Compactifications and Hyperspaces

This section describes the compactification associated with the hyper-uniformity generated by a totally bounded uniformity. Two preliminary subsections are needed before that can be done.

Stable Spaces and Near Compactifications

Stable spaces, which are a generalization of T_1 spaces, were defined in [2]. The results in this section, which are simple and undoubtedly known, are collected here for the convenience of the reader and for simple reference. Throughout this and the following subsections, given A, a subset of a topological space (X, τ), $c(A)$ (or $c_X(A)$ if the context involves several spaces) denotes the closure of A.

Definition R25.1.1 Let (X, τ) be a topological space. (X, τ) is stable provided, for all $x \in X$, $x \in O \in \tau$ implies $c(\{x\}) \subseteq O$.

Lemma R25.1.2 Every regular space is stable.

Proof: Let (X, τ) be a regular topological space and let $x \in O \in \tau$. By regularity there is $G \in \tau$ such that $x \in G \subseteq c(G) \subseteq O$ and so $c(\{x\}) \subseteq c(G) \subseteq O$.

Lemma R25.1.3 Let (X, τ) be a stable topological space and let $x, y \in X$. Then either $c(\{x\}) \cap c(\{y\}) = \emptyset$ or $c(\{x\}) = c(\{y\})$.

Proof: Suppose $c(\{x\}) \cap c(\{y\}) \neq \emptyset$. By the definition of stability, $x \in X - c(\{y\})$ would imply $c(\{x\}) \cap c(\{y\}) = \emptyset$. Thus $x \in c(\{y\})$ and so $c(\{x\}) \subseteq c(\{y\})$. Similarly $c(\{y\}) \subseteq c(\{x\})$.

The last lemma says that in a stable space (X, τ) the closures of singletons form a partition of X.

Definition R25.1.4 Let (X, τ) be a stable topological space. C (or $C(X)$ if the context involves several spaces) is the equivalence relation on X generated by the partition of closures of singletons.

Lemma R25.1.5 Let (X, τ) be a stable topological space and let $x, y \in X$. Then

i) The C-equivalence class of x is $c(\{x\})$.

ii) xCy if and only if $x \in c(\{y\})$.

Proof: When an equivalence relation is determined by a partition, the elements of the partition are the equivalence classes. Since $x \in c(\{x\})$, i) holds. For ii), xCy if and only if x, y are in the same element of the partition, i.e., by i) and R25.3, if and only if $c(\{x\}) = c(\{y\})$. Part ii) follows easily.

The fundamental fact about stable spaces is the following.

Proposition R25.1.6 Let (X, τ) be a stable topological space. Then the quotient space X/C is T_1.

Proof: Let π be the projection from X to X/C and let $p \in X/C$. Suppose p is the equivalence class of x. Then $\pi^{-1}[\{p\}]$ is the C-class of x, i.e., $c(\{x\})$. Since $\pi^{-1}[\{p\}]$ is closed in X, by a basic fact about the quotient topology, $\{p\}$ is closed in X/C. Thus X/C is T_1.

Lemma R25.1.7 Let (X, τ) be a topological space and let $G \subseteq X \times X$ be open. Assume $(a, b) \in G$. Then $c(\{a\}) \times c(\{b\}) \subseteq G$.

Proof: There exist $O_1, O_2 \in \tau$ such that $(a, b) \in O_1 \times O_2 \subseteq G$. The conclusion is now immediate by stability.

In what follows on stability, the focus will be on uniform spaces, which always generate stable topologies by P2.3 and R25.1.2.

In general, given a uniform space $(X,\mathcal{U})$ and an equivalence relation E on X, the quotient uniformity, i.e., the largest uniformity on X/E making the projection uniformly continuous, does not induce the quotient topology on X/E. However, it will be shown below that the equivalence relation C is an exception. $\mathcal{U}/C$ will denote the quotient uniformity.

Recall that a uniformity always contains a basis of symmetric entourages which are open in the product topology. For $S \subseteq X \times X$ and $x \in X$, the x-section of S is defined to be $S[x] = \{y : (x, y) \in S\}$.

Lemma R25.1.8 Let $(X,\mathcal{U})$ be a uniform space and let π denote the projection from X onto X/C. Then $\mathcal{U}/C = \{S \subseteq X/C \times X/C : (\pi \times \pi)[U] \subseteq S \text{ for some } U \in \mathcal{U}\}$.

Proof: Let $\mathcal{V} = \{S \subseteq X/C \times X/C : (\pi \times \pi)[U] \subseteq S \text{ for some } U \in \mathcal{U}\}$. For any $W \in \mathcal{U}/C$, by uniform continuity $(\pi \times \pi)^{-1}[W] \in \mathcal{U}$ and $(\pi \times \pi)[(\pi \times \pi)^{-1}[W]] \subseteq W$ so that $\mathcal{U}/C \subseteq \mathcal{V}$. Clearly, for any $S \in \mathcal{V}$, $(\pi \times \pi)^{-1}[S]$ is in $\mathcal{U}$. Thus, by definition of the quotient uniformity, it is sufficient to show that $\mathcal{V}$ is a uniformity. The first four requirements of definition P2.1 can be routinely verified for any equivalence relation. For the triangle inequality, let $S \in \mathcal{V}$ with $U \in \mathcal{U}$ such that $(\pi \times \pi)[U] \subseteq S$. There is an open, symmetric $W \in \mathcal{U}$ such $W \circ W \subseteq U$. Let $(\pi(a), \pi(b))$ be in $(\pi \times \pi)[W] \circ (\pi \times \pi)[W]$. Then there exist $c, d \in X$ such that $(a, c) \in W$, $(d, b) \in W$, and $\pi(c) = \pi(d)$. Since the projection maps a point to its C-equivalence class, by R25.1.5i $c(\{c\}) = c(\{d\})$. By R25.1.7 $c(\{a\}) \times c(\{c\}) \subseteq W$ and so $(a, d) \in W$. Thus $(a, b) \in U$ and so $(\pi(a), \pi(b)) \in S$, i.e., $(\pi \times \pi)[W] \circ (\pi \times \pi)[W] \subseteq S$.

Lemma R25.1.9 Let $(X,\mathcal{U})$ be a uniform space. Then $\tau(\mathcal{U}/C)$ is the quotient topology induced by $\tau(\mathcal{U})$ on X/C.

Proof: Since uniform continuity implies continuity, $\tau(\mathcal{U}/C)$ is contained in the quotient topology. Now let $G \subseteq X/C$ with $\pi^{-1}[G]$ in $\tau(\mathcal{U})$, where π is the projection. Let $\pi(x) \in G$ so that $x \in \pi^{-1}[G]$. There is $U \in \mathcal{U}$ symmetric and open in the product topology such that $U[x] \subseteq \pi^{-1}[G]$. By R25.1.8 $(\pi \times \pi)[U])$ is in $\mathcal{U}/C$. It is claimed that $(\pi \times \pi)[U][\pi(x)] \subseteq G$. Let $\pi(y) \in (\pi \times \pi)[U][\pi(x)]$. There is $(a, b) \in U$ such that $\pi(a) = \pi(x)$ and $\pi(b) = \pi(y)$. By definition of the projection, the image of a point is its equivalence class, i.e., $\pi(a)$ is $c(\{a\}) = c(\{x\})$ and $\pi(b)$ is $c(\{b\}) = c(\{y\})$. By R25.1.7 (x, y) is in U and so $y \in \pi^{-1}[G]$, i.e., $\pi(y) \in G$ as claimed. Thus $G \in \tau(\mathcal{U}/C)$ and so the quotient topology is contained in $\tau(\mathcal{U}/C)$.

