Disjoint Unions of Uniform Spaces

Most of this subsection is undoubtedly known but it is included here because I have no reference. The approach follows the pattern used for the disjoint union of topological spaces in [4]. As in R15.1.12, given an indexed family of sets $\{X_\alpha : \alpha \in \Delta\}$ with $\Delta \neq \emptyset$, the disjoint union is defined to be $\cup\{X_\alpha \times \{\alpha\} : \alpha \in \Delta\}$ and is denoted $\coprod\{X_\alpha : \alpha \in \Delta\}$.

Definition R24.1 Let $\{(X_\alpha,\mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty set of uniform spaces. For $U \in \mathcal{U}_\alpha$, U^* is defined to be $\{((x,\alpha),(y,\alpha)) : (x,y) \in U\}$.

The uniformity of the next definition appeared without the notation in R15.1.17.

Definition R24.2 Let $\{(X_\alpha,\mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty set of uniform spaces. Let $W \subseteq D \times D$, where for notational convenience D denotes $\coprod\{X_\alpha : \alpha \in \Delta\}$. W is in $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ if and only if there is $c \in \prod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ such that $\cup\{c(\alpha)^* : \alpha \in \Delta\} \subseteq W$.

The last definition contains two ambiguities which, it is hoped, will always be resolvable by context. (The alternative would be more notational complexity.) First, the definition intends to specify a set denoted $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$, which might be referred to as the coproduct uniformity, but the same notation might also indicate the disjoint union of the sets $\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. Secondly, the point c is in the product of the sets $\{\mathcal{U}_\alpha : \alpha \in \Delta\}$, not the product uniformity.

The first proposition justifies calling $\coprod\{X_\alpha : \alpha \in \Delta\}$ a uniformity and relates it to the disjoint union of the associated topological spaces $(\{(X_\alpha, \tau(\mathcal{U}_\alpha)) : \alpha \in \Delta\})$, the topology of which is denoted $\coprod\{(\tau(\mathcal{U}_\alpha) : \alpha \in \Delta\}$.

Proposition R24.3 Let $\{(X_\alpha,\mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty set of uniform spaces. Then

i) $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ ia a uniformity on $\coprod\{X_\alpha : \alpha \in \Delta\}$.

ii) $\tau(\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}) = \coprod\{(\tau(\mathcal{U}_\alpha) : \alpha \in \Delta\}$.

Proof: The routine proof of i) will be illustrated by a verification of the triangle inequality. Let W be in $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ and let c be in the product $\prod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ such that $\cup\{c(\alpha)^* : \alpha \in \Delta\} \subseteq W$. Define d in the product as follows: For $\alpha \in \Delta$ let $d(\alpha)$ be a symmetric entourage in $\mathcal{U}_\alpha$ such that $d(\alpha) \circ d(\alpha) \subseteq c(\alpha)$. Let $V = \cup\{d(\alpha)^* : \alpha \in \Delta\}$, which by definition is in $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. It is easy to check that $V \circ V \subseteq W$ as required. For ii) first let $p \in O \in \tau(\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\})$. There is W in the uniformity with $W[p] \subseteq O$ and by definition there is $c \in \prod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ such that $\cup\{c(\alpha)^* : \alpha \in \Delta\} \subseteq W$. There is $\delta \in \Delta$ and $x \in X_\delta$ such that $p = (x,\delta)$. There is $G \in \tau(\mathcal{U}_\delta)$ such that $x \in G \subseteq c(\delta)[x]$. Then $G \times \{\delta\}$ is in $\coprod\{(\tau(\mathcal{U}_\alpha) : \alpha \in \Delta\}$ and $p \in G \times \{\delta\} \subseteq c(\delta)^*[p] \subseteq W[p]$ and so O is in $\coprod\{(\tau(\mathcal{U}_\alpha) : \alpha \in \Delta\}$. For the converse containment, it is sufficient to show that every basic open set in$\coprod\{(\tau(\mathcal{U}_\alpha) : \alpha \in \Delta\}$ is also in $\tau(\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\})$. To that end, let $\delta \in \Delta$ and let $x \in G$, where $G \in \tau(\mathcal{U}_\delta)$. There is $U \in \mathcal{U}_\delta$ such that $U[x] \subseteq G$. Define c by $c(\delta) = U$ and $c(\alpha) = X_\alpha \times X_\alpha$ if $\alpha \neq \delta$. By definition $W = \cup\{c(\alpha)^* : \alpha \in \Delta\}$ is in the coproduct uniformity and it is easy to check that $W[(x,\delta)] \subseteq G \times \{\delta\}$. Thus the basic open set $G \times \{\delta\}$ is in $\tau(\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\})$ as required.

