldots,A_l,\ldots,A_m\}$ is re-indexed as $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_m$ and $\{A_1,\ldots,A_l\}$ as $C_1 \subseteq C_2 \subseteq \cdots \subseteq C_l$. As usual $B_0 = C_0 = \emptyset$. As a first case, assume $\alpha = m+1$ so that (y, α) must be in $\overline{f[X - B_m]} \times \{m+1\}$. Then there is a net $S : D \to X - B_m$ such that $f \circ S$ converges to y. For $d \in D$, $g_m \circ S(d) = (f \circ S(d), m+1)$ and so $g_m \circ S$ converges to $(y, m+1)$. By continuity $\phi_{ml} \circ g_m \circ S$ converges to $\phi_{ml}(y, m+1)$. Clearly $C_l \subseteq B_m$ so that S is always in $X - C_l$ and, as before, $g_l \circ S$ converges to $(y, l+1)$. Since $\phi_{ml} \circ g_m = g_l$ and limits are unique in a T_2 space, $\phi_{ml}((y, m+1)) = (y, l+1)$. Since $l \leq m$, $\beta = l+1 \leq m+1 = \alpha$. As a second case assume $\alpha \leq m$. If $C_l \subset B_\alpha$, since each C_t is some B_r, $l+1 \leq \alpha$. Also $B_{\alpha-1}$ cannot be a proper subset of C_l, since $B_{\alpha-1}$ and B_α are consecutive in $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_m$. Thus a net in $B_\alpha - B_{\alpha-1}$ is also in

$X - C_l$. By continuity as in the first case $\phi_{ml}((y,\alpha)) = (y, l+1)$ and $\beta = l+1 \leq \alpha$. Now assume $B_\alpha \subseteq C_l$. Since the sets are distinct, there is i such that $C_{i-1} \subset B_\alpha \subseteq C_i$ so that $i \leq \alpha$. Because $B_{\alpha-1}$ and B_α are consecutive, $B_{\alpha-1}$ cannot be a proper subset of C_{i-1}, i.e., $C_{i-1} \subseteq B_{\alpha-1}$. Thus a net in $B_\alpha - B_{\alpha-1}$ is also in $C_i - C_{i-1}$. By continuity as in the first case $\phi_{ml}((y,\alpha)) = (y,i)$ and $\beta = i \leq \alpha$.

Corollary R23.2.5 Let $z \in Z$. Then there is $y \in Y$ such that, for every n, $z(n) = (y, \alpha_n)$ for some $\alpha_n \in \{1, \ldots, n+1\}$.

Proof: $z(1) = (y, \alpha_1)$ for some $y \in Y$ and $\alpha_1 \in \{1, 2\}$. Let $n \in \mathbb{N}$. $z(n) = (w, \alpha_n)$ for some $w \in Y$ and $\alpha_n \in \{1, \ldots, n+1\}$. Since $z \in Z$, $\phi_{n1}(z(n)) = z(1)$, i.e., $\phi_{n1}((w, \alpha_n)) = (y, \alpha_1)$. By R23.2.4 $\phi_{n1}((w,\alpha_n)) = (w, \beta)$ where $\beta \leq \alpha_n$. Thus $w = y$ as required.

Corollary R23.2.6 Let $z \in Z$ and let y be the element in Y such that, for every n, $z(n) = (y, \alpha_n)$ for some $\alpha_n \in \{1, \ldots, n+1\}$. Then $\phi(z) = y$.

Proof: Pick any $n \in \mathbb{N}$. By R23.2.1, R23.2.2, and R23.2.3 $\phi(z) = \phi_n \circ \psi_n(z) = \phi_n(z(n)) = \phi_n((y, \alpha_n)) = y$.

To continue following the pattern used in R22.4, it is necessary to refine the definition of the relation described for an increasing sequence in R22.4.24. Recall that consecutive elements in a linearly ordered set are distinct by definition.

Definition R23.2.7 Let $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ be a countable chain of subsets of X with $A_0 = \emptyset$ and $A_i \neq A_j$ if $i \neq j$. Let $i, j \in \mathbb{N} \cup \{0\}$ and let $m \in \mathbb{N}$. A_i and A_j are consecutive at level m provided $i, j \leq m$ and, when the finite set $\{A_0, \ldots, A_m\}$ is arranged in increasing order, A_i and A_j are consecutive elements.

Definition R23.2.8 Let $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ be a countable chain of subsets of X with $A_0 = \emptyset$ and $A_i \neq A_j$ if $i \neq j$. Let h be a map from $\mathbb{N}$ to $\mathbb{N} \cup\{0\} \times \mathbb{N} \cup \{0\}$. h is $\mathcal{A}$-suitable provided, for every n, $h(n) = (i,j)$ where $i, j \leq n$ and either $i = j$ or A_i, A_j are consecutive at level n.

Recall the notation for the symmetric difference: $A \Delta B = (A - B) \cup (B - A)$.

Definition R23.2.9 Let $\mathcal{A} = \{A_n : n \in \{0\} \cup \mathbb{N}\}$ be a countable chain of subsets of X with $A_0 = \emptyset$ and $A_i \neq A_j$ if $i \neq j$. Let $S : D \to X$ be a net. Then S is $\mathcal{A}$-compatible if and only if S is eventually in $X - \cup_{i=1}^n A_i$ for every n or there exist $m \in \mathbb{N}$ and an $\mathcal{A}$-suitable map h such that $n \geq m$ implies S is eventually in $A_i \Delta A_j$ where $h(n) = (i, j)$.

The next lemma shows that R23.2.9 extends R22.4.10.

Lemma R23.2.10 Let $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ be an increasing sequence of subsets of X with $A_i \neq A_j$ if $i \neq j$ and let $S : D \to X$ be a net. Then S is $\mathcal{A}$-compatible in the sense of R23.2.9 if and only if S is $\mathcal{A}$-compatible in the sense of R22.4.10.

Proof: In the special case of an increasing sequence $X - \cup_{i=1}^n A_i = X - A_n$ for every n so that S is eventually in $X - \cup_{i=1}^n A_i$ if and only if S is eventually in $X - A_n$. Moreover the consecutive sets at level n are of the form A_{i-1}, A_i for some $i \leq n$, where $A_0 = \emptyset$ as usual and $A_{i-1} \Delta A_i = A_i - A_{i-1}$. Now assume S is $\mathcal{A}$-compatible in the sense of R23.2.9. If S is eventually in $X - \cup_{i=1}^n A_i$ for all n, then S is eventually in $X - A_n$ for all n. If there exist $m \in \mathbb{N}$ and an $\mathcal{A}$-suitable map h such that $n \geq m$ implies S is eventually in $A_i \Delta A_j$ where $h(n) = (i,j)$, then, for $n \geq m$, $A_i \Delta A_j$ is non-empty and so A_i, A_j are consecutive elements. By the initial remarks one can assume without loss of generality that $j = i - 1$ so that S is eventually in $A_i - A_{i-1}$. Thus in either case S is $\mathcal{A}$-compatible in the sense of R22.4.10. For the converse, assume S is $\mathcal{A}$-compatible in the sense of R22.4.10. If S

is eventually in $X - A_n$ for all n, then S is eventually in $X - \cup_{i=1}^{n} A_i$ for all n. If there exists i such that S is eventually in $A_i - A_{i-1}$, define h by $h(n) = (n, n)$ if $n < i$ and $h(n) = (i, i-1)$ if $n \geq i$. For $n \geq i$, since $\mathcal{A}$ is an increasing sequence, A_i, A_{i-1} are consecutive at level n and so h is $\mathcal{A}$-suitable. By the initial remarks, S is eventually in $A_i \Delta A_{i-1}$. Thus S is $\mathcal{A}$-compatible in the sense of R23.2.9.

The previous lemma also implies that the next definition extends R22.4.24.

Definition R23.2.11 Let $(X, \mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ be a countable chain of subsets of X with $A_i \neq A_j$ if $i \neq j$. Suppose $\mathcal{U}_e(\mathcal{A})$ corresponds to the compactification class $[(W, h)]$. Let $a, b \in W$. Then $aR_1^W(\mathcal{A})b$ if and only if there is a $\mathcal{U}$-Cauchy net $S : D \to X$ with two $\mathcal{A}$-compatible subnets, one whose h-image converges to a and the other whose h-image converges to b.

We now proceed with the compactification (Z, g) and the associated structures described at the beginning of this subsection. R_ϕ denotes the equivalence relation $\{(a, b) : \phi(a) = \phi(b)\}$.

Lemma R23.2.12 $R_1^Z(\mathcal{A}) \subseteq R_\phi$.

Proof: Let $aR_1^W(\mathcal{A})b$. Then there is a $\mathcal{U}$-Cauchy net $S : D \to X$ with two $\mathcal{A}$-compatible subnets, one whose g-image converges to a and the other whose g-image converges to b. Since S is $\mathcal{U}$-Cauchy, there is $y \in Y$ such that $f \circ S$ converges to y. Since ϕ is continuous, the $\phi \circ g$-image of the first subnet converges to $\phi(a)$. Since $\phi \circ g = f$, the f-image of the first subnet, which is a subnet of $f \circ S$, converges to $\phi(a)$. Since limits are unique in the T_2 space Y, $\phi(a) = y$. Similarly $\phi(b) = y$. Thus $aR_\phi b$.

Lemma R23.2.13 Let $z \in Z$ and let y be the element of Y such that, for every n, $z(n) = (y, \alpha_n)$ for some $\alpha_n \in \{1, \ldots, n+1\}$. If $\alpha_n = n+1$ for every n, then there is an $\mathcal{A}$-compatible net $S : D \to X$ such that $g \circ S$ converges to z.

Proof: This is essentially same as the proof of R22.4.15, but the details will be included here because of changes in terminology. As above, the complement of the largest among $A_1, \ldots, A_j$ is $X - \cup_{i=1}^{j} A_i$. Let E be the set of open Y-neighborhoods of y ordered by reverse inclusion, a directed set. For each $j \in \mathbb{N}$, since $\alpha_j = j+1$, y is in the Y-closure of $f[X - \cup_{i=1}^{j} A_i]$ and so let the net $S_j : E \to (X - \cup_{i=1}^{j} A_i)$ be constructed in the standard way: For $O \in E$, $O \cap f[X - \cup_{i=1}^{j} A_i] \neq \emptyset$ and so let $S_j(O)$ be any element of $X - \cup_{i=1}^{j} A_i$ with $f(S_j(O)) \in O$. As usual $f \circ S_j$ converges to y. Let $D = E \times \mathbb{N}$ with ordering defined by $(O, j) \preceq (G, k)$ if and only if $O \supseteq G$ and $j \leq k$. With this ordering D is a directed set. Define $S : D \to X$ by $S(O, j) = S_j(O)$. For $n \in \mathbb{N}$, if $n \leq j$, $X - \cup_{i=1}^{j} A_i \subseteq X - \cup_{i=1}^{n} A_i$. Thus $(O, j) \succeq (Y, n)$ implies $S(O, j) \in X - \cup_{i=1}^{n} A_i$, i.e., S is eventually in $X - \cup_{i=1}^{n} A_i$ and so S is an $\mathcal{A}$-compatible net. To see that $g \circ S$ converges to y, recall that convergence in Z is pointwise and so it is sufficient to verify $\psi_n \circ g \circ S$ converges to $\psi_n(z) = z(n)$ for all n. Let n be in $\mathbb{N}$. An open neighborhood of $z(n) = (y, n+1)$ must contain $(O \cap \overline{f[X - \cup_{i=1}^{n} A_i]}) \times \{n+1\}$ for some $O \in E$. Let $(G, j) \succeq (O, n)$ in D. Then $\psi_n \circ g \circ S(G, j) = g_n(S_j(G))$. As above, $S_j(G)$ is in $X - \cup_{i=1}^{n} A_i$ and so $g_n(S_j(G)) = (f(S_j(G)), n+1)$. Since $G \subseteq O$, $f(S_j(G)) \in O$ and so $g_n(S_j(G))$ is in $(O \cap \overline{f[X - \cup_{i=1}^{n} A_i]}) \times \{n+1\}$. Thus $\psi_n \circ g \circ S$ is eventually in the original open set, i.e., it converges to $z(n)$.