Corollary R25.1.10 Let $(X,\mathcal{U})$ be a uniform space Then $(X/C,\mathcal{U}/C)$ is separated.

Proof: The completely regular topology $\tau(\mathcal{U}/C)$ is T_1 by R25.1.9 and R25.1.6, and so it is $T_{3\frac{1}{2}}$. By P2.3 the conclusion follows.

Corollary R25.1.11 Let (X, τ) be a completely regular topological space. Then X/C with the quotient topology is $T_{3\frac{1}{2}}$.

Proof: By P2.3 there is a uniformity $\mathcal{U}$ such that $\tau(\mathcal{U}) = \tau$. By R25.1.9 and R25.1.6 the conclusion follows.

Proposition R25.1.12 Let $(X,\mathcal{U})$ be a uniform space. Then $(X/C,\mathcal{U}/C)$ is totally bounded if and only if $(X,\mathcal{U})$ is totally bounded.

Proof: Let π denote the projection from X onto X/C. A uniformly continuous image of a totally bounded space must be totally bounded and so, if $(X,\mathcal{U})$ is totally bounded, so is $(X/C,\mathcal{U}/C)$. Now assume $(X/C,\mathcal{U}/C)$ is totally bounded and let $U \in \mathcal{U}$. there is a symmetric, open $V \in U$ with $V \subseteq U$. By R25.1.8 $(\pi \times \pi)[V] \in \mathcal{U}/C$ and so there exist $a_1, a_2, \ldots, a_n \in X$ such that $\cup_{i=1}^{n}(\pi\times\pi)[V][\pi(a_i] = X/C$. It is claimed that $\cup_{i=1}^{n} U[a_i] = X$. Let $x \in X$. For some i, $(\pi(a_i), \pi(x)) \in (\pi \times \pi)[V]$ so that there is $(y,z) \in V$ such that $\pi(y) = \pi(a_i)$ and $\pi(z) = \pi(x)$. By definition of the projection, the image of a point is its equivalence class, i.e., $\pi(y)$ is $c(\{y\}) = c(\{a_i\})$ and $\pi(z)$ is $c(\{z\}) = c(\{x\})$. By R25.1.7 (a_i, x) is in V and so in U. Thus $x \in U[a_i]$ and the claim holds.

Proposition R25.1.13 Let $(X,\mathcal{U})$ be a uniform space. Then $(X/C,\mathcal{U}/C)$ is complete if and only if $(X,\mathcal{U})$ is complete.

Proof: Let π denote the projection from X onto X/C. Assume $(X/C,\mathcal{U}/C)$ is complete and let $S : D \to X$ be a $\mathcal{U}$-Cauchy net. Since π is uniformly continuous, $\pi \circ S$ is $\mathcal{U}/C$-Cauchy and by completeness it converges to some $\pi(x)$. It is claimed that S converges to x. Let $x \in O \in \tau(\mathcal{U})$, let U be in $\mathcal{U}$ with $U[x] \subseteq O$, and pick a symmetric, open V in $\mathcal{U}$ with $V \subseteq U$. By R25.1.8 $(\pi \times \pi)[V]$ is in $\mathcal{U}/C$ and so there exists $d_0 \in D$ such that $d \geq d_0$ implies $\pi(S(d)) \in (\pi \times \pi)[V][\pi(x)]$. Let $d \geq d_0$. There is $(y,z) \in V$ such that $(\pi(y), \pi(z)) = (\pi(x), \pi(S(d))$. Then $x \in c(\{y\})$ and $S(d) \in c(\{z\})$ so that by R25.1.7 $(x, S(d)) \in V$, i.e., $S(d) \in V[x] \subseteq O$ and so the claim holds. For the converse suppose $T : E \to \mathcal{U}/C$ is $\mathcal{U}/C$-Cauchy. Pick any $T_1 : E \to X$ such that $\pi \circ T_1 = T$. Now the claim is that T_1 is $\mathcal{U}$-Cauchy. Let $U \in \mathcal{U}$ and pick a symmetric, open $V \in \mathcal{U}$ with $V \subseteq U$. Since $(\pi \times \pi)[V] \in \mathcal{U}/C$, there is $e_0 \in E$ such that $e, e' \geq e_0$ implies $(\pi \circ T_1(e), \pi \circ T_1(e')) \in (\pi \times \pi)[V]$. Let $e, e' \in E$ with $e, e' \geq e_0$. There is $(y,z) \in V$ with $(\pi(y), \pi(z)) = (\pi \circ T_1(e), \pi \circ T_1(e'))$. As above $T_1(e) \in c(\{y\})$ and $T_1(e') \in c(\{z\})$. By R25.1.7 again $(T_1(e), T_1(e')) \in V \subseteq U$ and the claim is verified. Since $(X,\mathcal{U})$ is complete, there is $x \in X$ such that T_1 converges to x. Since π is uniformly continuous, $\pi(x)$ is the limit of $\pi \circ T_1 = T$. Thus $(X/C,\mathcal{U}/C)$ is complete.

An example illustrating the next definition will occur below in R25.3.3. Note that $(Y,\mathcal{V})$ is not assumed to be separated.

Definition R25.1.14 Let $(X,\mathcal{U})$ be separated and totally bounded. Let $(Y,\mathcal{V})$ be totally bounded and complete. Assume $f : X \to Y$ is a uniform embedding and $f[X]$ is dense in Y. The triple $(Y, f, \mathcal{V})$ is a near compactification corresponding to $\mathcal{U}$.

Proposition R25.1.15 Let $(X,\mathcal{U})$ be separated and totally bounded and let $(Y, f, \mathcal{V})$ be a near compactification corresponding to $\mathcal{U}$. Let π be the projection from Y onto $Y/C(Y)$. Then $(Y/C(Y), \pi \circ f)$ is in the compactification class corresponding to $\mathcal{U}$ and the unique uniformity for $Y/C(Y)$ is $\mathcal{V}/C(Y)$.

Proof: By the results above $(Y/C(Y), \mathcal{V}/C(Y))$ is complete, totally bounded, and separated. By general facts $\pi \circ f$ is uniformly continuous and $\pi \circ f[X]$ is dense in $Y/C(Y)$. Now assume $a, b \in X$ with $a \neq b$. There is $U \in \mathcal{U}$ such that $(a,b) \notin U$. Since f is a uniform embedding, there is $V \in \mathcal{V}$ such that $(f \times f)[U] = (f[X] \times f[X]) \cap V$. If $\pi \circ f(a) = \pi \circ f(b)$, $f(b) \in c_Y(\{f(a)\}) \subseteq V[f(a)]$ so that $(f(a), f(b)) \in V$ and so, since f is one-to-one, $(a,b) \in U$, a contradiction. Thus $\pi \circ f$ is one-to-one. To see that $\pi \circ f$ is a uniform embedding, let $U \in \mathcal{U}$ and let $V \in \mathcal{V}$ with $(f \times f)[U] = (f[X] \times f[X]) \cap V$.