Proposition R24.4 Let $\{(X_\alpha,\mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty set of uniform spaces. $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ is separated if and only if $\mathcal{U}_\alpha$ is separated for every $\alpha \in \Delta$.

Proof: Assume $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ is separated, let $\delta \in \Delta$, and let $x \neq y$ be in X_δ. Pick W in $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ such that $((x,\delta),(y,\delta))$ is not in W. Pick c such that W contains $\cup\{c(\alpha)^* : \alpha \in \Delta\}$. By definition $((x,\delta),(y,\delta))$ not in $c(\delta)^*$ implies $(x,y) \notin c(\delta)$. Thus $\mathcal{U}_\delta$

is separated. Now assume $\mathcal{U}_\alpha$ is separated for every $\alpha \in \Delta$ and let $(x, \beta) \neq (y, \gamma)$ be in $\coprod\{X_\alpha : \alpha \in \Delta\}$. If $\beta \neq \gamma$, let $c(\alpha) = X_\alpha \times X_\alpha$ for every $\alpha \in \Delta$. Clearly $((x, \beta), (y, \gamma))$ is not in $\cup\{c(\alpha)^* : \alpha \in \Delta\}$. If $\beta = \gamma$, $x \neq y$. Pick U in $\mathcal{U}_\beta$ such that $(x, y) \notin U$ and let $c(\beta) = U$ and $c(\alpha) = X_\alpha \times X_\alpha$ for $\alpha \neq \beta$. Then $((x, \beta), (y, \beta))$ is not in $c(\beta)^*$ and so not in $\cup\{c(\alpha)^* : \alpha \in \Delta\}$. Thus $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ is separated.

Lemma R24.5 Let $\{(X_\alpha,\mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty set of uniform spaces. Let $S : D \to \coprod\{X_\alpha : \alpha \in \Delta\}$ be a $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$-Cauchy net. Then there is $\delta \in \Delta$ such that S is eventually in $X_\delta \times \{\delta\}$.

Proof: Let $c(\alpha) = X_\alpha \times X_\alpha$ for every $\alpha \in \Delta$. By definition $W = \cup\{c(\alpha)^* : \alpha \in \Delta\}$ is in $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. Since S is Cauchy, there is $d_0 \in D$ such that $d, e \geq d_0$ implies $(S(d), S(e)) \in W$. Let $s(d_0) = (x_0, \delta)$. For $d \geq d_0$, there is $\alpha \in \Delta$ such that $(S(d), S(d_0))$ is in $c(\alpha)^*$, which implies that $\alpha = \delta$ and $S(d)$ is in $X_\delta \times \{\delta\}$.

Proposition R24.6 Let $\{(X_\alpha,\mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty set of uniform spaces. $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ is complete if and only if $\mathcal{U}_\alpha$ is complete for every $\alpha \in \Delta$.