Extending the previous lemma to other cases is a bit different from R22.4.

Lemma R23.2.14 Let $z \in Z$ and let y be the element of Y such that, for every n,

$z(n) = (y, \alpha_n)$ for some $\alpha_n \in \{1, \ldots, n+1\}$. If $\alpha_n \le n$ for some n, then $\alpha_k \le k$ for all $k \ge n$.

Proof: Assume $\alpha_n \le n$, let $k \ge n$, and suppose $\alpha_k = k+1$. Then $(y, k+1)$ is in $\overline{f[X - \cup_{i=1}^k A_i]} \times \{k+1\}$ and so there is a net $S : D \to X - \cup_{i=1}^k A_i$ such that $f \circ S$ converges to y. Since $k \ge n$, $X - \cup_{i=1}^k A_i \subseteq X - \cup_{i=1}^n A_i$ and so S is always in $X - \cup_{i=1}^n A_i$. Thus, for every $d \in D$, $g_n \circ S(d) = (f \circ S(d), n+1)$ so that $g_n \circ S$ converges to $(y, n+1)$ in Z_n. Similarly $g_k \circ S$ converges to $(y, k+1) = z(k)$ in Z_k. Since ϕ_{kn} is continuous, $\phi_{kn} \circ g_k \circ S$ converges to $\phi_{kn}(z(k))$, which equals $z(n)$ since $z \in Z$. Also $\phi_{kn} \circ g_k \circ S = g_n \circ S$. Since limits are unique in a T_2 space, $z(n) = (y, n+1)$, a contradiction.

Lemma R23.2.15 Let $n, k \in \mathbb{N}$ with $n \le k$. Assume A_i, A_j are consecutive at level n and A_r, A_s are consecutive at level k. Then $(A_i \Delta A_j) \cap (A_r \Delta A_s) = \emptyset$ or $A_r \Delta A_s \subseteq A_i \Delta A_j$.

Proof: Assume $\{A_1, \ldots, A_n\}$ reordered as a chain is $B_1 \subseteq \cdots \subseteq B_n$ and proceed by induction on k. When needed, $B_0 = \emptyset$. If $k = n$, then $A_i \Delta A_j$ and $A_r \Delta A_s$ are elements of $\{B_t - B_{t-1} : 1 \le t \le n\}$, which is a partition of B_n, and so the conclusion holds. Now assume the conclusion holds for some k and that $\{A_1, \ldots, A_k\}$ reordered as a chain is $C_1 \subseteq \cdots \subseteq C_k$. Again $C_0 = \emptyset$. There are two possibilities when A_{k+1} is included. If $C_k \subseteq A_{k+1}$ then $A_r \Delta A_s$ is either $A_{k+1} - C_k$ or $C_u - C_{u-1}$ for some $1 \le u \le k$. The first is disjoint from each $B_t - B_{t-1}$ and the induction hypothesis applies to the second. Thus the conclusion holds. If there is v such that $C_{v-1} \subseteq A_{k+1} \subseteq C_v$, then $A_r \Delta A_s$ is either $A_{k+1} - C_{v-1}$, $C_v - A_{k+1}$, or $C_u - C_{u-1}$ where $0 \le u \le v-1$ or $v+1 \le u \le k$. Since the first two are contained in $C_v - C_{v-1}$, the conclusion easily follows from the induction hypothesis.

Note the previous implicitly shows that $A_r \Delta A_s$ may be disjoint from all $A_i \Delta A_s$, e.g., when $n < k$, $\mathcal{A}$ is an increasing sequence, and $A_r \Delta A_s = A_k - A_{k-1}$.

Corollary R23.2.16 Let $n, k \in \mathbb{N}$ with $n \le k$. Assume A_i, A_j are consecutive at level n and A_r, A_s are consecutive at level k. If $(A_i \Delta A_j) \cap (A_r \Delta A_s) \ne \emptyset$, then $A_r \Delta A_s \subseteq A_i \Delta A_j$.

Proof: Immediate from R23.2.15.

Lemma R23.2.17 Let $z \in Z$ and let y be the element of Y such that, for every n, $z(n) = (y, \alpha_n)$ for some $\alpha_n \in \{1, \ldots, n+1\}$. If there is m such that $\alpha_m \le m$, then there is an $\mathcal{A}$-compatible net $S : D \to X$ such that $g \circ S$ converges to z.

Proof: Define $h : \mathbb{N} \to \mathbb{N} \cup \{0\} \times \mathbb{N} \cup \{0\}$ as follows: If $n < m$, $h(n) = (n, n)$. If $n \ge m$, by R23.2.14 $\alpha_n \le n$ so that there exist A_i and A_j consecutive at level n with $(y, \alpha_n) \in \overline{f[A_i \Delta A_j]} \times \{\alpha_n\}$. Let $h(n) = (i, j)$. By R23.2.8 h is $\mathcal{A}$-suitable. Now let E be the set of open neighborhoods of $y \in Y$ ordered by reverse inclusion, a directed set. For each $n \ge m$, with $h(n) = (i, j)$, by the meaning of closure there is $S_n : E \to A_i \Delta A_j$ such that $f(S_n(e)) \in e$ for all $e \in E$. As usual, it follows that $f \circ S_n$ converges to y. Also, since S_n is always in $A_i \Delta A_j$, $g_n \circ S_n$ converges to (y, α_n). As a final preliminary observation, let $k \ge n \ge m$. Since ϕ_{kn} is continuous and $z \in Z$, $\phi_{kn} \circ g_k \circ S_k$ converges to $\phi_{kn}(y, \alpha_k) = \phi_{kn}(z(k)) = z(n)$.Thus, since $\phi_{kn} \circ g_k = g_n$, $g_n \circ S_k$ converges to $z(n) = (y, \alpha_n)$. Now define S from $E \times \{n : n \ge m\}$, a directed set with the product ordering, into X by $S(e, n) = S_n(e)$. Now we verify that S is $\mathcal{A}$-compatible. First let $n \ge m$ be given and assume $h(n) = (i, j)$. Let $k \ge n$. By construction S_k is always in $A_r \Delta A_s$, where $h(k) = (r, s)$. Since $X - \cup_{i=1}^n A_i$ and the symmetric differences of consecutive pairs at level n partition X, by R23.2.15 $A_r \Delta A_s$ is a subset of exactly one element of this partition.

Thus there is $\beta \in \{1, \ldots, n+1\}$ such that, for every $e \in E$, $g_n \circ S_k(e) = (f \circ S_k(e), \beta)$ so that $g_n \circ S_k$ converges to (y, β). As above, $g_n \circ S_k$ converges to (y, α_n), i.e., $\beta = \alpha_n$. By R23.2.15 $A_r \Delta A_s \subseteq A_i \Delta A_j$. From this observation it follows that, for $(e,k) \geq (Y,n)$, $S(e,k) \in A_i \Delta A_j$, i.e., S is eventually in $A_i \Delta A_j$. By definition S is $\mathcal{A}$-compatible. It is now sufficient to show that $g \circ S$ converges to z, i.e., that $\psi_n \circ g \circ S$ converges to $z(n)$ for all n. As a first case suppose $n \geq m$. Let $h(n) = (i,j)$. An open set in Z_n containing (y, α_n) must include an open set of the form $(e_0 \cap \overline{f[A_i \Delta A_j]}) \times \{\alpha_n\}$, where $e_0 \in E$. Let $(e,k) \geq (e_0, n)$, i.e., $e \subseteq e_0$ and $k \geq n$, and let $h(k) = (r,s)$. Since $g_n \circ S_k$ converges to (y, α_n), $g_n \circ S_k$ is eventually in $(e_0 \cap \overline{f[A_i \Delta A_j]}) \times \{\alpha_n\}$. Pick $\tilde{e} \in E$ such that $g_n \circ S_k(\tilde{e})$ is in $(e_0 \cap \overline{f[A_i \Delta A_j]}) \times \{\alpha_n\}$. By definition of g_n, $S_k(\tilde{e})$ is in $A_i \Delta A_j$ so that, since S_k maps into $A_r \Delta A_s$, $(A_i \Delta A_j) \cap (A_r \Delta A_s) \neq \emptyset$. By R23.2.15 $A_r \Delta A_s \subseteq A_i \Delta A_j$. Thus $g_n \circ S_k(e)$ is in $(e_0 \cap \overline{f[A_i \Delta A_j]}) \times \{\alpha_n\}$. Now $\psi_n \circ g \circ S(e,k) = g_n \circ S(e,k) = g_n \circ S_k(e)$ and so this says $\psi_n \circ g \circ S$ is eventually in $(e_0 \cap \overline{f[A_i \Delta A_j]}) \times \{\alpha_n\}$. Thus $\psi_n \circ g \circ S$ converges to $z(n)$. For the second case assume $n < m$. Because of R23.2.2, the equation $\phi_{mn}(w(m)) = w(n)$ for all $w \in Z$ implies $\phi_{mn} \circ \psi_m = \psi_n$. Thus $\psi_n \circ g \circ S = \phi_{mn} \circ \psi_m \circ g \circ S$. By the first case $\psi_m \circ g \circ S$ converges to $z(m)$ and so, since ϕ_{mn} is continuous, $\psi_n \circ g \circ S$ converges to $\phi_{mn}(z(m)) = z(n)$ as claimed.

Corollary R23.2.18 Let $z \in Z$. There is an $\mathcal{A}$-compatible net $S : D \to X$ such that $g \circ S$ converges to z.

Proof: By R23.2.6 there is $y \in Y$ such that, for every $n \in \mathbb{N}$, $z(n) = (y, \alpha_n)$ for some $\alpha_n \in \{1, \ldots, n+1\}$. If there is m such that $\alpha_m \leq m$, by R23.2.17 the conclusion holds. Otherwise apply R23.2.13.

The next lemma combines, with appropriate modifications, several lemmas from R22.

Lemma R23.2.19 Let $a, b \in Z$ and assume $\phi(a) = \phi(b)$. Then there is a $\mathcal{U}$-Cauchy net $S^* : D^* \to X$ with two $\mathcal{A}$-compatible subnets, one whose g-image converges to a and one whose g-image converges to b.

Proof: By R23.2.18 there exist $\mathcal{A}$-compatible nets $S : D \to X$ and $T : E \to X$ such that $g \circ S$ converges to a and $g \circ T$ converges to b. As observed just before R22.3.6 one can assume $D \cap E = \emptyset$. Let D^* be the union of $\{(d,(d,e)) : (d,e) \in D \times E\}$ and $\{(e,(d,e)) : (d,e) \in D \times E\}$. Define $\geq^*$ on D^* by $(x,(d,e)) \geq^* (y,(f,g))$ if and only if $d \geq_D f$ and $e \geq_E g$. By R22.3.4 with this ordering D^* is a directed set. Define $S^* : D^* \to X$ by $S^*(d,(d,e)) = S(d)$ and $S^*(e,(d,e)) = T(e)$. S^* is a function since $D \cap E = \emptyset$. As a final preliminary, let $y = \phi(a) = \phi(b)$ and note that, since $\phi \circ g = f$, both $f \circ S$ and $f \circ T$ converge to y. Now begin: Since g is a unimorphism onto $g[X]$, S and T are both $\mathcal{U}_e(\mathcal{A})$-Cauchy and so also Cauchy relative to the smaller uniformity $\mathcal{U}$. Next it will be shown that S^* is $\mathcal{U}$-Cauchy. Let $U \in \mathcal{U}$ with $U = U^{-1}$. There exist $d_0 \in D$ and $e_0 \in E$ such that $p, q \in D$ with $p \geq_D d_0, q \geq_D d_0$ implies $(S(p), S(q)) \in U$ and $r, s \in E$ with $r \geq_E e_0, s \geq_E e_0$ implies $(T(r), T(s)) \in U$. Since f is a uniform embedding, there is V in the unique uniformity for Y with $(f \times f)[U] = (f[X] \times f[X]) \cap V$. Pick $W = W^{-1}$ in the unique uniformity for Y with $W \circ W \subseteq V$. By convergence there exist $d_1 \in D$ and $e_1 \in E$ such that $d \geq_D d_1$ implies $(f(S(d)), y) \in W$ and $e \geq_E e_1$ implies $(f(T(e)), y) \in W$. Then $d \geq_D d_1$ and $e \geq_E e_1$ implies $(f(S(d)), f(T(e))) \in V$ and so $(S(d), T(e)) \in U$. Pick $d_2 \in D$ larger than both d_0, d_1 and $e_2 \in E$ larger than both e_0, e_1. It is easy to check by cases that, if $(x,(p,r))$ and $(z,(q,s))$ are both larger in D^* than $(d_2,(d_2,e_2))$, then

$(S^*(x,(p,r)), S^*(z,(q,s))) \in U$. Thus S^* is $\mathcal{U}$-Cauchy.