There is $W \in \mathcal{V}$ such that $W = W^{-1}$ and $W \circ W \circ W \subseteq V$. Let $(\pi \circ f(x_1), \pi \circ f(x_2))$ be in $(\pi \times \pi)[W] \cap (\pi \circ f[X] \times \pi \circ f[X])$. There is $(y_1, y_2) \in W$ such that $\pi(y_i) = \pi(f(x_i))$, i.e., $c(\{y_i\}) = c(\{f(x_i)\})$ for $i = 1, 2$. Then $y_i \in c(\{f(x_i)\}) \subseteq W[f(x_i)]$ so that $(f(x_1), y_1)$ and $(f(x_2), y_2)$ are both in W. Using the symmetry and $(y_1, y_2) \in W$, we see that $(f(x_1), f(x_2))$ is in $W \circ W \circ W$ and so in V. Thus, since f is one-to-one, $(x_1, x_2) \in U$ and so $(\pi\circ f(x_1), \pi\circ f(x_2))$ is in $(\pi\circ f\times\pi\circ f)[U]\cap(\pi\circ f[X]\times\pi\circ f[X])$. By R25.1.8 $(\pi\times\pi)[W]$ is in $\mathcal{V}/C(Y)$ and so the containment implies $(\pi \circ f \times \pi \circ f)[U] \cap (\pi \circ f[X] \times \pi \circ f[X])$ is in the subspace uniformity for $\pi \circ f[X]$. Thus $\pi \circ f$ is a uniform embedding and the conclusions follow from R1.6a.

Hyperspaces

In what follows, given a topological space (X, τ), $\hat{X}$ denotes the power set of X and 2^X denotes the collection of closed subsets of X, both including $\emptyset$, which is an isolated point in the topologies considered here. The literature, e.g., Michael [4], more typically excludes $\emptyset$, probably because of the focus on selections. Subspaces of $\hat{X}$ are considered below and may include or exclude $\emptyset$.

Most of the results in this subsection can be found in Caufield [1], which also contains references to earlier sources. Proofs are included for the convenience of the reader.

Definition R25.2.1 Let X be a set and let $S \subseteq X \times X$. $H(S)$ is defined to be $\{(A, B) \in \hat{X} \times \hat{X} : A \subseteq S[B] \text{ and } B \subseteq S[A]\}$.

Lemma R25.2.2 Let $(X,\mathcal{U})$ be a uniform space. Then $\{H(U) : U \in \mathcal{U}\}$ is a basis for a uniformity on $\hat{X}$.

Proof: Let $U, V \in \mathcal{U}$ and let $W \in \mathcal{U}$ with $W \circ W \subseteq U$. It is easy to check that the diagonal of $\hat{X}$ is contained in $H(U)$, that $H(U \cap V) \subseteq H(U) \cap H(V)$, that $H(U)$ is symmetric, and that $H(W) \circ H(W) \subseteq H(U)$. The conclusion follows.

Definition R25.2.3 Let $(X,\mathcal{U})$ be a uniform space. $\hat{\mathcal{U}}$ is the uniformity for $\hat{X}$ with basis $\{H(U) : U \in \mathcal{U}\}$.

The previous definition is motivated by the following fact, which will not be needed here: If $\mathcal{U}$ is generated by a pseudo-metric ρ, then $\hat{\mathcal{U}}$ is generated by the Hausdorff pseudo-metric determined from ρ.

Lemma R25.2.4 Let $(X,\mathcal{U})$ be a uniform space and let $i_X : X \to \hat{X}$ by $i_X(x) = \{x\}$. Then i_X is a unimorphism from $(X,\mathcal{U})$ onto $i_X[X]$ with the subspace uniformity from $\hat{\mathcal{U}}$.

Proof: Clearly i_X is one-to-one and onto $i_X[X]$. Let $U \in \mathcal{U}$. It is easy to check that $(U \cap U^{-1}) \subseteq (i_X \times i_X)^{-1}[H(U)]$ and so i_X is uniformly continuous. It is also routine to check that $H(U \cap U^{-1}) \cap (i_X[X] \times i_X[X]) \subseteq (i_X \times i_X)[U]$. The conclusion follows.

Definition R25.2.5 Let X, Y be sets and let $f : X \to Y$. The map $\hat{f} : \hat{X} \to \hat{Y}$ is defined by $\hat{f}(A) = f[A]$.

Lemma R25.2.6 Let $(X,\mathcal{U})$ and $(Y, \mathcal{V})$ be uniform spaces and let $f : X \to Y$. Then $f : (X,\mathcal{U}) \to (Y, \mathcal{V})$ is uniformly continuous if and only if $\hat{f} : (\hat{X}, \hat{\mathcal{U}}) \to (\hat{Y}, \hat{\mathcal{V}})$ is uniformly continuous.

Proof: First assume $\hat{f}$ is uniformly continuous. It is easy to check that $f = i_Y^{-1} \circ \hat{f} \circ i_X$ and so f is also uniformly continuous. Now assume the uniform continuity of f. For $V \in \mathcal{V}$, it is routine to verify that $H((f \times f)^{-1}[V]) \subseteq (\hat{f} \times \hat{f})^{-1}[H(V)]$, from which the uniform continuity of $\hat{f}$ follows.

Proposition R25.2.7 Let $(X,\mathcal{U})$ be a uniform space. Then $(X,\mathcal{U})$ is totally bounded if and only if $(\hat{X},\hat{\mathcal{U}})$ is totally bounded.

Proof: First assume $(\hat{X},\hat{\mathcal{U}})$ is totally bounded, Then each of its subspaces is also totally bounded. By R25.2.4 $(X,\mathcal{U})$ is unimorphic to a subspace of $(\hat{X},\hat{\mathcal{U}})$ so that $(X,\mathcal{U})$ is totally bounded. Now assume $(X,\mathcal{U})$ is totally bounded. Let $H(U)$, where $U \in \mathcal{U}$, be a basic entourage in $\hat{\mathcal{U}}$. Let $V = U \cap U^{-1}$. There are $x_1, \ldots, x_n$ in X such that $X = \cup_{i=1}^{n} V[x_i]$. Let $\mathcal{S}$ be the finite set $\mathcal{P}(\{x_1, \ldots, x_n\})$. It is sufficient to show that $\hat{X} \subseteq \cup\{H(U)[S] : S \in \mathcal{S}\}$. Let $A \in \hat{X}$ and let $S = \{x_i : V[x_i] \cap A \neq \emptyset\}$,which is in $\mathcal{S}$. If $a \in A$, pick x_j such that $(x_j, a) \in V$. Then $x_j \in S$ and $a \in V[S]$. Thus $A \subseteq V[S] \subseteq U[S]$. Since V is symmetric, by definition of S, $S \subseteq V[A] \subseteq U[A]$. By definition $(S, A) \in H(U)$ and so the needed containment holds.

For the separated property, the situation is less straightforward. The following lemma is well-known but I don't have a specific reference.

Lemma R25.2.8 Let $(X,\mathcal{U})$ be a uniform space, let $\mathcal{B}$ be a basis for $\mathcal{U}$, and let $A \subseteq X$. Then $c(A) = \cap\{B[A] : B \in \mathcal{B}\}$.