Proof: First assume $\mathcal{U}_\alpha$ is complete for every $\alpha \in \Delta$ and let $S : D \to \coprod\{X_\alpha : \alpha \in \Delta\}$ be a $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$-Cauchy net. By the lemma there is δ and $d_0 \in D$ such that $d \geq d_0$ implies $S(d) \in X_\delta \times \{\delta\}$. Let D_1 be the directed set $\{d \in D : d \geq d_0\}$ with the order inherited from D. Define $T : D_1 \to X_\delta$ by $T(d) = x$ where $S(d) = (x, \delta)$. For $U \in \mathcal{U}_\delta$, let $c(\delta) = U$ and, for $\alpha \neq \delta$, $c(\alpha) = X_\alpha \times X_\alpha$. Using the entourage $\cup\ c(\alpha)^*$ and the fact that S is Cauchy, one easily sees that T is $\mathcal{U}_\delta$-Cauchy. By assumption T has a limit $x_0 \in X_\delta$. It is claimed that S has (x_0, δ) as a limit. Let W be in the coproduct uniformity and pick c such that $\cup\{c(\alpha)^* : \alpha \in \Delta\} \subseteq W$. There is $d_1 \geq d_0$ such $d \geq d_1$ implies $T(d) \in c(\delta)[x_0]$. Thus $d \geq d_1$ implies $S(d) \in c(\delta)^*[(x_0, \delta)] \subseteq W[(x_0, \delta)]$ and so the claim holds. Conversely assume $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ is complete, let $\beta \in \Delta$, and let $S : D \to X_\beta$ be $\mathcal{U}_\beta$-Cauchy. Define $T : D \to \coprod\{X_\alpha : \alpha \in \Delta\}$ by $T(d) = (S(d), \beta)$. Given W in the coproduct uniformity, pick c with $\cup\{c(\alpha)^* : \alpha \in \Delta\} \subseteq W$. There is $d_0 \in D$ such that $d, e \geq d_0$ implies $(S(d), S(e)) \in c(\beta)$. It follows that $d, e \geq d_0$ implies $(T(d), T(e)) \in c(\beta)^* \subseteq W$ and so T is $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$-Cauchy. By assumption T has a limit (x_0, γ). First $\gamma = \beta$ (and so $x_0 \in X_\beta$) will be verified. Define c_1 by $c_1(\alpha) = X_\alpha \times X_\alpha$ for every $\alpha \in \Delta$. There is $d_1 \in D$ such that $d \geq d_1$ implies $T(d) \in \cup\{c_1(\alpha)^* : \alpha \in \Delta\}[(x_0, \gamma)] = c_1(\gamma)^*[(x_0, \gamma]$. Since $T(d) = (S(d), \beta)$, the claim holds. Finally it will be shown that S has x_0 as a limit. Let U be in $\mathcal{U}_\beta$. Define $e(\beta) = U$ and $e(\alpha) = X_\alpha \times X_\alpha$ for $\alpha \neq \beta$. $W = \cup\{e(\alpha)^* : \alpha \in \Delta\}$ is in the coproduct uniformity and so there is $d_2 \in D$ such that $d \geq d_2$ implies $T(d) \in W[(x_0, \beta)]$. Since $W[(x_0, \beta)] = e(\beta)^*[(x_0, \beta)] = U^*[(x_0, \beta)]$. It follows that $d \geq d_2$ implies $S(d) \in U[x_0]$, i.e., S has x_0 as a limit. Thus $\mathcal{U}_\beta$ is complete.

Proposition R24.7 Let $\{(X_\alpha,\mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty set of non-empty uniform spaces. $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ is totally bounded if and only if Δ is finite and $\mathcal{U}_\alpha$ is totally bounded for every $\alpha \in \Delta$.

Proof: First assume $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ is totally bounded and suppose Δ is infinite. Define $d(\alpha) = X_\alpha \times X_\alpha$ for every $\alpha \in \Delta$ and let $W = \cup\{d(\alpha)^* : \alpha \in \Delta\}$, which is in the coproduct uniformity. For any finite set of points $\{(x_i, \alpha_i) : 1 \leq i \leq n\}$ in $\coprod\{X_\alpha : \alpha \in \Delta\}$, pick $\delta \in \Delta - \{\alpha_1, \ldots, \alpha_n\}$. Since $W[(x_i, \alpha_i)] = X_{\alpha_i} \times \{\alpha_i\}$, no point of $X_\delta \times \{\delta\}$ is in $\cup\{W[(x_i, \alpha_i)] : 1 \leq i \leq n\}$, which contradicts the total-boundedness. Next fix $\gamma \in \Delta$ and let $U \in \mathcal{U}_\gamma$. Define $e(\gamma) = U$ and $e(\alpha) = X_\alpha \times X_\alpha$ if $\alpha \neq \gamma$. Let $V = \cup\{e(\alpha)^* : \alpha \in \Delta\}$.