Lastly the required subnets will be constructed. Note that $D \times E$ is a directed set with ordering defined by $(m,r) \geq (n,s)$ if and only if $m \geq_D n$ and $r \geq_E s$. Define $u : D \times E \to D^*$ by $u(d,e) = (d,(d,e))$. For any $(x,(m,r)) \in D^*$, if $(d,e) \geq (m,r)$, $u(d,e) \geq^* u(m,r)$. Thus u is a finalizing map, i.e., it has the subnet property, i.e., $S^* \circ u$ is a subnet of S^*. In addition, $S^* \circ u(d,e) = S(d)$. Since $g \circ S$ converges to a, it is easily checked by fixing an arbitrary element of E that $g \circ S^* \circ u$ also converges to a. If S is eventually in $X - \cup_{i=1}^n A$ for every n, in much the same way $S^* \circ u$ is eventually in $X - \cup_{i=1}^n A_i$ for every n. If there exist $m \in \mathbb{N}$ and an $\mathcal{A}$-suitable map h such that $n \geq m$ implies S is eventually in $A_i \Delta A_j$ where $h(n) = (i,j)$, likewise $n \geq m$ implies $S^* \circ u$ is eventually in $A_i \Delta A_j$ where $h(n) = (i,j)$. Thus $S^* \circ u$ is $\mathcal{A}$-compatible. Similarly, for $v : D \times E \to D^*$ by $v(d,e) = (e,(d,e))$, $S^* \circ v$ is an $\mathcal{A}$-compatible subnet and $g \circ S^* \circ v$ converges to b.

Corollary R23.2.20 $R_1^Z(\mathcal{A}) = R_\phi$ and so $R_1^Z(\mathcal{A})$ is an equivalence relation.

Proof: By R23.2.12 $R_1^Z(\mathcal{A}) \subseteq R_\phi$ and R23.2.19 says $R_1^Z(\mathcal{A}) \supseteq R_\phi$. The second claim is immediate because R_ϕ is an equivalence relation.

The proof of the next proposition is almost identical to the proof of R22.3.10.

Proposition R23.2.21 Let π denote the canonical projection from $Z \to Z/R_1^Z(\mathcal{A})$. Then $[(Z/R_1^Z(\mathcal{A}), \pi \circ g)]$ is the T_2 compactification class corresponding to $\mathcal{U}$.

Proof: By the previous corollary $R_1^Z(\mathcal{A}) = R_\phi$. By general theory, since ϕ must be a closed map, the compact, T_2 space Y is homeomorphic to the quotient space Z/R_ϕ via the map ψ, which maps y to the R_ϕ-equivalence class $\phi^{-1}[\{y\}]$. It is easy to check that $\psi \circ f = \pi \circ g$. R22.3.9 applies and so $[(Z/R_1^Z(\mathcal{A}), \pi \circ g)] = [(Y,f)]$, i.e., $[(Z/R_1^Z(\mathcal{A}), \pi \circ g)]$ is the T_2 compactification class corresponding to $\mathcal{U}$.

The previous proposition can, unsurprisingly, be modified to apply to an arbitrary representation of the compactification class corresponding to $\mathcal{U}_e(\mathcal{A})$. The rest of this subsection fills in the routine details using definition R23.2.11.

Lemma R23.2.22 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A}$ be a countably infinite chain of subsets of X. Suppose $\mathcal{U}_e(\mathcal{A})$ corresponds to the compactification class $[(W,h)]$. Then $R_1^W(\mathcal{A})$ is an equivalence relation.

Proof: Using any one-to-one indexing of $\mathcal{A}$, construct (Z,g) as above. Since (Z,g) is in the compactification class $[(W,h)]$, there is a homeomorphism $\psi : W \to Z$ such that $\psi \circ h = g$. Let $a, b \in W$ with $aR_1^W(\mathcal{A})b$. By definition there is a $\mathcal{U}$-Cauchy net $S : D \to X$ with $\mathcal{A}$-compatible subnets T_1 and T_2 such that $h \circ T_1$ converges to a and $h \circ T_2$ converges to b. By continuity $\psi \circ h \circ T_1 = g \circ T_1$ converges to $\psi(a)$ and $\psi \circ h \circ T_2$ converges to $\psi(b)$. Thus $\psi(a)R_1^Z(\mathcal{A})\psi(b)$. Similarly, for $c, d \in Z$, $cR_1^Z(\mathcal{A})d$ implies $\psi^{-1}(c)R_1^W(\mathcal{A})\psi^{-1}(d)$. Since $R_1^Z(\mathcal{Z})$ is an equivalence relation by R23.2.20, it follows easily that $R_1^W(\mathcal{A})$ is also an equivalence relation.

Proposition R23.2.23 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A}$ be a countably infinite chain of subsets of X. Suppose $\mathcal{U}_e(\mathcal{A})$ corresponds to the compactification class $[(W,h)]$. Let ρ denote the canonical projection from W to $W/R_1^W(\mathcal{A})$. Then $(W/R_1^W(\mathcal{A}), \rho \circ h)$ is a T_2 compactification of $(X, \tau(\mathcal{U}))$ and $[(W/R_1^W(\mathcal{A}), \rho \circ h)]$ is the T_2 compactification class corresponding to $\mathcal{U}$.

Proof: Using any one-to-one indexing of $\mathcal{A}$, construct (Z,g) as above. By R22.3.9

and R23.2.21 it is sufficient to find a homeomorhism $\overline{\psi}$ from $W/R_1^W(\mathcal{A})$ to $Z/R_1^Z(\mathcal{A})$ such that $\overline{\psi} \circ \rho \circ h = \pi \circ g$, where π is the canonical projection from Z to $Z/R_1^Z(\mathcal{A})$. Since (Z, g) is in the compactification class $[(W, h)]$, there is a homeomorphism $\psi : W \to Z$ such that $\psi \circ h = g$. Define $\overline{\psi} : W/R_1^W(\mathcal{A}) \to Z/R_1^Z(\mathcal{A})$ by $\overline{\psi}([w]_W) = [\psi(w)]_Z$. As in the proof of R23.2.22 $aR_1^W(\mathcal{A})b$ if and only if $\psi(a)R_1^Z(\mathcal{A})\psi(b)$. From this fact it follows easily that $\overline{\psi}$ is in fact a function and is one-to-one. Since ψ is onto, $\overline{\psi}$ is also onto. For $w \in W$, $\overline{\psi} \circ \rho(w) = \overline{\psi}([w]_W) = [\psi(w)]_Z = \pi \circ \psi(w)$, i.e., $\overline{\psi} \circ \rho = \pi \circ \psi$, which is continuous. Since ρ is a quotient map, this implies $\overline{\psi}$ is continuous. Since $\psi \circ h = g$, it also implies $\overline{\psi} \circ \rho \circ h = \pi \circ g$. Lastly, since the domain is compact and the image T_2, $\overline{\psi}$ is also closed. Thus $\overline{\psi}$ is the required homeomorphism.

Extensions and Stone-Čech Compactifications

In this subsection partial answers are given about what can said if $\mathcal{U}$ (or $\mathcal{U}_e(\mathcal{A})$) corresponds to the Stone-Čech compactification of its underlying topological space. The first fact is quite simple. Note that discrete case is one when the hypothesis of extension by clopen sets holds.

Lemma R23.3.1 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A}$ be a non-empty collection of subsets of X. Assume each $A \in \mathcal{A}$ is $\tau(\mathcal{U})$-clopen. If $\mathcal{U}$ corresponds to the Stone-Čech compactification, then $\mathcal{U} = \mathcal{U}_e(\mathcal{A})$.

Proof: By definition $\mathcal{U}_e(\mathcal{A}) = \mathcal{U} \vee (\vee\{\mathcal{U}_{E(A)} : A \in \mathcal{A}\}$ and so by P2.14 $\tau(\mathcal{U}_e(\mathcal{A})) = \tau(\mathcal{U}) \vee (\vee\{\tau(\mathcal{U}_{E(A)}) : A \in \mathcal{A}\}$. For each $A \in \mathcal{A}$, $\tau(\mathcal{U}_{E(A)}) = \{\emptyset, A, X - A, X\}$ and, since A is $\tau(\mathcal{U})$-clopen, $\tau(\mathcal{U}_{E(A)}) \subseteq \tau(\mathcal{U})$. Thus $\tau(\mathcal{U}_e(\mathcal{A})) = \tau(\mathcal{U})$. If $\mathcal{U}$ corresponds to the Stone-Čech compactification, then as noted in [2] $\mathcal{U}$ is the largest totally bounded uniformity generating $\tau(\mathcal{U})$ and so, since $\mathcal{U} \subseteq \mathcal{U}_e(\mathcal{A})$, the equality follows.

The lemma provides admittedly trivial examples when both $\mathcal{U}$ and $\mathcal{U}_e(\mathcal{A})$ correspond to the Stone-Čech compactification. Its contrapositive yields the following more interesting example, in which the larger uniformity corresponds to the Stone-Čech compactification but the smaller does not.

Example R23.3.2 Let (X, τ) be a discrete space with X infinite. Let $\mathcal{U}$ be the totally bounded uniformity generating τ such that $\mathcal{U}$ corresponds to the one point compactification, which is not the Stone-Čech compactification. Let $\mathcal{A}$ be the power set of X. As noted in [3] each two-point compactification of X corresponds to $\mathcal{U} \vee \mathcal{U}_{E(A)}$ for some $A \subseteq X$ with both A and $X - A$ infinite. Thus $\mathcal{U}_e(\mathcal{A})$ corresponds to the supremum of the two-point compactifications, which by R6.1.8 is the Stone-Čech compactification.

Lemma R23.3.3 Let X be a non-empty set and let $\mathcal{W} = \vee\{\mathcal{U}_{E(A)} : A \subseteq X\}$. Then $\mathcal{W}$ is the largest totally bounded uniformity generating the discrete topology on X and $\mathcal{W}$ corresponds to the Stone-Čech compactification of X with the discrete topology.

Proof: The set $\{E(\{x\}) : x \in X\}$ is a basis for $\mathcal{U}_m$, the uniformity corresponding to the one-point compactification of X with the discrete topology. Thus $\mathcal{U}_m \subseteq \mathcal{W}$. By P2.13 $\mathcal{W}$ is totally bounded and by R5.1.10 and R5.2.4 it contains every uniformity corresponding to a two-point compactification of X with the discrete topology. The conclusion is now immediate from R6.1.8.

Example R23.3.4 Let X be a non-empty set and let $\mathcal{W} = \vee\{\mathcal{U}_{E(A)} : A \subseteq X\}$. Let $\mathcal{A}$ be the power set of X. For any separated totally bounded uniformity $\mathcal{U}$ on X, clearly $\mathcal{U}_e(\mathcal{A}) = \mathcal{U} \vee \mathcal{W}$, which corresponds to the Stone-Čech compactification of X with the discrete topology. For infinite X one can choose $\mathcal{U}$ corresponding to the Stone-Čech compactification of $(X, \tau(\mathcal{U}))$ with $\tau(\mathcal{U})$ not discrete. This provides non-trivial examples when both $\mathcal{U}$ and $\mathcal{U}_e(\mathcal{A})$ correspond to the Stone-Čech compactification.