Proof: Let $t \in c(A)$. For any $B \in \mathcal{B}$, since $\mathcal{B} \subseteq \mathcal{U}$, $B \cap B^{-1} \in \mathcal{U}$ and $(B \cap B^{-1})[t]$ is a neighborhood of t, $A \cap (B \cap B^{-1})[t] \neq \emptyset$, i.e., there is $a \in A$ such that $(t, a) \in B \cap B^{-1}$. Then $t \in (B \cap B^{-1})[a] \subseteq B[A]$. Thus the closure is contained in the intersection. Now let $x \in \cap\{B[A] : B \in \mathcal{B}\}$ and let $x \in O \in \tau(\mathcal{U})$. There is $U \in \mathcal{U}$ such that $U[x] \subseteq O$. Since $\mathcal{B}$ is a basis, there is $B \in \mathcal{B}$ with $B \subseteq U \cap U^{-1}$. Then $x \in B[A] \subseteq (U \cap U^{-1})[A]$, i.e., there is $a \in A$ such that $(a, x) \in U \cap U^{-1}$. Then $a \in U[x]$, i.e., $O \cap A \neq \emptyset$. Thus $x \in c(A)$.

Corollary R25.2.9 Let $(X,\mathcal{U})$ be a uniform space and let $A, B \in \hat{X}$. Then (A, B) is in $\cap\{H(U) : U \in \mathcal{U}\}$ if and only if $c(A) = c(B)$.

Proof: Let $(A, B) \in \cap\{H(U) : U \in \mathcal{U}\}$. Then, for every $U \in \mathcal{U}$, $A \subseteq U[B]$ and so by the previous lemma $A \subseteq c(B)$. Thus $c(A) \subseteq c(B)$. Similarly $c(B) \subseteq c(A)$. For the converse assume $c(A) = c(B)$ and let $U \in \mathcal{U}$. By the lemma $A \subseteq c(B) \subseteq U[B]$ and $B \subseteq c(A) \subseteq U[A]$. By definition $(A, B) \in H(U)$. Thus $(A, B) \in \cap\{H(U) : U \in \mathcal{U}\}$.

Definition R25.2.10 Let $(X,\mathcal{U})$ be a uniform space and let $\mathcal{S} \subseteq \hat{X}$. $\hat{\mathcal{U}}(\mathcal{S})$ is the subspace uniformity on $\mathcal{S}$ from $\hat{\mathcal{U}}$.

Proposition R25.2.11 Let $(X,\mathcal{U})$ be a uniform space and let $\mathcal{S} \subseteq \hat{X}$. Then $(\mathcal{S},\hat{\mathcal{U}}(\mathcal{S}))$ is separated if and only if $A, B \in \mathcal{S}$ with $A \neq B$ implies $c(A) \neq c(B)$.

Proof: Recall the definition: $\hat{\mathcal{U}}(\mathcal{S})$ is separated if and only if $\cap\{W : W \in \hat{\mathcal{U}}(\mathcal{S})\}$ is $\Delta_{\mathcal{S}}$, the diagonal of $\mathcal{S}$. First assume the space is separated and let $A, B \in \mathcal{S}$ with $A \neq B$. There is $W \in \hat{\mathcal{U}}(\mathcal{S})$ such that $(A, B) \notin W$. Since $\hat{\mathcal{U}}(\mathcal{S})$ is a subspace uniformity, there is $U \in \mathcal{U}$ such that $H(U) \cap (\mathcal{S} \times \mathcal{S}) \subseteq W$. By R25.2.9 $(A, B) \notin H(U)$ implies $c(A) \neq c(B)$. For the converse, let $A, B \in \hat{S}$ with $A \neq B$. By assumption $c(A) \neq c(B)$. By R25.2.9 there is $U \in \mathcal{U}$ such that $(A, B) \notin H(U)$ and so (A, B) is not in $H(U) \cap (\mathcal{S} \times \mathcal{S})$, an element of $\hat{\mathcal{U}}(\mathcal{S})$. Thus $(A, B) \notin \cap\{W : W \in \hat{\mathcal{U}}(\mathcal{S})\}$. This shows $\cap\{W : W \in \hat{\mathcal{U}}(\mathcal{S})\} \subseteq \Delta_{\mathcal{S}}$. The reverse containment is automatic since every entourage contains the diagonal.

Corollary R25.2.12 Let $(X,\mathcal{U})$ be a uniform space. Then $(2^X,\hat{\mathcal{U}}(2^X))$ separated.

Proof: Let A, B be in 2^X with $A \neq B$. Since $c(A) = A$ and $c(B) = B$, $c(A) \neq c(B)$.

Note that the previous two results do not require that $(X,\mathcal{U})$ be separated. If not, the resulting hyperspace may be quite trivial. For example, for nonempty X and $\mathcal{U}$ indiscrete, 2^X has exactly two elements.

Corollary R25.2.13 Let $(X,\mathcal{U})$ be a uniform space. Then $(\hat{X},\hat{\mathcal{U}})$ is separated if and only if $\tau(\mathcal{U})$ is the discrete topology.

Proof: Assume the space is separated. If $\tau(\mathcal{U})$ is not discrete, there is $A \in \hat{X}$ with $A \neq c(A)$. By R25.2.11 with $\mathcal{S} = \hat{X}$, this says $(\hat{X},\hat{\mathcal{U}})$ is not separated, a contradiction. Now assume the topology is discrete. Given $A, B \in \hat{X}$ with $A \neq B$, since $c(A) = A$ and $c(B) = B$, $c(A) \neq c(B)$. By R25.2.11, the space is separated.

Completeness also does not in general transfer to the hyperspace uniformity: [1] contains an example with $(X,\mathcal{U})$ complete but $(\hat{X},\hat{\mathcal{U}})$ not complete. For the purpose of compactification, a simple partial result (R25.2.24 below), which is a corollary of the sufficient conditions for hypercompleteness in [1], will suffice. It will be derived by a different method, based on an argument from [3].

The idea of universal nets, which corresponds to the concept of ultrafilters and appears in some older textbooks such as Wilansky [6; p.133], will be used. Since this notion is less frequently studied than ultrafilters, the proofs of some basic known facts are outlined here. The domain of a net is assumed to be a non-empty directed set.

Definition R25.2.14 Let X be a set, D a directed set, and $S : D \to X$ a net. S is a universal net provided, for every $A \subseteq X$, S is eventually in A or S is eventually in $X - A$.

Lemma R25.2.15 Let X be a set. Every net in X has a universal subnet.

Proof: Let $S : D \to X$ be a net in X. Let $\mathcal{F}$ be the collection of all A in $\hat{X}$ such that S is eventually in A. $\mathcal{F}$ is a filter and so is contained in an ultrafilter $\mathcal{G}$. Note that S is frequently in each element of $\mathcal{G}$. Let $E = \{(d, G) : S(d) \in G \in \mathcal{G}\}$ and order E as follows: $(d_1, G_1) \geq (d_2, G_2)$ if and only if $d_1 \geq_D d_2$ and $G_1 \subseteq G_2$. With this order E is a directed set. Define $T : E \to D$ by $T(d, G) = d$. For $d_0 \in D$ $(d, G) \geq (d_0, X)$ implies $T(d, G) \geq_D d_0$ and so T has the subnet property, i.e., $S \circ T$ is a subnet of S. $S \circ T$ is a universal net as follows: Let $A \in \hat{X}$. Since $\mathcal{G}$ is an ultrafilter, either $A \in \mathcal{G}$ or $X - A \in \mathcal{G}$. If $A \in \mathcal{G}$, pick d_0 such $S(d_0) \in A$. For $(d, G) \geq (d_0, A)$, $S(d) \in G \subseteq A$. Thus $S \circ T$ is eventually in A. Similarly, if $X - A \in \mathcal{G}$, $S \circ T$ is eventually in $X - A$.