By hypothesis, $\coprod\{X_\alpha : \alpha \in \Delta\}$ contains a finite set F such that $V[F] = \coprod\{X_\alpha : \alpha \in \Delta\}$. Since $V[(x,\alpha)] = e(\alpha)^*[(x,\alpha)] = e(\alpha)[x] \times \{\alpha\}$ and $X_\gamma \neq \emptyset$, $S = \{x : (x,\gamma) \in F\}$, which is finite, is clearly non-empty and $U[S] = X_\gamma$. Thus $\mathcal{U}_\gamma$ is totally bounded. For the converse it is sufficient to consider a basic entourage, say $W = \cup\{c(\alpha)^* : \alpha \in \Delta\}$. For each α there is a finite set F_α such that $c(\alpha)[F_\alpha] = X_\alpha$. Since Δ is finite, the set $F = \cup\{F_\alpha \times \{\alpha\} : \alpha \in \Delta\}$ is finite and it follows easily that $W[F] = \coprod\{X_\alpha : \alpha \in \Delta\}$.

A less old-fashioned approach to the disjoint union involves a universal property of morphisms, which allows an easy way to recognize a space as isomorphic to a disjoint union. What follows is a specialization of the categorical presentation of coproducts (e.g., see pp.62-64 of [1]). Obviously similar modifications could be made for disjoint unions of sets or topological spaces.

Definition R24.8 Let $\{(X_\alpha,\mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty set of uniform spaces. For each $\gamma \in \Delta$, $i_\gamma : X_\gamma \to \coprod\{X_\alpha : \alpha \in \Delta\}$ is defined by $i_\gamma(x) = (x,\gamma)$.

Lemma R24.9 Let $\{(X_\alpha,\mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty set of uniform spaces and let $\gamma \in \Delta$. Then $i_\gamma : (X_\gamma,\mathcal{U}_\gamma) \to (\coprod\{X_\alpha : \alpha \in \Delta\}, \coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\})$ is uniformly continuous.

Proof: Let $W \in \coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. Then there is a collection $\{U_\alpha : \alpha \in \Delta\}$ with each $U_\alpha \in \mathcal{U}_\alpha$ such that $\cup\{U_\alpha^* : \alpha \in \Delta\} \subseteq W$. Clearly $(i_\gamma \times i_\gamma)^{-1}[U_\gamma^*] = U_\gamma$. Thus $U_\gamma \subseteq (i_\gamma \times i_\gamma)^{-1}[W]$ and so i_γ is uniformly continuous.

Lemma R24.10 Let $\{(X_\alpha,\mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty set of uniform spaces. Let $(Z,\mathcal{W})$ be a uniform space and assume for each $\gamma \in \Delta$ there is $h_\gamma : (X_\gamma,\mathcal{U}_\gamma) \to (Z,\mathcal{W})$ uniformly continuous. There is a unique $H : (\coprod\{X_\alpha : \alpha \in \Delta\}, \coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}) \to (Z,\mathcal{W})$ uniformly continuous such that, for every $\alpha \in \Delta$, $h_\alpha = H \circ i_\alpha$.

Proof: Define $H : \coprod\{X_\alpha : \alpha \in \Delta\} \to Z$ by $H(t,\gamma) = h_\gamma(t)$. Clearly $H \circ i_\alpha = h_\alpha$ for all $\alpha \in \Delta$. For $W \in \mathcal{W}$, $(H \times H)^{-1}[W]$ contains $\cup\{((h_\alpha \times h_\alpha)^{-1}[W])^* : \alpha \in \Delta\}$, which is in the coproduct uniformity. Thus H is uniformly continuous. Lastly suppose G is uniformly continuous and $G \circ i_\alpha = h_\alpha$ for all α. For $(t,\gamma) \in \coprod\{X_\alpha : \alpha \in \Delta\}$, $G((t,\gamma)) = G \circ i_\gamma(t) = h_\gamma(t) = H \circ i_\gamma(t) = H((t,\gamma))$. Thus $G = H$.

The last three items describe the universal property mentioned above. The next proposition shows that it determines the disjoint union up to unimorphism. The proof is a version of a standard categorical argument.

Proposition R24.11 Let $\{(X_\alpha,\mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty set of uniform spaces. Let $(Y,\mathcal{V})$ be a uniform space and $\{f_\alpha : X_\alpha \to Y : \alpha \in \Delta\}$ be a collection of uniformly continuous maps. Assume that for any uniform space $(Z,\mathcal{W})$ and for any collection $\{h_\alpha : X_\alpha \to Z : \alpha \in \Delta\}$ of uniformly continuous maps there is a unique uniformly continuous map $G : Y \to Z$ such that $G \circ f_\alpha = h_\alpha$. Then $(\coprod\{X_\alpha : \alpha \in \Delta\}, \coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\})$ is unimorphic to $(Y,\mathcal{V})$ by a unimorphism mapping (x,α) to $f_\alpha(x)$.