Lemma R23.3.5 (Pasting Lemma) Let $(X,\mathcal{U})$ be a uniform space. Assume $X = A_1 \cup A_2$ with $A_1 \cap A_2 = \emptyset$. Let $\mathcal{V}_1$ be a uniformity on A_1 and $\mathcal{V}_2$ a uniformity on A_2. Let $W = \{W \subseteq X \times X :$ for some $V_1 \in \mathcal{V}_1$ and $V_2 \in \mathcal{V}_2$, $V_1 \cup V_2 \subseteq W\}$. Then $\mathcal{W}$ is a uniformity for X. Moreover, if $\tau(\mathcal{V}_i)$ is contained in the subspace topology on A_i from $\tau(\mathcal{U})$ for $i = 1, 2$, then $\tau(\mathcal{W}) \subseteq \tau(\mathcal{U}_e(\{A_1\}))$.

Proof: For the first conclusion, the diagonal and superset requirements are clearly satisfied. Since $\mathcal{V}_1$ and $\mathcal{V}_2$ are uniformities and $V_1 \cup V_2 \subseteq W$ implies $V_1^{-1} \cup V_2^{-1} \subseteq W^{-1}$, the symmetry requirement follows easily. Since A_1 and A_2 are disjoint, for $U_i, V_i \in \mathcal{V}_i$, $(U_1 \cup U_2) \cap (V_1 \cup V_2) = (U_1 \cap V_1) \cup (U_2 \cap V_2)$, which easily yields the finite intersection property. Finally let $U_i \circ U_i \subseteq V_i$. Since A_1 and A_2 are disjoint, $(U_1 \cup U_2) \circ (U_1 \cup U_2) \subseteq V_1 \cup V_2$. The triangle inequality follows easily. Thus $\mathcal{W}$ is a uniformity for X. Now assume $\tau(\mathcal{V}_i)$ is contained in the subspace topology on A_i from $\tau(\mathcal{U})$ for $i = 1, 2$. Let $O \in \tau(\mathcal{W})$ and let $x \in O$. For some $W \in \mathcal{W}$, $W[x] \subseteq O$. There exist $V_i \in \mathcal{V}$ such that $V_1 \cup V_2 \subseteq W$. As a first case assume $x \in A_2$. By hypothesis there is $G \in \tau(\mathcal{U})$ such that $x \in G \cap A_2 \subseteq V_2[x]$. Since $x \notin A_1$, $(V_1 \cup V_2)[x] = V_2[x]$ and so $G \cap A_2 \subseteq W[x]$. Since $A_2 = X - A_1$ is in the topology generated by $\mathcal{U}_{E(A_1)}$ and $\tau(\mathcal{U}) \vee \tau(\mathcal{U}_{E(A_1)}) = \tau(\mathcal{U}_e(\{A_1\}))$, O is a $\tau(\mathcal{U}_e(\{A_1\}))$-neighborhood of x. Similarly, if $x \in A_1$, O is a $\tau(\mathcal{U}_e(\{A_1\}))$-neighborhood of x. Thus $O \in \tau(\mathcal{U}_e(\{A_1\}))$ as required.

Lemma R23.3.6 Let X be a set and let $\mathcal{U}, \mathcal{V}$ be totally bounded uniformities for X with $\tau(\mathcal{U}) \subseteq \tau(\mathcal{V})$. Let $\mathcal{W}$ be the largest totally bounded uniformity generating $\tau(\mathcal{V})$. Then $\mathcal{U} \subseteq \mathcal{W}$.

Proof: By P2.13 $\mathcal{U} \vee \mathcal{V}$ is totally bounded and by P2.14 $\tau(\mathcal{U} \vee \mathcal{V}) = \tau(\mathcal{U}) \vee \tau(\mathcal{V})$, which by hypothesis must be $\tau(\mathcal{V})$. The assumed property of $\mathcal{W}$ implies $\mathcal{U} \vee \mathcal{V}$ (and so $\mathcal{U}$) is contained in $\mathcal{W}$.

The previous lemma will be applied with $\mathcal{V}$ separated, in which case $\mathcal{W}$ corresponds to the Stone-Čech compactification of $(X, \tau(\mathcal{V}))$, as observed in R1.8. Note that $\mathcal{U}$ need not be separated.

The next results make use of the sets B_p as defined in R22.2.2.

Lemma R23.3.7 Let $(X,\mathcal{U})$ be a separated totally bounded uniform space and let $\mathcal{A}$ be a non-empty collection of subsets of X. Each $A \in \mathcal{A}$ is clopen in $(X, \tau(\mathcal{U}_e(\mathcal{A})))$ and each B_p is $\tau(\mathcal{U}_e(\mathcal{A}))$-closed. If $\mathcal{A}$ is finite, each B_p is $\tau(\mathcal{U}_e(\mathcal{A}))$-clopen.

Proof: For $A \in \mathcal{A}$, $E(A)$ is in $\mathcal{U}_e(\mathcal{A})$. For $x \in A$, $E(A)[x] = A$ and, for $x \in X - A$, $E(A)[x] = X - A$. It follows easily that both A and $X - A$ are $\tau(\mathcal{U}_e(\mathcal{A}))$-open. By definition each B_p is an intersection of closed sets and so closed. If $\mathcal{A}$ is finite, each B_p is a finite intersection of clopen sets and so clopen.

Proposition R23.3.8 Let $(X,\mathcal{U})$ be a separated totally bounded uniform space and let $\mathcal{A}$ be a finite, non-empty collection of subsets of X. Assume that, for at least one B_p generated from $\mathcal{A}$, the subspace uniformity for B_p from $\mathcal{U}$ does not correspond to the Stone-Čech compactification of B_p with the subspace topology from $\tau(\mathcal{U})$. Then $\mathcal{U}_e(\mathcal{A})$ does not correspond to the Stone-Čech compactification of $(X, \tau(\mathcal{U}_e(\mathcal{A})))$.

Proof: Let B_p be as hypothesized. Pick a sequence $\{V_n\}$ in the uniformity corresponding to the Stone-Čech compactification of B_p with the subspace topology from $\tau(\mathcal{U})$ such that each V_n is not in the subspace uniformity for B_p from $\mathcal{U}$ and, for all n, $V_n = V_n^{-1}$ and $V_{n+1} \circ V_{n+1} \subseteq V_n$. Let $\mathcal{V}_1 = \{W \subseteq B_p \times B_p : V_n \subseteq W \text{ for some } n\}$. With the properties of $\{V_n\}$ it is easy to check that $\mathcal{V}_1$ is a uniformity on B_p. Let $\mathcal{V}_2$ be the indiscrete uniformity on $X - B_p$. Since $\mathcal{A}$ is finite, B_p is $\tau(\mathcal{U}_e(\mathcal{A}))$-clopen. It follows that both $\tau(\mathcal{V}_1)$ and $\tau(\mathcal{V}_2)$ are contained in $\tau(\mathcal{U}_e(\mathcal{A}))$. Let $\mathcal{W}$ be the uniformity for X constructed as in the pasting lemma. By R23.3.5 $\tau(\mathcal{W}) \subseteq \tau(\mathcal{U}_e(\mathcal{A}))$ and so, by R23.3.6, $V_1 \cup ((X - B_p) \times (X - B_p))$ is in the uniformity corresponding to the Stone-Čech compactification of $(X, \tau(\mathcal{U}_e(\mathcal{A})))$. It is claimed that $V_1 \cup ((X - B_p) \times (X - B_p))$ is not in $\mathcal{U}_e(\mathcal{A})$. Deny the claim. Then there is $U \in \mathcal{U}$ such that $U \cap (\cap\{E(A) : A \in \mathcal{A}\} \subseteq V_1 \cup ((X - B_p) \times (X - B_p))$. Let $(x, y) \in U \cap B_p \times B_p$. By definition of B_p, for each $A \in \mathcal{A}$, x and y are both in A or both in $X - A$, i.e., $(x, y) \in E(A)$. Thus $(x, y) \in V_1 \cup ((X - B_p) \times (X - B_p))$ and, since x, y are both in B_p, $(x, y) \in V_1$. But $U \cap B_p \times B_p \subseteq V_1$ contradicts the choice of V_1. Thus the claim holds and the conclusion immediately follows.

Corollary R23.3.9 Let $(X, \mathcal{U})$ be a separated totally bounded uniform space. Suppose $A \subseteq X$ has the property that the subspace uniformity for A from $\mathcal{U}$ does not correspond to the Stone-Čech compactification of A with the subspace topology from $\tau(\mathcal{U})$. Then $\mathcal{U}_e(\{A\})$ does not correspond to the Stone-Čech compactification of $(X, \tau(\mathcal{U}_e(\{A\})))$.

Proof: Let $\mathcal{A} = \{A\}$. With $p(A) = A$, $B_p = A$ and R23.3.8 applies.

This result makes it easy to find examples in which a finite extension of $\mathcal{U}$ does not correspond to the Stone-Čech compactification, whether or not $\mathcal{U}$ does, as the following illustrates.

Example R23.3.10 Let $X = [0, 1]$ and let $\mathcal{U}$ be the usual uniformity for X. Since $(X, \tau(\mathcal{U}))$ is compact, it is its Stone-Čech compactification, which corresponds to the uniformity $\mathcal{U}$. Let $A = [0, 1)$ with the subspace topology. The usual uniformity on A (the subspace uniformity from $\mathcal{U}$) corresponds to the one-point compactification $[0, 1]$, which is not the Stone-Čech compactification, and so by R23.3.9 $\mathcal{U}_e(\{A\})$ is not the uniformity corresponding to the Stone-Čech compactification of $(X, \tau(\mathcal{U}_e(\{A\}))$. As a variation, let $Y = [0, 2)$, let $\mathcal{V}$ be the usual uniformity on Y, and let A be as before. In this case $\mathcal{V}$ does not correspond to the Stone-Čech compactification. In the same way, $\mathcal{V}_e(\{A\})$ is not the uniformity corresponding to the Stone-Čech compactification of $(Y, \tau(\mathcal{V}_e(\{A\}))$.

Finite-point Compactifications and Extensions

The first proposition is a general result which applies to arbitrary extensions.

Proposition R23.4.1 Let $\mathcal{U}, \mathcal{V}$ be separated, totally bounded uniformities for a set X with $\mathcal{U} \subseteq \mathcal{V}$. Let $\mathcal{U}$ correspond to the compactification class $[(Y, f)]$ and $\mathcal{V}$ to the class $[(Z, g)]$. If (Z, g) is a finite-point compactification of $(X, \tau(\mathcal{V}))$, then (Y, f) is a finite-point compactification of $(X, \tau(\mathcal{U}))$ and $|Y - f[X]| \leq |Z - g[X]|$.

Proof: By R13.1.2 $\mathcal{U} \subseteq \mathcal{V}$ implies $[(Y, f)] \leq [(Z, g)]$ and so by definition there is a continuous surjection $\phi : Z \to Y$ such that $\phi \circ g = f$. First it will be shown that $Y - f[X] \subseteq \phi[Z - g[X]]$. Let $y \in Y - f[X]$ and pick Z such that $\phi(z) = y$. Suppose $z = g(x)$ for some x. Then $y = \phi(z) = \phi(g(x)) = f(x)$, which contradicts the choice of y. Thus the claim holds. Since $|\phi[Z - g[X]| \leq |Z - g[X]|$ and $Z - g[X]$ is given to be finite,

$Y - f[X]$ is also finite and $|Y - f[X]| \leq |Z - g[X]|$.

Corollary R23.4.2 Let $\mathcal{U}$ be a separated, totally bounded uniformity for a set X and let $\mathcal{A}$ be a non-empty collection of subsets of X. If $\mathcal{U}_e(\mathcal{A})$ corresponds to a finite-point compactification class for $(X, \tau(\mathcal{U}_e(\mathcal{A}))$, then, for every $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{U}_e(\mathcal{B})$ corresponds to a finite-point compactification class for $(X, \tau(\mathcal{U}_e(\mathcal{B})))$.

Proof: Since $\mathcal{U}_e(\mathcal{B}) \subseteq \mathcal{U}_e(\mathcal{A})$, this follows immediately from the previous proposition.

Corollary R23.4.3 Let $\mathcal{U}$ be a separated, totally bounded uniformity for a set X and let $\mathcal{A}$ be a non-empty collection of subsets of X. If $\mathcal{U}_e(\mathcal{A})$ corresponds to a finite-point compactification class for $(X, \tau(\mathcal{U}_e(\mathcal{A}))$, then $\mathcal{U}$ corresponds to a finite-point compactification class for $(X, \tau(\mathcal{U}))$.