Lemma R25.2.16 Let (X, τ) be a topological space. Then the following are equivalent:

i) (X, τ) is compact.
ii) Every universal net in X converges.
iii) Every net in X has a convergent subnet.
iv) Every net in X has a cluster point.

Proof: By R25.2.15 ii) implies iii). Assume iii), let $S : D \to X$ and let $S \circ T$, where $T : E \to D$, be a subnet of S converging to x_0. Let $x_0 \in O \in \tau$ and let $d_0 \in D$. There is e_0 in E such that $e \geq_E e_0$ implies $S \circ T(e) \in O$. There is $e \geq_E e_0$ such that $T(e) \geq_D d_0$. Then $S(T(e)) \in O$. Thus S is frequently in O and x_0 is a cluster point of S. Next assume iv). Let $\mathcal{C}$ be a non-empty collection of closed sets with the finite intersection property and let $\mathcal{F}$ be the set of all finite intersections of sets from $\mathcal{C}$. Let D be the non-empty set $\{(x, F) : F \in \mathcal{F}$ and $x \in F\}$. With ordering $(x_1, F_1) \geq (x_2, F_2)$ if and only if $F_1 \subseteq F_2$, D is a directed set. Define $S : D \to X$ by $S(x, F) = x$. Let x_0 be a cluster point of the net S, let $C \in \mathcal{C}$, and let $x_0 \in O \in \tau$. For any $x \in C$, $(x, C) \in D$ and, since S is frequently in O, there is $(y, F) \in D$ with $(y, F) \geq (x, C)$ and $S(y, F) \in O$. Since $y \in F \subseteq C$, $C \cap O \neq \emptyset$. Thus $x_0 \in c(C) = C$ and so $x_0 \in \cap\, \mathcal{C}$. Part i) follows. Finally, assume the space is compact and let $S : D \to X$ be a universal net. Deny ii). For each $x \in X$, there is $O_x \in \tau$ with $x \in O_x$ and S frequently (and so eventually by universality) in $X - O_x$. For any finite subcover, use the directed set property to see that S is eventually in $\emptyset$, a contradiction.

The next definition can be used to describe Kuratowski's topology of lower semi-continuity, which will not be used here.

Definition R25.2.17 Let X be a set and let $A \subseteq X$. A^- is defined to be the collection $\{B \in \hat{X} : A \cap B \neq \emptyset\}$.

Definition R25.2.18 Let (X, τ) be a topological space and let $S : D \to \hat{X}$ be a net. The set $\underline{lim}\ S$ is $\{x \in X : x \in O \in \tau \Rightarrow S$ is eventually in $O^-\}$ and $\overline{lim}\ S$ is $\{x \in X : x \in O \in \tau \Rightarrow S$ is frequently in $O^-\}$.

Lemma R25.2.19 Let (X, τ) be a topological space and let $S : D \to \hat{X}$ be a net. Then

i) $\underline{lim}\ S \subseteq \overline{lim}\ S$.

ii) Both $\underline{lim}\ S$ and $\overline{lim}\ S$ are closed.

Proof: Part i) holds since 'eventually' implies 'frequently.'. Now let $x \in c(\underline{lim}\ S)$ and let $O \in \tau$ with $x \in O$. Then there is $t \in O \cap \underline{lim}\ S$ and so S is eventually in O. Thus $x \in \underline{lim}\ S$. Similarly $\overline{lim}\ S$ is closed.

Lemma R25.2.20 Let (X, τ) be a compact topological space and let $S : D \to \hat{X}$ be a net. Let N be a neighborhood of the diagonal in $X \times X$. Then eventually $\underline{lim}\ S \subseteq N[S(d)]$.

Proof: Assume $\underline{lim}\ S \neq \emptyset$. For each $x \in \underline{lim}\ S$ there is $O(x) \in \tau$ such that $x \in O(x)$ and $O(x) \times O(x) \subseteq N$. The closed $\underline{lim}\ S$ is compact and so there are $x_1, \ldots, x_n \in \underline{lim}\ S$ such that $\underline{lim}\ S \subseteq \cup_{i=1}^n O(x_i)$. For each i, S is eventually in $O(x_i)^-$, i.e. there is $d_i \in D$ such that $d \geq d_i$ implies $S(d) \cap O(x_i) \neq \emptyset$. By the directed set property there is $d_0 \geq d_i$ for all i. Let $d \geq d_0$ and let x be in $\underline{lim}\ S$. There is j such that $x \in O(x_j)$. Since $d \geq d_j$, there is t in $S(d) \cap O(x_j)$. Then $(t, x) \in O(x_j) \times O(x_j)$ and so $x \in N[t] \subseteq N[S(d)]$. The conclusion follows.

Lemma R25.2.21 Let (X, τ) be a compact topological space and let $S : D \to \hat{X}$ be a net. Let N be an open neighborhood of the diagonal in $X \times X$. Then eventually $S(d) \subseteq N[\overline{lim}\ S]$.

Proof: Since N is open in $X \times X$, $N[\overline{lim}\ S]$ is open in X and its complement is closed and so compact. For each $x \notin N[\overline{lim}\ S]$, x is not in $\overline{lim}\ S$ and so there is $O(x)$ such that $x \in O(x)$ and it is false that S is frequently in $O(x)^-$, i.e., there is $d(x) \in D$ such that $d \geq d(x)$ implies $S(d) \cap O(x) = \emptyset$. By compactness there exist $x_1, \ldots, x_n$ such that $X - N[\overline{lim}\ S] \subseteq \cup_{i=1}^n O(x_i)$. Pick $d_0 \in D$ with $d_0 \geq d(x_i)$ for $i = 1, \ldots, n$. For $d \geq d_0$, $S(d) \cap O(x_i) = \emptyset$ and so $S(d) \subseteq N[\overline{lim}\ S]$, i.e. the conclusion holds.

Lemma R25.2.22 Let (X, τ) be a topological space and let $S : D \to \hat{X}$ be a universal net. Then $\underline{lim}\ S = \overline{lim}\ S$.

Proof: By R25.2.19i it is sufficient to show $\overline{lim}\ S \subseteq \underline{lim}\ S$. Let $x \in \overline{lim}\ S$ and let $O \in \tau$ with $x \in O$. Since S is universal, either S is eventually in O^- or S is eventually in $\hat{X} - O^-$. Suppose the latter. Then there is $d_0 \in D$ such that $d \geq d_0$ implies $S(d) \notin O^-$, i.e., $S(d) \cap O = \emptyset$. But this contradicts $x \in \overline{lim}\ S$. Thus S is eventually in O^-. By definition $x \in \underline{lim}\ S$.