Proof: Apply the assumption to the collection $\{i_\alpha\}$ to obtain $G : Y \to \coprod\{X_\alpha : \alpha \in \Delta\}$ uniformly continuous with $G \circ f_\alpha = i_\alpha$ for all α. Apply R24.9 to the collection $\{f_\alpha\}$ to obtain $H : \coprod\{X_\alpha : \alpha \in \Delta\} \to Y$ uniformly continuous with $H \circ i_\alpha = f_\alpha$ for all α. Then $G \circ H \circ i_\alpha = i_\alpha$ for all α and by the uniqueness in R24.10 $G \circ H$ must be the identity map on $\coprod\{X_\alpha : \alpha \in \Delta\}$. Also $H \circ G \circ f_\alpha = f_\alpha$ for all α and by the uniqueness in the assumption $H \circ G$ must be the identity map on Y. Thus G and H are inverses of each other and so the spaces are unimorphic as required. The equation $H \circ i_\alpha = f_\alpha$ says that the unimorphism H maps (x,α) to $f_\alpha(x)$.

The previous proposition can be used to see that certain constructions used previously lead to spaces unimorphic to a disjoint union.

Corollary R24.12 Let $(X,\mathcal{U})$ be a uniform space and let, E be an equivalence relation on X with finitely many equivalence classes $\{C_1,\ldots,C_n\}$. Let $\mathcal{U}_j$ be the subspace uniformity from $\mathcal{U}$ on C_j. Then $(\coprod_{j=1}^n C_j, \coprod_{j=1}^n \mathcal{U}_j)$ is unimorphic to $(X,\mathcal{U}\vee\mathcal{U}_E)$ by a unimorphism mapping mapping (x,j) to x.

Proof: For each j, let ι_j be the inclusion map from C_j to X. The fact that ι_j is uniformly continuous from $(C_j,\mathcal{U}_j)$ to $(X,\mathcal{U}\vee\mathcal{U}_E)$ follows immediately from the hypothesis and $(\iota_j\times\iota_j)^{-1}[U\cap E] = U\cap(C_j\times C_j)$. Now let $(Z,\mathcal{W})$ be a uniform space and, for each j, let $f_j : (C_j,\mathcal{U}_j)\to(Z,\mathcal{W})$ be uniformly continuous. Define $F : X\to Z$ by $F(x) = f_j(x)$ if $x\in C_j$. Clearly $F\circ\iota_j = f_j$. For each j pick $U_j\in\mathcal{U}_j$ such that $(f_j\times f_j)^{-1}[W] = U_j\cap(C_j\times C_j)$. It is easy to check that $(\cap_{j=1}^n U_j)\cap E\subseteq(F\times F)^{-1}[W]$ and so F is uniformly continuous from $(X,\mathcal{U}\vee\mathcal{U}_E)$ to $(Z,\mathcal{W})$. The uniqueness of F is immediate from the functional equations. Since each ι_j is inclusion, by R24.11 the conclusion holds.

In a similar way, the uniformity constructed in R23.3.5 can be recognized as unimorphic to a disjoint union. The next two corollaries also relate to prior results. For $\mathcal{A}$ a collection of subsets of a set X, the equivalence relation associated with $\mathcal{A}$, denoted $E(\mathcal{A})$, was defined in R23.4.8. The sets $\mathcal{C}(\mathcal{A})$ and B_p were defined in R22.2.2.

Corollary R24.13 Let $(X,\mathcal{U})$ be a uniform space and let $\mathcal{A}$ be a finite collection of subsets of X. For each $p\in\mathcal{C}(\mathcal{A})$, let $\mathcal{U}_p$ be the subspace uniformity from $\mathcal{U}$ on B_p. Then $(\coprod\{B_p : p\in\mathcal{C}(\mathcal{A})\}, \coprod\{\mathcal{U}_p : p\in\mathcal{C}(\mathcal{A})\})$ is unimorhic to $(X,\mathcal{U}_e(\mathcal{A}))$ by a unimorphism mapping (x,p) to x.