Proof: Again, $\mathcal{U} \subseteq \mathcal{U}_e(\mathcal{A})$ and so this is immediate from R23.4.1.

The converse of the previous corollary is false, as is shown below, but positive results can be obtained in certain cases. The next result and its proof refer to $\mathcal{C}(\mathcal{A})$ and the sets B_p,where $p \in \mathcal{C}(\mathcal{A})$, which are generated from the collection $\mathcal{A}$ as defined in R22.2.2.

Proposition R23.4.4 Let $\mathcal{U}$ be a separated, totally bounded uniformity for a set X and let $\mathcal{A}$ be a finite, non-empty collection of subsets of X. Let $\mathcal{U}$ correspond to the compactification class $[(Y, f)]$ and, for $S \subseteq Y$, let $\overline{S}$ denote the closure of S in Y. Let $\mathcal{U}_e(\mathcal{A})$ correspond to the compactification class $[(W, h)]$. Then (W, h) is a finite-point compactification of $(X, \tau(\mathcal{U}_e(\mathcal{A})))$ if and only if $\overline{f[B_p]} - f[B_p]$ is finite for every p.

Proof: By R22.2.6 (W, h) is equivalent to (Z, g), where $Z = \coprod\{\overline{f[B_p]} : p \in \mathcal{C}(\mathcal{A})\}$ and $g : X \to Z$ by $g(x) = (f(x), p)$ for $x \in B_p$. It is easy to check that the set $g[X]$ is $\cup\{\{(f(x), p) : x \in B_p\} : p \in \mathcal{C}(\mathcal{A})\}$ and so $Z - g[X] = \cup\{\overline{f[B_p]} - f[B_p] : p \in \mathcal{C}(\mathcal{A})\}$. Since $\mathcal{C}(\mathcal{A})$ is finite, $Z - g[X]$ is finite if and only if $\overline{f[B_p]} - f[B_p]$ is finite for every p. By equivalence (W, h) is a finite-point compactification if and only if (Z, g) is, i.e., if and only if $Z - g[X]$ is finite.

The previous proposition easily yields many examples showing that the converse of R23.4.3 is false, one of which is the following.

Example R23.4.5 Let X be the set of reals and $\mathcal{U}$ be the uniformity for X which corresponds to the one-point compactification class for X with the usual topology. Let $[(Y, f)]$ be the class of the one-point compactification and let A be the set of rational numbers. Since $\overline{f[A]} - f[A] \supseteq f[X] - f[A]$, by R23.4.3 $\mathcal{U}_e(\mathcal{A})$ does not correspond to a finite-point compactification.

The method of the previous example provides a necessary condition for an extension to correspond to a finite-point compactification class.

Proposition R23.4.6 Let $\mathcal{U}$ be a separated, totally bounded uniformity for a set X and let $\mathcal{A}$ be a non-empty collection of subsets of X. Let c denote the closure operator for $(X, \tau(\mathcal{U}))$. If $\mathcal{U}_e(\mathcal{A})$ corresponds to a finite-point compactification class for $(X, \tau(\mathcal{U}_e(\mathcal{A}))$, then, for every A in $\mathcal{A}$, both $c(A) - A$ and $c(X - A) - (X - A)$ are finite.

Proof: Let $A \in \mathcal{A}$ and let $\mathcal{B} = \{A\}$. By R23.4.2 $\mathcal{U}(\mathcal{B})$ corresponds to a finite-point compactification class. For the singleton $\mathcal{B}$ there are exactly two B_p sets, A and $X - A$. Letting $[(Y, f)]$ denote the compactification class corresponding to $\mathcal{U}$ and $\overline{S}$ denote the closure of S contained in Y, one has by R23.4.4 that $\overline{f[A]} - f[A]$ is finite. Since f is continuous, $f[c(A)] - f[A]$ is contained in $\overline{f[A]} - f[A]$. Since f is one-to-one, $f[c(A) - A] = c(A) - A$ and the latter is finite. Similarly, $c(X - A) - (X - A)$ is finite.

R23.4.4 also yields the following positive result.

Corollary R23.4.7 Let $\mathcal{U}$ be a separated, totally bounded uniformity for a set X and let $\mathcal{A}$ be a finite, non-empty collection of clopen subsets of X. Let $\mathcal{U}$ correspond to the compactification class $[(Y, f)]$ and let $\mathcal{U}_e(\mathcal{A})$ correspond to the compactification class $[(W, h)]$. Then (W, h) is a finite-point compactification of $(X, \tau(\mathcal{U}_e(\mathcal{A})))$ if and only if (Y, f) is a finite-point compactification of $(X, \tau(\mathcal{U}))$.

Proof: If (W, h) is a finite-point compactification, so is (Y, f) by R23.4.3. Conversely assume (Y, f) is a finite-point compactification and, for $S \subseteq Y$, let $\overline{S}$ denote the closure of S in Y. For each $p \in \mathcal{C}(\mathcal{A})$, since both A and $X - A$ are $\tau(\mathcal{U})$-closed for all $A \in \mathcal{A}$, by definition B_p is $\tau(\mathcal{U})$-closed and so $\overline{f[B_p]} - f[B_p] \subseteq Y - f[X]$, which is finite by hypothesis. By R23.4.4 (W, h) is a finite-point compactification.

Another necessary condition can be developed using the equivalence relation associated with a collection $\mathcal{A}$, which is defined next. Recall from R22.2.3 that the non-empty elements of $\{B_p : p \in \mathcal{C}(\mathcal{A})\}$ form a partition of X.

Definition R23.4.8 Let X be a set and let $\mathcal{A}$ be a non-empty collection of subsets of X. The equivalence relation associated with $\mathcal{A}$, denoted $E(\mathcal{A})$, is the relation generated by the non-empty elements of $\{B_p : p \in \mathcal{C}(\mathcal{A})\}$.

For $B \subseteq X$, $E(B)$ has denoted $B \times B \cup (X - B) \times (X - B)$. Clearly $E(B) = E(\{B\})$.

Lemma R23.4.9 Let E be an equivalence relation on a set X and let $\mathcal{A}$ be the the equivalence classes of E. Then $E(\mathcal{A}) = E$.

Proof: For $\mathcal{A}$ as given, B_p is non-empty if and only there is $A_0 \in \mathcal{A}$ such that $p(A_0) = A_0$ and $p(A) = X - A$ for $A \neq A_0$, i.e., B_p is non-empty if and only there is $A_0 \in \mathcal{A}$ such that $B_p = A_0$. Since $E(\mathcal{A})$ and E have the same equivalence classes, the equality holds.

Lemma R23.4.10 Let $(X,\mathcal{U})$ be a uniform space and let $\mathcal{A}$ be a non-empty collection of subsets of X. Then $\mathcal{U}_e(\mathcal{A}) \subseteq \mathcal{U} \vee \mathcal{U}_{E(\mathcal{A})}$. If $\mathcal{A}$ is finite, equality holds.

Proof: Because of R22.2.4, $\vee\{\mathcal{U}_{E(A)} : A \in \mathcal{A}\} \subseteq \mathcal{U}_{E(\mathcal{A})}$ and so the first claim follows from R22.1.1, the definition of $\mathcal{U}_e(\mathcal{A})$. R22.2.5 implies equality if $\mathcal{A}$ is finite.

Lemma R23.4.11 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space. Let E be an equivalence relation on X with finitely many equivalence classes, at least n of which are not $\tau(\mathcal{U})$-compact. Let $\mathcal{U} \vee \mathcal{U}_E$ correspond to the compactification class $[(W, h)]$. Then $|W - h[X]| \geq n$.

Proof: Let $[(Y, f)]$ be the compactification class corresponding to $\mathcal{U}$ and, for $S \subseteq Y$, let $\overline{S}$ denote the closure of S in Y. Let $\{C_1, \ldots, C_k, C_{k+1}, \ldots, C_m\}$ denote the equivalence classes of E, where $C_{k+1}, \ldots, C_m$ are $\tau(\mathcal{U})$-compact and $C_1, \ldots, C_k$ are not. Let $Z = \coprod\{\overline{f[C_i]} : 1 \leq i \leq m\}$ and let $g : X \to Z$ by $g(x) = (f(x), i)$ where $x \in C_i$. By R15.1.19 $[(Z, g)]$ is the compactification class corresponding to $\mathcal{U} \vee \mathcal{U}_E$. Since (Z, g) is equivalent to (W, h), $|W - h[X]| = |Z - g[X]|$. For $k + 1 \leq i \leq m$, $f[C_i]$ is compact in Y and so $\overline{f[C_i]} = f[C_i]$. Thus $Z - g[X] = \cup_{i=1}^{k}((\overline{f[C_i]} - f[C_i]) \times \{i\})$ and, since the terms in that union are disjoint, $|Z - g[X]| = \Sigma_{i=1}^{k}|\overline{f[C_i]} - f[C_i]|$. For $1 \leq i \leq k$, $f[C_i]$ is not compact in Y and so not closed, which implies $|\overline{f[C_i]} - f[C_i]| \geq 1$. Thus $|Z - g[X]| \geq k$. By hypothesis $k \geq n$.

Lemma R23.4.12 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A}$ be a non-empty collection of subsets of X. Assume for every positive integer k there is a finite non-empty $\mathcal{B}$ contained in $\mathcal{A}$ such that $E(\mathcal{B})$ has at least k non-compact (relative to $\tau(\mathcal{U})$) equivalence classes. Let $\mathcal{U}_e(\mathcal{A})$ correspond to the compactification class $[(W,h)]$. Then W is not a finite-point compactification of $(X,\tau(\mathcal{U}_e(\mathcal{A}))$.

Proof: Suppose $|W-h[X]|=m$. Pick a finite non-empty $\mathcal{B}\subseteq\mathcal{A}$ such that $E(\mathcal{B})$ has at least $m+1$ non-compact (relative to $\tau(\mathcal{U})$) equivalence classes. Let $[(Z,g)]$ be the compactification class corresponding to $\mathcal{U}_e(\mathcal{B})$. Since $\mathcal{U}_e(\mathcal{B})\subseteq\mathcal{U}_e(\mathcal{A})$, by R23.4.1 $|Z-g[X]|\le|W-h[X]|$. But by R23.4.10 and R23.4.11 $|Z-g[X]|\ge m+1$, a contradiction.

The previous lemma easily yields a necessary condition for $\mathcal{U}_e(\mathcal{A})$ to correspond to a finite-point compactification class.

Corollary R23.4.13 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A}$ be a non-empty collection of subsets of X. Let $\mathcal{U}_e(\mathcal{A})$ correspond to the compactification class $[(W,h)]$. If W is a finite-point compactification of $(X,\tau(\mathcal{U}_e(\mathcal{A}))$, then there is a positive integer M such that, for every finite non-empty $\mathcal{B}$ contained in $\mathcal{A}$, $E(\mathcal{B})$ has at most M non-compact (relative to $\tau(\mathcal{U})$) equivalence classes.

Proof: This is the contrapositive of R23.4.12.

The next corollary is a special case.

Corollary R23.4.14 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A}=\{A_n : n\in\mathbb{N}\}$ be a countable collection of subsets of X with $A_n\subseteq A_{n+1}$ for all n. Let $\mathcal{U}_e(\mathcal{A})$ correspond to the compactification class $[(W,h)]$. If W is a finite-point compactification of $(X,\tau(\mathcal{U}_e(\mathcal{A}))$, then eventually $A_{n+1}-A_n$ is $\tau(\mathcal{U})$-compact.