Proposition R25.2.23 Let $(X, \mathcal{U})$ be a uniform space and assume $(X, \tau(\mathcal{U}))$ is compact. Then $(\hat{X}, \tau(\hat{\mathcal{U}}))$ is compact.

Proof: Let $S : D \to \hat{X}$ be a universal net. By R25.2.16 it is sufficient to show that S converges to $\overline{lim}\ S = \underline{lim}\ S$, which will be denoted L in this argument. A basic neighborhood of L has the form $H(N)[L]$, where N is in $\mathcal{U}$ and is open in $X \times X$, i.e., N is an open neighborhood of the diagonal. By R25.2.20 there is $d_1 \in D$ such that $d \geq d_1$ implies $L \subseteq N[S(d)]$. By R25.2.21 there is $d_2 \in D$ such that $d \geq d_2$ implies $N[S(d)] \subseteq L$. For $d_0 \in D$ with $d_0 \geq d_i$, $i = 1, 2$, $d \geq d_0$ implies $S(d)$ is in $H(N)[L]$, which yields the needed convergence.

Corollary R25.2.24 Let $(X,\mathcal{U})$ be a complete uniform space and assume $(X, \tau(\mathcal{U}))$ is compact. Then $(\hat{X},\hat{\mathcal{U}})$ is complete.

Proof: The uniformity of a compact space must be complete.

Compactifications

In this subsection, given a totally bounded and separated uniform space $(X,\mathcal{U})$, the compactification of a separated subspace of $(\hat{X},\hat{\mathcal{U}})$ is described.

Lemma R25.3.1 Let X and Y be sets and let $f : X \to Y$ be one-to-one. Then $\hat{f}$ is one-to-one.

Proof: Suppose $A, B \in \hat{X}$ with $\hat{f}(A) = \hat{f}(B)$, i.e., $f[A] = f[B]$. If $x \in A$, then there is $b \in B$ such that $f(x) = f(b)$. Since f is one-to-one, $x = b$ and so $x \in B$. Thus $A \subseteq B$ and similarly $B \subseteq A$, i.e., $A = B$ as needed.

Lemma R25.3.2 Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be uniform spaces and assume $f : X \to Y$ is one-to-one and uniformly open onto $f[X]$. Let $\mathcal{S} \subseteq \hat{X}$. Then $\hat{f}|_{\mathcal{S}}$ is uniformly open onto its image.

Proof: Let $U \in \mathcal{U}$. It is sufficient to show that $(\hat{f}|_{\mathcal{S}} \times \hat{f}|_{\mathcal{S}})[H(U)]$ is in the subspace uniformity from $\hat{\mathcal{V}}$ on $\hat{f}|_{\mathcal{S}}[\mathcal{S}]$. By the assumption of uniformly open for f there is $V \in \mathcal{V}$ with $(f[X] \times f[X]) \cap V \subseteq (f \times f)[U]$. Let $(\hat{f}|_{\mathcal{S}}(A), \hat{f}|_{\mathcal{S}}(B))$, i.e., $(f[A], f[B])$ with $A, B \in \mathcal{S}$, be in $(\hat{f}|_{\mathcal{S}}[\mathcal{S}] \times \hat{f}|_{\mathcal{S}}[\mathcal{S}]) \cap H(V)$. By definition $f[A] \subseteq V[f[B]]$ and $f[B] \subseteq V[f[A]]$, from which it easily follows that $A \subseteq U[B]$ and $B \subseteq U[A]$, i.e., $(A, B) \in H(U)$. Thus $(\hat{f}|_{\mathcal{S}}(A), \hat{f}|_{\mathcal{S}}(B))$ is in $(\hat{f}|_{\mathcal{S}} \times \hat{f}|_{\mathcal{S}})[H(U)]$. The latter, as a superset of an element in the subspace uniformity, is also in the subspace uniformity.

The last two lemmas will be applied in a context which begins with a separated, totally bounded uniform space $(X,\mathcal{U})$. R25.2.11 gives a criterion for $\mathcal{S} \subseteq \hat{X}$ to be a separated subspace of $(\hat{X},\hat{\mathcal{U}})$. (By R25.2.12 $\mathcal{S} = 2^X$, the most often studied hyperspace, is a separated subspace.)

Proposition R25.3.3 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space, let (Y, f) be in the compactification class corresponding to $\mathcal{U}$, and let $\mathcal{V}$ be the unique uniformity for the topology of Y. Let $\mathcal{S} \subseteq \hat{X}$ and assume $(\mathcal{S},\hat{\mathcal{U}}(\mathcal{S}))$ is separated. Let $c_{\hat{Y}}$ denote the closure in $(\hat{Y}, \tau(\hat{\mathcal{V}}))$. Then the triple $(c_{\hat{Y}}(\hat{f}|_{\mathcal{S}}[\mathcal{S}]), \hat{f}|_{\mathcal{S}}, \hat{\mathcal{V}}(c_{\hat{Y}}(\hat{f}|_{\mathcal{S}}[\mathcal{S}]))$ is a near compactification corresponding to $\hat{\mathcal{U}}(\mathcal{S})$.

Proof: Since a subspace of a totally bounded space is totally bounded, by R25.2.7 $(\mathcal{S},\hat{\mathcal{U}}(\mathcal{S}))$ is totally bounded as well as separated. The set $\hat{f}|_{\mathcal{S}}[\mathcal{S}]$ is dense in its closure. By R25.2.6 $\hat{f}$ is uniformly continuous and so the restriction $\hat{f}|_{\mathcal{S}}$ is also. This, R25.3.1, and R25.3.2 imply $\hat{f}|\mathcal{S}$ is a uniform embedding. By R25.2.23 $(\hat{Y}, \hat{\mathcal{V}})$ is totally bounded and complete, as is the closed subspace $(c_{\hat{Y}}(\hat{f}|_{\mathcal{S}}[\mathcal{S}]), \hat{\mathcal{V}}(c_{\hat{Y}}(\hat{f}|_{\mathcal{S}}[\mathcal{S}]))$. By definition R25.1.14 the conclusion holds.

Corollary R25.3.4 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space, let (Y, f) be in the compactification class corresponding to $\mathcal{U}$, and let $\mathcal{V}$ be the unique uniformity for the topology of Y. Let $\mathcal{S} \subseteq \hat{X}$ and assume $(\mathcal{S},\hat{\mathcal{U}}(\mathcal{S}))$ is separated. Let $c_{\hat{Y}}$ denote the closure in $(\hat{Y}, \tau(\hat{\mathcal{V}}))$. Let π denote the projection from $c_{\hat{Y}}(\hat{f}|_{\mathcal{S}}[\mathcal{S}])$ onto the quotient space $c_{\hat{Y}}(\hat{f}|_{\mathcal{S}}[\mathcal{S}])/C(c_{\hat{Y}}(\hat{f}|_{\mathcal{S}}[\mathcal{S}]))$. Then $(c_{\hat{Y}}(\hat{f}|_{\mathcal{S}}[\mathcal{S}])/C(c_{\hat{Y}}(\hat{f}|_{\mathcal{S}}[\mathcal{S}])), \pi \circ \hat{f}|_{\mathcal{S}})$ is in the compactification class corresponding to $\hat{\mathcal{U}}(\mathcal{S})$.