Proof: Since $\mathcal{A}$ is finite, by R23.4.10 $\mathcal{U}_e(\mathcal{A}) = \mathcal{U}\vee\mathcal{U}_{E(\mathcal{A})}$. $E(\mathcal{A})$ has finitely many equivalence classes, which are the non-empty B_p. The conclusion now follows from R24.12.

The next is equivalent to the converse of R23.3.8.

Corollary R24.14 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{A}$ be a finite collection of subsets of X. For each $p\in\mathcal{C}(\mathcal{A})$ assume the subspace uniformity on B_p corresponds to the Stone-Čech compactification of B_p with the subspace topology from $\tau(\mathcal{U})$. Then $\mathcal{U}_e(\mathcal{A})$ corresponds to the Stone-Čech compactification of $(X,\tau(\mathcal{U}_e(\mathcal{A})))$.

Proof: Since the Stone-Čech compactification is characterized by the extendibility of all continuous maps to compact T_2 spaces, let $h : (X,\tau(\mathcal{U}_e(\mathcal{A})))\to Z$ be continuous, where Z is a compact T_2 space. By R7.1.1 it is sufficient to show that h is uniformly continuous from $(X,\mathcal{U}_e(\mathcal{A}))$ to Z. For each $p\in\mathcal{C}(\mathcal{A})$, let h_p be h restricted to B_p and let $\mathcal{U}_p$ be the subspace uniformity on B_p from $\mathcal{U}$. By hypothesis and R7.1.1 h_p is uniformly continuous from $(B_p,\mathcal{U}_p)$ to Z. By R24.10 there is a unique uniformly continuous map H from $(\coprod\{B_p : p\in\mathcal{C}(\mathcal{A})\}, \coprod\{\mathcal{U}_p : p\in\mathcal{C}(\mathcal{A})\})$ to Z such that $h_p = H\circ\iota_p$ for all p. Let ψ be the unimorphism of R24.13, i.e., $\psi(x,p) = x$. It is easy to check that $h = H\circ\psi^{-1}$ and so h is uniformly continuous as required.

The proof of R24.14 can be done directly. With notation as in the last proof, let V be in the unique uniformity for Z. For each p there is U_p in $\mathcal{U}$ such that $U_p\cap B_p = (h_p\times h_p)^{-1}[V]$. The finite intersection $W = \cap\{U_p : p\in\mathcal{C}(\mathcal{A})\}$ is in $\mathcal{U}$ and so $W\cap(\cup\{B_p\times B_p : p\in\mathcal{C}(\mathcal{A})\}$ is in $\mathcal{U}_e(\mathcal{A})$. It is easy to check that $W\cap(\cup\{B_p\times B_p : p\in\mathcal{C}(\mathcal{A})\}\subseteq(h\times h)^{-1}[V]$, i.e., h is uniformly continuous.

A simple example shows that R24.14 may not hold if $\mathcal{A}$ is infinite: Let X be infinite with the discrete topology, let $\mathcal{U}$ be the uniformity corresponding to the one point compactification of X, and let $\mathcal{A}$ be the collection of all singletons from X. Since $E_{\{x\}} \in \mathcal{U}$ for all x, $\mathcal{U}_e(\mathcal{A}) = \mathcal{U}$, which does not correspond to the Stone-Čech compactification of X.

Proposition R24.15 Let $\{(X_i, \mathcal{U}_i) : 1 \leq i \leq n\}$ be a finite set of separated, totally bounded uniform spaces. Assume, for each i, that $\mathcal{U}_i$ corresponds to the compactification class $[(Y_i, f_i)]$. Let $g : \coprod_{i=1}^n X_i \to \coprod_{i=1}^n Y_i$ by $g((x,i)) = (f_i(x), i)$. Then

i) $(\coprod_{i=1}^n X_i, \coprod_{i=1}^n \mathcal{U}_i)$ is separated and totally bounded.

ii)$(\coprod_{i=1}^n Y_i, g)$ is a T_2 compactification of $(\coprod_{i=1}^n X_i, \tau(\coprod_{i=1}^n \mathcal{U}_i))$.

iii) $\coprod_{i=1}^n \mathcal{U}_i$ corresponds to the compactification class $[(\coprod_{i=1}^n Y_i, g)]$.