Proof: Assume $W-h[X]$ is finite and for notational convenience let $A_0=\emptyset$. By R23.4.13 there is an upper bound to the number of non-compact (relative to $\tau(\mathcal{U})$) equivalence classes of $E(\mathcal{B})$, where $\mathcal{B}$ is a finite subset of $\mathcal{A}$. Thus there is an integer m which is the largest number of non-compact equivalence classes of $E(\{A_1,\ldots,A_n\})$ over all n. As a first case, assume there is k such that $\{A_i-A_{i-1} : 1\le i\le k\}$ (which by R22.4.1 are $E(\{A_1,\ldots,A_k\})$ equivalence classes) has exactly m non-$\tau(\mathcal{U})$-compact elements. For $n\ge k$ these are also classes of $E(\{A_1,\ldots,A_{n+1}\}$ and so the additional class $A_{n+1}-A_n$ must be compact since the number of non-compact classes cannot increase. Now suppose the first case does not hold. Pick j such that $E(\{A_1,\ldots,A_j\})$ has m non-$\tau(\mathcal{U})$-compact classes. In this case, by R22.4.1 $X-A_j$ must be non-compact with the remaining $m-1$ non-compact classes in $\{A_i-A_{i-1} : 1\le i\le j\}$. For $n\ge j$ these are also classes of $E(\{A_1,\ldots,A_{n+1}\}$ and so the additional class $A_{n+1}-A_n$ must be compact since case one does not hold.

This subsection concludes with results related to the general question of when a uniformity $\mathcal{U}\vee\mathcal{U}_E$, where $(X,\mathcal{U})$ is separated and totally bounded and E is an equivalence relation on X with finitely many equivalence classes, corresponds to a finite-point extension. R23.4.9 and R23.4.10 relate this to the topic of this subsection.

Proposition R23.4.15 Let $\mathcal{U}$ be a separated, totally bounded uniformity for a set X. Assume $\mathcal{U}$ corresponds to the compactification class $[(Y,f)]$ with $|Y-f[X]|=k$. Let E be an equivalence relation with finitely many equivalence classes. Let $\mathcal{U}\vee\mathcal{U}_E$ correspond to the compactification class $[(W,h)]$. Assume that the non-$\tau(\mathcal{U})$-compact E-equivalence classes are $\{C_1,\ldots,C_n\}$. Let $\aleph_i=|c(C_i)-C_i|$, where c denotes the closure operator in $(X,\tau(\mathcal{U}))$. Then $\Sigma_{i=1}^n\aleph_i\le|W-h[X]|\le kn+\Sigma_{i=1}^n\aleph_i$.

Proof: For $S\subseteq Y$ let $\overline{S}$ denote the closure of S in Y and let the $\tau(\mathcal{U})$-compact E-

equivalence classes be $\{C_{n+1}, \ldots, C_m\}$. By R15.1.19 $\mathcal{U} \vee \mathcal{U}_E$ corresponds to $[(Z, g)]$, where $Z = \coprod_{i=1}^m \overline{f[C_i]}$ and $g : X \to Z$ by $g(x) = (f(x), i)$ if $x \in C_i$. Since (W, h) is equivalent to (Z, g), $|W - h[X]| = |Z - g[X]|$. Since $Z - g[X] = \cup_{i=1}^m((\overline{f[C_i]} - f[C_i]) \times \{i\})$ and the terms in that union are disjoint, $|Z - g[X]| = \Sigma_{i=1}^m|\overline{f[C_i]} - f[C_i]|$. For $n+1 \leq i \leq m$, since C_i is $\tau(\mathcal{U})$-compact, $f[C_i]$ is compact and so closed in Y, i.e., $\overline{f[C_i]} - f[C_i] = \emptyset$. Thus $|Z - g[X]| = \Sigma_{i=1}^n|\overline{f[C_i]} - f[C_i]|$. Moreover, for $1 \leq i \leq n$, $f[c(C_i) - C_i] \subseteq \overline{f[C_i]} - f[C_i] \subseteq (Y - f[X]) \cup f[c(C_i) - C_i]$ and so, since f is one-to-one, $\aleph_i \leq |\overline{f[C_i]} - f[C_i]| \leq k + \aleph_i$. The conclusion is now immediate.

In R5.1.1 it is shown that a T_2 space (X, τ) has an n-point compactification if and only if it is locally compact and has an n-star, i.e., a collection of n disjoint open sets $\{G_1, \ldots, G_n\}$ such that $K = X - \cup_{i=1}^n G_i$ is compact and $K \cup G_i$ is non-compact for every i. From the proof of R5.1.1 an n-star determines an n-point compactification class and by R5.1.2 each n-point compactification is in a compactification class determined by an n-star.

Note that $\{K, G_1, \ldots, G_n\}$ is a partition of X and so generates an equivalence relation E. It is natural to ask whether $\mathcal{U}_m \vee \mathcal{U}_E$, where $\mathcal{U}_m$ is the smallest uniformity for the locally compact τ, corresponds to the compactification class determined by the n-star or indeed to any finite-point compactification class. R5.2.4 gives a positive answer to both if τ is discrete, but, in general, the answer to the first question is negative: $\tau(\mathcal{U}_m \vee \mathcal{U}_E)$ may be strictly larger than τ and the compactification class determined by an n-star may not correspond to a uniformity of the form $\mathcal{U}_m \vee \mathcal{U}_E$. (See the addendum to [R5].) Results from this subsection provide the following straightforward answer for the second question.

Corollary R23.4.16 Let $\{G_1, \ldots, G_n\}$ be an n-star for the locally compact T_2 space (X, τ) and let the partition $\{G_1, \ldots, G_n, X - \cup_{i=1}^n G_i\}$ generate the equivalence relation E. Let $\mathcal{U}_m$ be the smallest uniformity for the locally compact τ and let $\mathcal{U}_m \vee \mathcal{U}_E$ correspond to the compactification class $[(W, h)]$. Let c denote the $\tau(\mathcal{U})$-closure operator. Then $W - h[X]$ is finite if and only if $c(G_i) - G_i$ is finite for $1 \leq i \leq n$.

Proof: Let $\aleph_i = |c(G_i) - G_i|$. R23.4.15 shows that $W - h[X]$ is infinite if even one $\aleph_i \geq \aleph_0$ and, conversely, $W - h[X]$ is finite if $\aleph_i < \aleph_0$ for $1 \leq i \leq n$.

The following easy examples show some possibilities.

Example R23.4.17 Let $X = [0, 1)$ with the usual topology. In this case $\mathcal{U}_m$ is the usual uniformity for X because it is the subspace uniformity from $[0, 1]$ with its usual uniformity. Let $K_1 = \{0\} \cup \{1/(n+1) : n \in \mathbb{N}\}$, which is compact in X by sequence convergence. Let $G_1 = X - K_1$. Note that $\{G_1\}$ is a 1-star for X. Let E_1 be generated by the partition $\{G_1, K_1\}$. If $\mathcal{U}_m \vee \mathcal{U}_{E_1}$ corresponds to the class $[(W, h)]$, by R23.4.15 $W - h[X]$ is countably infinite. Now consider the 1-star $\{G_2\}$, where $G_2 = (0, 1)$. Let E_2 be generated by the partition $\{G_2, \{0\}\}$. $\mathcal{U}_m \vee \mathcal{U}_{E_2}$ corresponds to a class of two-point compactifications. The topologies being compactified are non-homeomorphic and strictly larger than the usual topology for X.

Zero Dimensional Extensions of Discrete Spaces

The results in this subsection are refinements of material in [3] and [5].

Lemma R23.5.1 Let (X, τ) be discrete with X infinite. Let (Y, f) be a T_2 compactification of (X, τ) with $Y - f[X] = \{y_1, \ldots, y_n\}$. Assume $O_1, \ldots, O_n$ are pairwise disjoint open sets in Y with $y_i \in O_i$. Let y_j be in O, an open set in Y. Then $O_j - O$ is finite.

Proof: Let S be a subset of O_j with $y_j \in S$. It is claimed that S is closed in Y. Let t be in the Y-closure of S. If $t \in f[X]$, since $f[X]$ is open in Y and discrete, $\{t\}$ is open in Y and so $\{t\} \cap S \neq \emptyset$, i.e., $t \in S$. If $t = y_i$ for some i, $i \neq j$ cannot occur since, for $i \neq j$, $y_i \in O_i$ and $O_i \cap S \subseteq O_i \cap O_j = \emptyset$. Hence $t = y_j$, which is in S. Thus S is closed. In particular, O_j is closed and O is open so that $O_j - O$ is a closed subset of the compact set Y. Clearly $O_j - O \subseteq f[X]$ and, since the only compact subsets of a discrete space are finite, the conclusion holds.

Recall that a space with a finite-point compactification must be locally compact so that it has a one-point compactification,which determines a unique compactification class. For the rest of this subsection $\mathcal{U}_m$ will denote the uniformity corresponding to the class of the one-point compactification.

As noted in the addendum to [3], a finite-point compactification class need not correspond to a uniformity of the form $\mathcal{U}_m \vee \mathcal{U}_E$ for some equivalence relation E. By contrast, in discrete spaces the following holds.

Proposition R23.5.2 Let (X, τ) be discrete with X infinite and let (Y, f) be a finite-point compactification of (X, τ). Then there is an equivalence relation E with finitely many equivalence classes such that $\mathcal{U}_m \vee \mathcal{U}_E$ corresponds to $[(Y, f)]$.

Proof: Let $Y - f[X]$ have n distinct points, $y_1, \ldots, y_n$. Since Y is T_2, there is a pairwise disjoint collection of open sets $O_1, \ldots, O_n$ with $Y_i \in O_i$. Let $G_i = f^{-1}[O_i]$ and let $K = f^{-1}[Y - \cup_{i=1}^n O_i]$. Note that $\{G_1, \ldots, G_n\}$ is an n-star for X. Let E be the equivalence on X determined by the partition $G_1, \ldots, G_n, K$. Let $Y_E = X \cup \{p_1, \ldots, p_n\}$, where $p_i \notin X$ for all i and $i \neq j$ implies $p_i \neq p_j$. As in [3] let $\tau(E)$ be defined as $\{G \subseteq Y_E : p_i \in G \Rightarrow (X - G) \cap G_i \text{ is finite}\}$ and let $\iota_E : X \to Y_E$ by $\iota_E(x) = x$. As shown in [3], $\tau(E)$ is a compact, T_2 topology for Y_E, (Y_E, ι_E) is an n-point compactification of (X, τ), and, by R5.2.4, $\mathcal{U}_m \vee \mathcal{U}_E$ corresponds to $[(Y_E, \iota_E)]$. Thus to verify the conclusion, it is sufficient to show that (Y_E, ι_E) is equivalent to (Y, f). Define $\phi : Y_E \to Y$ by $\phi(x) = f(x)$ and $\phi(p_i) = y_i$. By definition $\phi \circ \iota_E = f$ and clearly ϕ is onto. Since f is one-to-one, ϕ is one-to-one. To see that ϕ is continuous, let O be open in Y and suppose $p_j \in \phi^{-1}[O]$. Then $\phi(p_j) = y_j \in O$ and by R23.5.1 $O_j - O$ is finite. Since f is one-to-one, $f^{-1}[O_j - O]$ is also finite. It is easy to check that $(X - \phi^{-1}[O]) \cap G_j = f^{-1}[O_j - O]$ and so $\phi^{-1}[O] \in \tau(E)$. Thus ϕ is continuous and, since both spaces are compact and T_2, a homeomorphism as required for equivalence.

Proposition R23.5.3 Let (X, τ) be discrete with X infinite and let $\mathcal{U}_m$ be the smallest totally bounded uniformity generating τ. Let (Y, f) be a compactification of (X, τ). Then Y is zero-dimensional if and only if $[(Y, f)]$ corresponds to an extension of $\mathcal{U}_m$ by some collection of subsets of X.