Proof: This follows from the previous proposition and R25.1.15.

It might be asked if the quotient space in the last corollary is necessary, i.e., does the equivalence relation identify distinct points? The next few results show that, in the cases of most interest, the quotient is required.

Lemma R25.3.5 Let $(X,\mathcal{U})$ be a uniform space and let $A \in \hat{X}$. Then $c_{\hat{X}}(\{A\}) = \{B \in \hat{X} : c_X(B) = c_X(A)\}$.

Proof: By R25.2.8 $c_{\hat{X}}(\{A\}) = \cap\{H(U)[\{A\}] : U \in \mathcal{U}\}$. Then $B \in c_{\hat{X}}(\{A\})$ if and only if $B \in H(U)[A]$ for every $U \in \mathcal{U}$, i.e., if and only if $B \subseteq \cap\{U[A] : U \in \mathcal{U}\}$ and $A \subseteq \cap\{U[B] : U \in \mathcal{U}\}$, i.e., $B \subseteq c_X(A)$ and $A \subseteq c_X(B)$. The conclusion follows easily.

Lemma R25.3.6 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let (Y, f) be in the compactification class corresponding to $\mathcal{U}$. Let $\mathcal{S} \subseteq \hat{X}$ and assume $(\mathcal{S},\hat{\mathcal{U}}(\mathcal{S}))$ is separated. Suppose for some $A \in \mathcal{S}$, $f[A]$ is not closed in Y. Then the equivalence relation $C(c_{\hat{Y}}(\hat{f}|_{\mathcal{S}}[\mathcal{S}]))$ is not equality.

Proof: Pick any A in $\mathcal{S}$ with $f[A]$ not closed in Y. Let B be the Y-closure of $f[A]$. By the previous lemma B is in $c_{\hat{Y}}(\{f[A]\})$, which, since $f[A] \in \hat{f}|_{\mathcal{S}}[\mathcal{S}]$, is contained in $c_{\hat{Y}}(\hat{f}|_{\mathcal{S}}[\mathcal{S}])$. Since the equivalence classes of $C(c_{\hat{Y}}(\hat{f}|_{\mathcal{S}}[\mathcal{S}]))$ are the closures of singletons, the class of $\{f[A]\}$ contains $B \neq f[A]$ and so $C(c_{\hat{Y}}(\hat{f}|_{\mathcal{S}}[\mathcal{S}]))$ is not equality.

Corollary R25.3.7 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let (Y, f) be in the compactification class corresponding to $\mathcal{U}$. Let $\mathcal{S} \subseteq \hat{X}$ and assume $(\mathcal{S},\hat{\mathcal{U}}(\mathcal{S}))$ is separated. Suppose, for some $A \in \mathcal{S}$, A is not compact in X. Then the equivalence relation $C(c_{\hat{Y}}(\hat{f}|_{\mathcal{S}}[\mathcal{S}]))$ is not equality.

Proof: Since f is a homeomorphism onto its image, $f[A]$ is not compact in $f[X]$ and so not in Y. Thus $f[A]$ is not closed in Y and the previous lemma applies.

Corollary R25.3.8 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let (Y, f) be in the compactification class corresponding to $\mathcal{U}$. Assume $(X, \tau(\mathcal{U}))$ is not compact. Then the equivalence relation $C(c_{\hat{Y}}(\hat{f}|_{2^X}[2^X]))$ is not equality.

Proof: By R25.2.12 $(2^X,\hat{\mathcal{U}}(2^X))$ is separated. Since the non-compact X is in 2^X, the previous lemma applies with $\mathcal{S} = 2^X$.

The next few results identify a simpler representation of the compactification in R25.3.4.

Lemma R25.3.9 Let $(Z, \mathcal{W})$ be a uniform space and let $\mathcal{P} \subseteq \hat{Z}$.
Then $c_{\hat{Z}}(\mathcal{P})/C(c_{\hat{Z}}(\mathcal{P})$ is unimorphic to $c_{\hat{Z}}(\mathcal{P}) \cap 2^Z$ by the map $[A] \mapsto c_Z(A)$, where $[A]$ is the equivalence class of A.

Proof: Let $A \in c_{\hat{Z}}(\mathcal{P})$. Then $c_{\hat{Z}}(\{A\})$ is contained in $c_{\hat{Z}}(\mathcal{P})$ and so by R25.3.5 $c_Z(A)$ is in $c_{\hat{Z}}(\mathcal{P}) \cap 2^Z$. In addition, R25.1.5i and R25.3.5 show that the rule described is singlevalued. Thus the rule defines a function. Call it h. If $h([A]) = h([B])$, i.e., $c_Z(A) = c_Z(B)$, again by R23.1.5i and R25.3.5 $[A] = [B]$ and so h is one-to-one. If $F \in c_{\hat{Z}}(\mathcal{P}) \cap 2^Z$, then $h([F]) = F$ and so h is onto. Now let π denote the projection from $c_{\hat{Z}}(\mathcal{P})$ onto $c_{\hat{Z}}(\mathcal{P})/C(c_{\hat{Z}}(\mathcal{P}))$. Note that $h \circ \pi(A) = c_Z(A)$. First the uniform openness of h will be verified. By R25.1.8, a basic entourage in the quotient uniformity on $c_{\hat{Z}}(\mathcal{P})/C(c_{\hat{Z}}(\mathcal{P})$ has the form $(\pi \times \pi)[H(W) \cap (c_{\hat{Z}}(\mathcal{P}) \times c_{\hat{Z}}(\mathcal{P}))]$ for some $W \in \mathcal{W}$. Let $U(W)$ denote the set $H(W) \cap ((c_{\hat{Z}}(\mathcal{P}) \cap 2^Z) \times (c_{\hat{Z}}(\mathcal{P}) \cap 2^Z))$, a basic entourage in the subspace uniformity on $c_{\hat{Z}}(\mathcal{P}) \cap 2^Z$. Let (A, B) be in $U(W)$ and note that $H(W) \cap (c_{\hat{Z}}(\mathcal{P}) \times c_{\hat{Z}}(\mathcal{P}))$ also contains (A, B). Then $(h \times h)(\pi \times \pi)(A, B) = (c_Z(A), c_Z(B)) = (A, B)$ so that $U(W)$ is contained in $(h \times h)[(\pi \times \pi)[H(W) \cap (c_{\hat{Z}}(\mathcal{P}) \times c_{\hat{Z}}(\mathcal{P})]]$. Thus h is uniformly open as required. Lastly, the uniform continuity of h will be checked. Let $W \in \mathcal{W}$ and pick $W_1 \in \mathcal{W}$ such that $W_1 = W_1^{-1}$ and $W_1 \circ W_1 \subseteq W$. With $U(W)$ as above, it is claimed that $(\pi \times \pi)[H(W_1) \cap (c_{\hat{Z}}(\mathcal{P}) \times c_{\hat{Z}}(\mathcal{P})]$ is contained in $(h \times h)^{-1}[U(W)]$. Let $(\pi(A), \pi(B))$ be in $(\pi \times \pi)[H(W_1) \cap (c_{\hat{Z}}(\mathcal{P}) \times c_{\hat{Z}}(\mathcal{P})]$, where $(A, B) \in H(W_1) \cap (c_{\hat{Z}}(\mathcal{P}) \times c_{\hat{Z}}(\mathcal{P})$. From above, $(h \times h)(\pi(A), \pi(B)) = (c_Z(A), c_Z(B))$. Then $c_Z(A) \subseteq W_1[A] \subseteq W_1[W_1[B]] \subseteq W[B]$, with the last contained in $W[c_Z(B)]$, and similarly $c_Z(B) \subseteq W[c_Z(A)]$. Thus $(c_Z(A), c_Z(B))$ is in $U(W)$ and so h is uniformly continuous.