Proof: R24.4 and R24.7 yield i). $\coprod_{i=1}^n Y_i$ is compact and T_2 by R15.1.16. Let $\mathcal{V}_i$ be the unique uniformity for Y_i. By R15.1.17 $\coprod_{i=1}^n \mathcal{V}_i$ is the unique uniformity for $\coprod_{i=1}^n Y_i$. For each j let $\kappa_j : Y_j \to \coprod_{i=1}^n Y_i$ and $\iota_j : X_j \to \coprod_{i=1}^n X_i$ be defined as in R24.8. Apply R24.10 to the uniformly continuous maps $\{\kappa_j \circ f_j\}$ to obtain a unique uniformly continuous map $H : \coprod_{i=1}^n X_i \to \coprod_{i=1}^n Y_i$ such that $H \circ \iota_i = \kappa_i \circ f_i$ for all i. It is easy to check that $H = g$ and so g is uniformly continuous. Since each f_i is one-to-one, g is also one-to-one. Now let $\cup_{i=1}^n U_i^*$, where $U_i \in \mathcal{U}_i$, be a basic entourage in $\coprod_{i=1}^n \mathcal{U}_i$. For each i there is $V_i \in \mathcal{V}_i$ such that $(f_i \times f_i)[U_i] = V_i \cap f_i[X_i] \times f_i[X_i]$. By definition $\cup_{i=1}^n V_i^* \in \coprod_{i=1}^n \mathcal{V}_i$ and it is easy to check that $(g \times g)[\cup_{i=1}^n U_i^*] \supseteq (\cup_{i=1}^n V_i^*) \cap (g[\coprod_{i=1}^n X_i] \times g[\coprod_{i=1}^n X_i])$. Thus g is a unimorphism onto its image. For density of the image, let $\cup_{i=1}^n W_i^*$, where $W_i \in \mathcal{V}_i$, be chosen and let $(y, j) \in \coprod_{i=1}^n Y_i$. There is $x \in X_j$ such that $f_j(x) \in W_j[y]$. Since $W_j^*[(y,j)] = W_j[y] \times \{j\}$, $g((x,j)) \in (\cup_{i=1}^n W_i^*)[(y,j)]$, which implies the required density. Thus ii) holds and conclusion iii) follows from R1.6a.

Note that R15.1.19 is a corollary of R24.15 and R15.1.18, since given an equivalence relation E with finitely many classes $\{C_1, \ldots, C_n\}$, by R24.12 $(X, \mathcal{U} \vee \mathcal{U}_E)$ is unimorphic to $(\coprod_{i=1}^n C_i, \coprod_{i=1}^n \mathcal{U}_i)$ where $\mathcal{U}_i$ is the subspace uniformity from $\mathcal{U}$ on C_i.

Lemma R24.16 Let $\{(X_\alpha, \mathcal{U}_\alpha) : \alpha \in \Delta\}$ be a non-empty collection of uniform spaces. Assume that for every $\alpha \in \Delta$ there is a collection $\{\mathcal{V}_\gamma : \gamma \in \Gamma_\alpha\}$ of uniformities such that $\mathcal{U}_\alpha = \vee\{\mathcal{V}_\gamma : \gamma \in \Gamma_\alpha\}$. Then $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\} \supseteq \vee\{\coprod\{\mathcal{V}_{p(\alpha)} : \alpha \in \Delta\} : p \in \Pi\{\Gamma_\alpha : \alpha \in \Delta\}\}$, with equality if Δ is finite.