Proof: Assume Y is zero-dimensional. By R9.3.3 (Y, f) is a supremum of finite-point compactifications. By R23.5.2 the class of each of those compactifications corresponds to $\mathcal{U}_m \vee \mathcal{U}_E$ for some equivalence relation E on X with finitely many equivalence classes. Let $\mathcal{E}$ denote the collection of such relations. It follows from R13.1.2 that $[(Y, f)]$ corresponds to $\vee\{\mathcal{U}_m \vee \mathcal{U}_E : E \in \mathcal{E}\}$. For each $E \in \mathcal{E}$, let $\mathcal{C}(E)$ be the set of equivalence classes of E. By R23.4.9 $E(\mathcal{C}(E)) = E$ so that by R23.4.10 $\mathcal{U}_m \vee \mathcal{U}_E = \mathcal{U}_m \vee \mathcal{U}_{E(\mathcal{C}(E))} = (\mathcal{U}_m)_e(\mathcal{C}(E))$. Thus the uniformity corresponding to $[(Y, f)]$ is$\vee\{(\mathcal{U}_m)_e(\mathcal{C}(E)) : E \in \mathcal{E}\}$. Now let $\mathcal{B} =$

$\cup\{\mathcal{C}(E) : E \in \mathcal{E}\}$. It is easy to check that $\vee\{(\mathcal{U}_m)_e(\mathcal{C}(E)) : E \in \mathcal{E}\} = (\mathcal{U}_m)_e(\mathcal{B})$, i.e., $[(Y,f)]$ corresponds to the extension of $\mathcal{U}_m$ by $\mathcal{B}$. Conversely assume $[(Y,f)]$ corresponds to the extension of $\mathcal{U}_m$ by $\mathcal{A}$, a collection of subsets of X. For each $A \in \mathcal{A}$, $E(A)$ has two equivalence classes, A and $X - A$. By R5.1.10 $E(A)$ is either 1- or 2-compatible and so by R5.2.4 $\mathcal{U}_m \vee \mathcal{U}_{E(A)}$ corresponds to a finite point compactification class. Since $[(Y,f)]$ is the supremum of these finite point classes, by R9.3.3 Y is zero-dimensional.

Lemma R23.5.4 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A}$, $\mathcal{B}$ be collections of subsets of X. Then $\mathcal{U}_e(\mathcal{A}\cup\mathcal{B}) = (\mathcal{U}_e(\mathcal{A}))_e(\mathcal{B})$.

Proof: $\mathcal{U}_e(\mathcal{A}\cup\mathcal{B}) = \vee\{\mathcal{U}\vee\mathcal{U}_{E(S)} : S \in \mathcal{A}\cup\mathcal{B}\}$ by definition and so contains $\mathcal{U}_e(\mathcal{A})$ and, for every $B \in \mathcal{B}$, $\mathcal{U}_{E(B)}$. As an upper bound of $\{\mathcal{U}_e(\mathcal{A})\vee\mathcal{U}_{E(B)} : B \in \mathcal{B}\}$, it contains the supremum, $(\mathcal{U}_e(\mathcal{A}))_e(\mathcal{B})$. Conversely, $(\mathcal{U}_e(\mathcal{A}))_e(\mathcal{B})$ clearly contains $\mathcal{U}\vee\mathcal{U}_{E(S)}$ for every $S \in \mathcal{A}\cup\mathcal{B}$ and so contains the supremum, $\mathcal{U}_e(\mathcal{A}\cup\mathcal{B})$.

Corollary R23.5.5 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space with X infinite and $\tau(\mathcal{U})$ discrete and let $\mathcal{U}$ correspond to $[(Y,f)]$. Let $\mathcal{A}$ be a non-empty collection of subsets of X and let $\mathcal{U}_e(\mathcal{A})$ correspond to $[(Z,g)]$. If Y is zero-dimensional, then Z is zero-dimensional.

Proof: Assume Y is zero-dimensional. By R23.5.3 there is $\mathcal{B}$, a collection of subsets of X, such that $\mathcal{U} = (\mathcal{U}_m)_e(\mathcal{B})$. Then by R23.5.4 $\mathcal{U}_e(\mathcal{A}) = (\mathcal{U}_m)_e(\mathcal{B}\cup\mathcal{A})$, which, by a second application of R25.5.3, shows that Z is zero-dimensional.

Example R23.5.6 In R9.3.7 a uniform space $(X,\mathcal{V})$ with X infinite and $\tau(\mathcal{V})$ discrete is constructed with the property that $\mathcal{V}$ corresponds to a compactification class $[(Y,f)]$, where Y is not zero-dimensional. By R23.5.3 $\mathcal{V}$ is not an extension of $\mathcal{U}_m$ by any collection of subsets of X. In addition, the Stone-Čech compactification of X is the supremum of the two-point compactificaions of X by R6.1.8 and so $\mathcal{V}_e(\mathcal{P}(X))$ corresponds to the class of the Stone-Čech compactification. Thus an extension of $\mathcal{V}$ corresponds to a zero-dimensional compactification even though $\mathcal{V}$ does not.

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Added Subsection 2014

The notation of R23.1 will be used here. Throughout k denotes a fixed positive integer greater than or equal to 2.

The main motivation of R23.1 and R23.2 was the wish to describe an element of the compactification class corresponding to $\mathcal{U}(\prec_k)$. Here that is done more explicitly by calculating R_1. (For notational convenience the superscript of R23.2.11, which here would be $\mathbf{R}_k$, is omitted.) By R23.2.23 the desired compactification is $(\mathbf{R}_k/R_1, \rho \circ f_k)$, where ρ denotes the projection onto the quotient space.

Lemma R23.Add.1 Let $\mathcal{F} <_k \mathcal{G}$ be in $\mathbf{R}_k$. Assume $\mathcal{F}$ and $\mathcal{G}$ are not $<_k$-consecutive. Then $\mathcal{F} <_k f_k(r) <_k f_k(s) <_k f_k(t) <_k \mathcal{G}$ for some $r, s, t \in \mathbb{N}$.

Proof: By R19.1.13 there is $\mathcal{H}$ not in the countable $f_k[\mathbf{Z}]$ such that $\mathcal{F} <_k \mathcal{H} <_k \mathcal{G}$. By R19.1.19 $\mathcal{H}$ is not part of any $<_k$- consecutive pair and so the open $<_k$-interval from $\mathcal{H}$ to $\mathcal{G}$ is non-empty. By the density of $f_k[\mathbb{N}]$ in $\mathbf{R}_k$, there is t in $\mathbb{N}$ such that $\mathcal{H} <_k f_k(t) <_k \mathcal{G}$. The open $<_k$-interval from $\mathcal{H}$ to $f_k(t)$ is non-empty and the density can be used again to select s in $\mathbb{N}$ with $\mathcal{H} <_k f_k(s) <_k f_k(t)$. Finally the open $<_k$-interval from $\mathcal{H}$ to $f_k(s)$ is non-empty and so again by density there is $r \in \mathbb{N}$ such that $\mathcal{H} <_k f_k(r) <_k f_k(s)$.

Lemma R23.Add.2 Let $\mathcal{F} \neq \mathcal{G}$ be in $\mathbf{R}_k$. Assume $\mathcal{F}$ and $\mathcal{G}$ are not $<_k$-consecutive. Then $\mathcal{F}$ is not R_1-related to $\mathcal{G}$.

Proof: Without loss of generality assume $\mathcal{F} <_k \mathcal{G}$. Deny the conclusion. Then there is a $\mathcal{U}(\prec_k)$-Cauchy net $S : D \to \mathbb{N}$ which has two $\mathcal{A}$-compatible subnets, one with f_k-image converging to $\mathcal{F}$ and the other with f_k-image converging to $\mathcal{G}$. Here it will be clearer with the subnets described explicitly: Let $T_1 : E \to D$ and $T_2 : F \to D$, where E, F are directed sets, have the subnet property, i.e., they are finalizing maps. $S \circ T_i$ is $\mathcal{A}$-compatible for $i = 1, 2$, $f_k \circ S \circ T_1$ converges to $\mathcal{F}$, and $f_k \circ S \circ T_2$ converges to $\mathcal{G}$. By the previous lemma there exist $r, s, t \in \mathbb{N}$ such that $\mathcal{F} <_k f_k(r) <_k f_k(s) <_k f_k(t) <_k \mathcal{G}$. By definition of the ordering $\prec_k$, $r \prec_k s \prec_k t$. Let $\mathcal{I}$ be the $\prec_k$-open cover $\{(-\infty, s)_k, (r, t)_k, (s, \infty)_k\}$. By definition $R(\mathcal{I})$ is in $\mathcal{U}(\prec_k)$. Since S is Cauchy, there is $d_0 \in D$ such that $d, d' \geq_D d_0$ implies $(S(d), S(d')) \in R(\mathcal{I})$. By the subnet property and convergence of the first subnet, there is $e_0 \in E$ such that $e \geq_E e_0$ implies $T_1(e) \geq_D d_0$ and $f_k \circ S \circ T_1(e) <_k f_k(r)$. Since $S \circ T_1(e_0) \in \mathbb{N}$, $S \circ T_1(e_0)$ is in $(-\infty, s)_k$ and not in $(r, t)_k$ or $(s, \infty)_k$. Similarly, there is $f_0 \in F$ such that $f \geq_F f_0$ implies $T_2(f) \geq_D d_0$ and $f_k \circ S \circ T_2(f) >_k f_k(t)$ Thus $S \circ T_2(f_0)$ is in $(s, \infty)_k$ but not in $(r, t)_k$ or $(-\infty, s)_k$. By definition of $R(\mathcal{I})$, $(S \circ T_1(e_0), S \circ T_2(f_0))$ is not in $R(\mathcal{I})$, a contradiction.

Corollary R23.Add.3 Let $\mathcal{F}$ be in $\mathbf{R}_k$. Assume $\mathcal{F}$ is not part of a $<_k$-consecutive pair. Then the R_1-equivalence class of $\mathcal{F}$ is $\{\mathcal{F}\}$.

Proof: For any $\mathcal{G}$ in $\mathbf{R}_k$ with $\mathcal{G} \neq \mathcal{F}$, by the lemma $\mathcal{G}$ is not R_1-related to $\mathcal{F}$ and so is not in the R_1-equivalence class of $\mathcal{F}$. Since R_1 is reflexive, the conclusion follows.

Lemma R23.Add.4 Let $\mathcal{F}$ be in $\mathbf{R}_k$. Then $\mathcal{F}$ is part of at most one $<_k$-consecutive pair.

Proof: Assume $\mathcal{F}$ is the smaller of a $<_k$-consecutive pair. By R19.1.19 $\mathcal{F} = f_k(-j)$ for some $j \in \mathbb{N}$. Now assume $\mathcal{F}$ is also the larger of some other $<_k$-consecutive pair. By R19.1.19 again $\mathcal{F} = f_k(l)$ for some $l \geq 2$ in $\mathbb{N}$, a contradiction since f_k is one-to-one on $\mathbf{Z}$. The argument is similar if $\mathcal{F}$ is the larger of a $<_k$-consecutive pair.

Corollary R23.Add.5 There is no $<_k$-consecutive triple in $\mathbf{R}_k$.

Proof: This is immediate from R23.Add.4.

Corollary R23.Add.6 Let $\mathcal{F}$ be in $\mathbf{R}_k$. Assume $\mathcal{F}$ is part of a $<_k$-consecutive pair and let $\mathcal{G} \in \mathbf{R}_k$ be such that $\mathcal{F}, \mathcal{G}$ are $<_k$-consecutive. Then the R_1-equivalence class of $\mathcal{F}$ is contained in $\{\mathcal{F}, \mathcal{G}\}$.

Proof: For any $\mathcal{H}$ in $\mathbf{R}_k$ with $\mathcal{H} \notin \{\mathcal{F}, \mathcal{G}\}$, $\mathcal{F}$ and $\mathcal{H}$ are not $<_k$-consecutive and so by R23.Add.2 $\mathcal{F}$ and $\mathcal{H}$ are not R_1-related.

To show that consecutive pairs are related, the following analog of R22.3.5 is needed.

Lemma R23.Add.7 Let $\mathcal{F}, \mathcal{G}$ be in $\mathbf{R}_k$. Assume $\mathcal{F}$ and $\mathcal{G}$ are $<_k$-consecutive. Let $T_1 : D \to \mathbb{N}$ and $T_2 : E \to \mathbb{N}$ be nets such that $f_k \circ T_1$ converges to $\mathcal{F}$ and $f_k \circ T_2$ converges to $\mathcal{G}$. Assume $D \cap E = \emptyset$ and $D^* = \{(d,(d,e)) : (d,e) \in D \times E\} \cup \{(e,(d,e)) : (d,e) \in D \times E\}$ with order $\leq^*$ as in R22.3.4. Let $S : D^* \to \mathbb{N}$ be defined by $S(x,(d,e)) = T_1(d)$ if $x = d$ and $S(x,(d,e)) = T_2(e)$ if $x = e$. Then S is a $\mathcal{U}_k$-Cauchy net.