Proof: Let $\mathcal{W} = \vee\{\coprod\{\mathcal{V}_{p(\alpha)} : \alpha \in \Delta\} : p \in \Pi\{\Gamma_\alpha : \alpha \in \Delta\}\}$ for notational convenience. Given $p \in \Pi\{\Gamma_\alpha : \alpha \in \Delta\}$, since $\mathcal{V}_{p(\alpha)} \subseteq \mathcal{U}_\alpha$, $\coprod\{\mathcal{V}_{p(\alpha)} : \alpha \in \Delta\} \subseteq \coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. Thus $\mathcal{W} \subseteq \coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. Now assume Δ is finite and let $\cup\{U_\alpha^* : \alpha \in \Delta\}$, where each $U_\alpha \in \mathcal{U}_\alpha$, be a basic entourage in $\coprod\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. For each $\delta \in \Delta$ there is F_δ, a finite subset of Γ_δ, and a set $\{V_t : t \in F_\delta\}$ with each $V_t \in \mathcal{V}_t$ such that $\cap\{V_t : t \in F_\delta\} \subseteq U_\delta$. For each t let $\tilde{V}_t = V_t^* \cup \{(X_\alpha \times X_\alpha)^* : \alpha \neq \delta, \alpha \in \Delta\}$ and let $\tilde{U}_\delta = U_\delta^* \cup \{(X_\alpha \times X_\alpha)^* : \alpha \neq \delta, \alpha \in \Delta\}$. With $p(\delta) = t$ and $p(\alpha)$ arbitrary in Γ_α for $\alpha \neq \delta$, $\tilde{V}_t$ is in $\coprod\{\mathcal{V}_{p(\alpha)} : \alpha \in \Delta\}$ and so in $\mathcal{W}$. Since the finite intersection $\cap\{\tilde{V}_t : t \in F_\delta\}$ is contained in $\tilde{U}_\delta$, $\tilde{U}_\delta$ is in $\mathcal{W}$. Clearly $\cap\{\tilde{U}_\alpha : \alpha \in \Delta\}$ is contained in $\cup\{U_\alpha^* : \alpha \in \Delta\}$ and so the latter, since Δ is finite, is in $\mathcal{W}$. Thus, when Δ is finite, equality holds as asserted.

As [3] suggests, a pair (Y, f), where Y is compact and T_2, f is one-to-one into Y, and the image of f is dense in Y, can be considered a T_2 compactification associated with the set X, which is the domain of f. The topology (or uniformity) for X need not be explicitly mentioned because they are determined by Y and f. To say that the class of (Y, f) is the supremum of the classes of a collection $\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$ implicitly assumes that each (Y_α, f_α) is also associated with X. This language is used in the last two results.

Lemma R24.17 For $1 \leq i \leq n$, let (Y_i, f_i) be a T_2 compactification associated with a set X_i. Let $g : \coprod_{i=1}^n X_i \to \coprod_{i=1}^n Y_i$ by $g(x,i) = (f_i(x), i)$. Then $(\coprod_{i=1}^n Y_i, g)$ is a T_2 compactification associated with $\coprod_{i=1}^n X_i$.

Proof: This is simply a restatement of R24.15ii in compactification-focused language.

Corollary R24.18 For $1 \leq i \leq n$, let (Y_i, f_i) be a T_2 compactification associated with a set X_i. Assume for each i there is a non-empty family of T_2 compactifications $\{(Y_\alpha^i, f_\alpha^i) : \alpha \in \Delta_i\}$ such that the class of (Y_i, f_i) is the supremum of the classes of $\{(Y_\alpha^i, f_\alpha^i) : \alpha \in \Delta_i\}$. Then the class of $(\coprod_{i=1}^n Y_i, g)$, where $g : \coprod_{i=1}^n X_i \to \coprod_{i=1}^n Y_i$ by $g(x,i) = (f_i(x), i)$ is the supremum of the classes $\{(\coprod_{i=1}^n Y_{p(i)}^i, g_p) : p \in \Pi_{i=1}^n \Delta_i\}$, where $g_p(x,i) = (f_{p(i)}^i(x), i)$ as in R24.17.

Proof: For each i and each $\alpha \in \Delta_i$, let $\mathcal{U}_\alpha^i$ be the separated, totally bounded uniformity associated with $[(Y_\alpha^i, f_\alpha^i)]$. By R13.1.7 $\mathcal{U}_i = \vee\{\mathcal{U}_\alpha^i : \alpha \in \Delta_i\}$ is the uniformity associated with $[(Y_i, f_i)]$. By R24.15iii $\coprod_{i=1}^n \mathcal{U}_i$ is the uniformity associated with $[(\coprod_{i=1}^n Y_i, g)]$. For each $p \in \Pi_{i=1}^n \Delta_i$, again by R24.15iii $\coprod_{i=1}^n \mathcal{U}_{p(i)}^i$ is the uniformity associated with $[(\coprod_{i=1}^n Y_{p(i)}^i, g_p)]$. The conclusion now follows from R13.1.7 and R24.16.

A somewhat careless but more concise version of the last conclusion:

$$(\coprod_{i=1}^n Y_i, g) = \bigvee\{(\coprod_{i=1}^n (Y_{p(i)}, g_p) : p \in \Pi_{i=1}^n \Delta_i\}$$

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http://www.susanjkleinart.com/compactification/

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