Proof: Without loss of generality, assume $\mathcal{F} <_k \mathcal{G}$. By R19.1.19 there exist $j, l \in \mathbb{N}$ with $l \geq 2$ such that $\mathcal{F} = f_k(-j)$ and $\mathcal{G} = f_k(l)$. S is a function since $D \cap E = \emptyset$ and D^* is a directed set by R22.3.4. Let $R(\mathcal{I})$ be a basic entourage in $\mathcal{U}_k$, i.e., $\mathcal{I} = \{I_1, \ldots, I_n\}$ is a finite cover of $\mathbb{N}$ by intervals in $\tau(\prec_k)$. There is t such that $l \in I_t$. Since $\mathbb{N}$ contains no $\prec_k$-consecutive pairs and l is not the smallest element, by R23.1.2 l is not an endpoint of I_t and there exist $p, q \in I_t$ such that $p \prec_k l \prec_k q$. By definition $f_k(p) <_k f_k(l) <_k f_k(q)$ and, since $f_k(-j)$ and $f_k(l)$ are $<_k$-consecutive and f_k is one-to-one, $f_k(p) <_k f_k(-j) <_k f_k(l)$. By convergence there is d_0 such that $d \geq_D d_0$ implies $f_k(p) <_k f_k \circ T_1(d) <_k f_k(q)$ and there is e_0 such that $e \geq_E e_0$ implies $f_k(p) <_k f_k \circ T_2(e) <_k f_k(q)$. Since T_1 and T_2 map into $\mathbb{N}$, $d \geq_D d_0$ implies $p \prec_k T_1(d) \prec_k q$ and so $T_1(d) \in I_t$ and similarly $e \geq_E e_0$ implies $T_2(e) \in I_t$. By definition of S, $(x,(d,e)) \geq^* (d_0,(d_0,e_0))$ implies $S(x,(d,e)) \in I_t$. It follows easily that $S \times S$ is eventually in $I_t \times I_t \subseteq R(\mathcal{I})$ and so S is $\mathcal{U}_k$-Cauchy.

By the metrizability of $\mathbf{R}_k$ the nets chosen at the beginning of the next proof could be sequences but that does not seem advantageous in this context. To conform to the notation of R23.2, the following argument uses A_n to denote the $\prec_k$-interval $[n,\infty)_k$ and $\mathcal{A}$ to denote $\{A_n : n \in \mathbb{N}\}$.

Lemma R23.Add.8 Let $\mathcal{F}, \mathcal{G}$ be in $\mathbf{R}_k$. Assume $\mathcal{F}$ and $\mathcal{G}$ are $<_k$-consecutive. Then $\mathcal{F}$ and $\mathcal{G}$ are R_1-related.

Proof: Without loss of generality, assume $\mathcal{F} <_k \mathcal{G}$. By R19.1.19 there exist j, l in $\mathbb{N}$ with $l \geq 2$ such that $\mathcal{F} = f_k(-j)$ and $\mathcal{G} = f_k(l)$. Since $\mathbb{N}$ is dense, a net $T_1 : D \to \mathbb{N}$ exists such that $f_k \circ T_1$ converges to $f_k(-j)$. Let E be a non-empty directed set with $E \cap D = \emptyset$ and let T_2 be the constant net on E defined by $T_2(e) = l$. Clearly $f_k \circ T_2$ converges to $f_k(l)$. Let D^* be the directed set described in R22.3.4 and let S be as in R23.Add.7. By that lemma S is $\mathcal{U}_k$-Cauchy. Recall that $D \times E$ is a directed set with the ordering $(d,e) \geq_{D \times E} (d_1, e_1)$ if and only if $d \geq_D d_1$ and $e \geq_E e_1$. Define $u : D \times E \to D^*$ by $u(d,e) = (d,(d,e))$. For $(x,(d_0,e_0)) \in D^*$, $(d,e) \geq_{D \times E} (d_0,e_0)$ implies $u(d,e) \geq_{D^*} (x,(d_0,e_0))$ and so u has the subnet property. Similarly $v : D \times E \to D^*$ by $v(d,e) = (e,(d,e))$ has the subnet property. Since $S \circ u(d,e) = T_1(d)$, it is easy to check that $f_k \circ S \circ u$, i.e., the f_k-image of the subnet $S \circ u$, converges to $f_k(-j)$. Similarly, the f_k-image of the subnet $S \circ v$ converges to $f_k(l)$. To verify the conclusion, it remains to show that the two subnets are $\mathcal{A}$-compatible. Since $\mathbb{N}$ with order $\prec_k$ has smallest element 1 and $l \geq 2$, $1 \prec_k l$ so that A_l is a proper subset of A_1. For each $n \in \mathbb{N}$ with $n \geq l$, when $A_1, \ldots, A_n$ are arranged in increasing order by containment, A_l is not the largest element and so there is $p(n) \in \{1, \ldots, n\}$ such that $A_{p(n)}$ and A_l are consecutive at level n with $A_{p(n)}$ the larger. Now define h_1 on $\mathbb{N}$ by $h_1(n) = (n, n)$ if $n < l$ and $h_1(n) = (p(n), l)$ for $n \geq l$. By definition R23.2.8 h_1 is $\mathcal{A}$-suitable. Note that, for $n \geq l$, $A_{p(n)} \Delta A_l = [p(n), l)_k$. Now fix $n \geq l$. Since $p(n) \prec_k l$, $f_k(p(n)) <_k f_k(l)$ and, since $f_k(-j)$ and $f_k(l)$ are $<_k$-consecutive with $f_k(-j)$ smaller, $f_k(p(n)) <_k f_k(-j) <_k f_k(l)$. By convergence $f_k \circ S \circ u$ is eventually in the $\mathbf{R}_k$ -open $<_k$-interval from $f_k(p(n))$ to $f_k(l)$. By definition of $\prec_k$, since $S \circ u$ maps into $\mathbb{N}$, $S \circ u$ is eventually in $(p(n), l)_k$, which is contained in $A_{p(n)} \Delta A_l$. Thus $S \circ u$ is $\mathcal{A}$-compatible with associated map h_1. Define h_2 on $\mathbb{N}$ by $h_2(n) = (r, s)$ where A_r, A_s are consecutive at level n and $l \in A_r \Delta A_s$. Note that such r, s exist since $l \in A_1 = \mathbb{N}$. Then clearly h_2 is $\mathcal{A}$-suitable and, since $S \circ v$ is a constant subnet with value l, $S \circ v$ is $\mathcal{A}$-compatible with associated map h_2.

Corollary R23.Add.9 Let $\mathcal{F}, \mathcal{G}$ be in $\mathbf{R}_k$. Assume $\mathcal{F}$ and $\mathcal{G}$ are $<_k$-consecutive. Then the R_1-equivalence class of $\mathcal{F}$ is $\{\mathcal{F}, \mathcal{G}\}$.

Proof: This is immediate from R23.Add.6 and R23.Add.8.

The simplicity of the quotient space raises the question of whether the compactification could be calculated directly, without the machinery of R23.2. That issue will not be pursued here.

This description of the quotient space allows easy examples to show that addition and multiplication in $\mathbf{R}_k$, as well as the tools of [12], do not transfer via the quotient map. For example, assume $k \geq 3$. By R19.1.21 and the above, the equivalence classes of $f_k(2)$ and $f_k(3)$ are $\{f_k(2), f_k(-(k-1))\}$ and $\{f_k(3), f_k(-(k-2))\}$. Attempting to add by picking representatives from the classes yields three possible results - $f_k(5), f_k(-k+4)$, and $f_k(-2k+3)$ - which obviously do not determine a single equivalence class.

Added Reference

12. This website, R20: p-adic Tools for the Remnant Rings

Added 2018

This note points out that R23.1.5 yields a description of $\mathcal{V}_k$ for $k \geq 2$, where $\mathcal{V}_k$ is the separable totally bounded uniformity on $\mathbf{Z}$ corresponding to the compactification class of $\mathbf{R}_k$. The description is more straightforward than that of $\tilde{\mathcal{V}}_k$ given in R23.1.9.

Fix $k \in \mathbb{N}$ with $k \geq 2$. For notational convenience let $A = f_k[\mathbf{Z}\,]$ and let $<_{k,A}$ be the restriction of $<_k$ to A, i.e., $\mathcal{F} <_{k,A} \mathcal{G}$ if and only if $\mathcal{F}, \mathcal{G}$ are in A and $\mathcal{F} <_k \mathcal{G}$. Let $\mathcal{U}^A(<_k)$ be the subspace uniformity on A from $\mathcal{U}(<_k)$. By R21.36 $\mathcal{U}(<_{k,A}) \subseteq \mathcal{U}^A(<_k)$. The next lemma shows that in this example the opposite containment also holds.

Lemma R23.Add.10 With notation as above, $\mathcal{U}(<_{k,A}) = \mathcal{U}^A(<_k)$.

Proof: An element of $\mathcal{U}^A(<_k)$ must be of the form $U \cap (A \times A)$ for some $U \in \mathcal{U}(<_k)$. There is a finite cover $\mathcal{I}$ consisting of intervals in $\tau(<_k)$ such that $R(\mathcal{I}) \subseteq U$. By R23.1.5 there is $\mathcal{J}$, a finite cover of $\mathbf{R}_k$ by non-empty intervals in $\tau(<_k)$, such that $\mathcal{J}$ refines $\mathcal{I}$ and all the endpoints of intervals in $\mathcal{J}$ are in $f_k[\mathbf{Z}\,] = A$. Clearly, for each $J \in \mathcal{J}$, $J \cap A$ is a $<_{k,A}$-interval. Since A is dense and, by R19.1.19, contains all consecutive pairs of $\mathbf{R}_k$, R21.30 applies: the subspace topology on A is $\tau(<_{k,A})$ and so, for each $J \in \mathcal{J}$, $J \cap A$ is in $\tau(<_{k,A})$. Then $\mathcal{J}_1 = \{J \cap A : J \in \mathcal{J}\}$ is a cover of A by $<_{k,A}$-intervals in $\tau(<_{k,A})$ so that

$R(\mathcal{J}_1)$ is in $\mathcal{U}(<_{k,A})$. Since $\mathcal{J}$ refines $\mathcal{I}$, $R(J) \subseteq R(\mathcal{I})$ and so $R(\mathcal{J}_1) \subseteq U \cap (A \times A)$, which implies $U \cap (A \times A) \in \mathcal{U}(<_{k,A})$. Thus $\mathcal{U}^A(<_k) \subseteq \mathcal{U}(<_{k,A})$ and the result follows.

Lemma R23.Add.11 Let Y be a set linearly ordered by $<$, let X be a set, and let $g : X \to Y$ be one-to-one and onto. Let $<_X$ be defined by $a <_X b$ if and only if $a, b \in X$ and $g(a) < g(b)$. Then $<_X$ is a linear order on X and $g : (X, \mathcal{U}(<_X)) \to (Y, \mathcal{U}(<))$ is a unimorphism.

Proof: This is a routine transference argument.

Corollary R23.Add.12 Let $\prec_k$ be defined on $\mathbf{Z}$ by $z \prec_k w$ if and only if $z, w \in \mathbf{Z}$ and $f_k(z) <_k f_k(w)$. Then $\mathcal{V}_k = \mathcal{U}(\prec_k)$.

Proof: By R19.1.7 $\tau(<_k)$ is the topology for the compact space $\mathbf{R}_k$ and so by R21.15ii $\mathcal{U}(<_k)$ is the unique uniformity for $\mathbf{R}_k$. By definition $\mathcal{V}_k$ is the unique uniformity which makes $f_k : \mathbf{Z} \to f_k[\mathbf{Z}]$ a unimorphism, where the image space has the subspace uniformity from $\mathcal{U}(<_k)$. The conclusion follows easily from the two preceding lemmas.

Comment: R21.Add.25 is a generalized version of R23.Add.10. In R21.Add.30 it is noted that the Dedekind compactification of the order space $(\mathbf{Z}, \prec_k)$ is equivalent to the compactification $(\mathbf{R}_k, f_k)$.