The next lemma is included for clarity, although it might be called obvious and unnecessary.

Lemma R25.3.10 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let (Y, f) be in the compactification class corresponding to $\mathcal{U}$. Let $u : Y \to Z$ be a unimorphism. Then (Y, f) and $(Z, u \circ f)$ are equivalent compactifications of $(X, \tau(\mathcal{U}))$.

Proof: As a unimorphism u is also a homeomorphism of the underlying topological spaces. This easily yields that $(Z, u \circ f)$ is a compactification. The map u itself is the homeomorphism needed to show equivalence.

Corollary R25.3.11 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let (Y, f) be in the compactification class corresponding to $\mathcal{U}$. Let $\mathcal{S} \subseteq \hat{X}$ and assume $(\mathcal{S},\hat{\mathcal{U}}(\mathcal{S}))$ is separated. Define g on $\mathcal{S}$ by $g(S) = c_Y(f[S])$. Then $g : \mathcal{S} \to c_{\hat{Y}}(\hat{f}[\mathcal{S}]) \cap 2^Y$ and $(c_{\hat{Y}}(\hat{f}[\mathcal{S}]) \cap 2^Y, g)$ is a T_2-compactification of $(\mathcal{S}, \tau(\hat{\mathcal{U}}(\mathcal{S}))$ in the class corresponding to $\hat{\mathcal{U}}(\mathcal{S})$.

Proof: Let $S \in \mathcal{S}$. Then $f[S] = \hat{f}(S)$ is in $\hat{f}[\mathcal{S}]$ and so $c_{\hat{Y}}(\{f[S]\})$ is contained in $c_{\hat{Y}}(\hat{f}[\mathcal{S}])$. By R25.3.5, $c_Y(f[S])$ is in $c_{\hat{Y}}(\hat{f}[\mathcal{S}])$, which verifies the assertion about the image of g. By R25.3.4 $(c_{\hat{Y}}(\hat{f}|_{\mathcal{S}}[\mathcal{S}])/C(c_{\hat{Y}}(\hat{f}|_{\mathcal{S}}[\mathcal{S}])), \pi \circ \hat{f}|_{\mathcal{S}})$ is in the compactification class corresponding to $\hat{\mathcal{U}}(\mathcal{S})$, where π is the projection onto the quotient. Let $h : c_{\hat{Y}}(\hat{f}|_{\mathcal{S}}[\mathcal{S}])/C(c_{\hat{Y}}(\hat{f}|_{\mathcal{S}}[\mathcal{S}])) \to c_{\hat{Y}}(\hat{f}[\mathcal{S}]) \cap 2^Y$ by $[A] \mapsto c_Y(A)$. By the preceding two lemmas, h is a unimorphism and $(c_{\hat{Y}}(\hat{f}[\mathcal{S}]) \cap 2^Y, h \circ \pi \circ \hat{f}|_{\mathcal{S}})$ is also in the compactification class corresponding to $\hat{\mathcal{U}}(\mathcal{S})$. It remains to verify that $h \circ \pi \circ \hat{f}|_{\mathcal{S}} = g$. Let $S \in \mathcal{S}$. Then $h(\pi(\hat{f}(S))) = h(\pi(f[S]) = h([f[S]]) = c_Y(f[S]) = g(S)$, i.e., the equality holds.

The above presentation of R25.3.11 follows the path by which it was derived. The result also appears to be directly verifiable, i.e. without the use of the intermediate quotient space, by a fairly routine argument, although the many details have not been carefully checked by me.

This subsection closes with several examples relating to suprema, finite-point compactifications, and Stone-Čech compactifications. The common theme is that these may not carry over at the hyperspace level.

First is an example showing that suprema of uniformities may not be preserved when passing to a hyperspace. This is not surprising since there exist many uniformities for $\hat{X}$ not of the form $\hat{\mathcal{U}}$ for some uniformity $\mathcal{U}$ on X.

Example R25.3.12 Let X be $\mathbb{N}$ with the discrete topology, let $\mathcal{U}_{min}$ be the uniformity on X corresponding the class of the one-point compactification, let $E(n)$ be the equivalence relation equivalence mod n on X, and let $\mathcal{U}_n = \mathcal{U}_{min} \vee \mathcal{U}_{E(n)}$ where $\mathcal{U}_{E(n)}$ (as defined In R5.2.1) consists of all supersets of $E(n)$. Since $E(2) \cap E(3) = E(6)$, $\mathcal{U}_2 \vee \mathcal{U}_3 = \mathcal{U}_6$. Obviously, $\widehat{\mathcal{U}_2} \vee \widehat{\mathcal{U}_3}$ is a subset of $\widehat{\mathcal{U}_6}$. What follows shows that the containment is proper. From the description of $\mathcal{U}_{min}$ in [8] it is easy to see that a base for $\mathcal{U}_{min}$ consists of all entourages $S(F)$, where F is a finite subset of X and $S(F) = \{(x,x) : x \in F\} \cup ((X - F) \times (X - F))$. Since $E(6) \in \mathcal{U}_6$, it is sufficient to show that $H(E(6))$ is not in $\widehat{\mathcal{U}_2} \vee \widehat{\mathcal{U}_3}$. Suppose the contrary. Then X has finite subsets F_1 and F_2 such that $H(S(F_1) \cap E(2)) \cap H(S(F_2) \cap E(3) \subseteq H(E(6))$. Pick $x_1 \in X$ such that $x_1 > t$ for every $t \in F_1 \cup F_2$. Let $A = \{x_1, x_1 + 1\}$ and let $B = \{x_1 + 3, x_1 + 4\}$. By the choice of x_1, A and B are both subsets of $X - (F_1 \cup F_2)$. Since both contain an even and an odd, $(A, B) \in H(S(F_1) \cap E(2))$. Since $x_1 \equiv x_1 + 3 \bmod 3$ and $x_1 + 1 \equiv x_1 + 4 \bmod 3$, $(A, B) \in H(S(F_2) \cap E(3))$. Since $x_1 \not\equiv x_1 + 3 \bmod 6$ and $x_1 \not\equiv x_1 + 4 \bmod 6$, $x_1 \notin E(6)[B]$. Thus $(A, B) \notin H(E(6))$, a contradiction.

The previous example yields another showing that suprema of compactifications may not be preserved when passing to the hyperspace level. The following uses mixed suprema as in [11], since it can be shown that $\tau(\widehat{\mathcal{U}_2}) \neq \tau(\widehat{\mathcal{U}_3})$.

Example R25.3.13 Continue the notation for $X, \mathcal{U}_n$ from the previous example. Since $\tau(\mathcal{U}_n)$ is discrete, $\hat{X} = 2^X$ and $\widehat{\mathcal{U}_n}$ is separable by R25.2.12. Let (Y_n, f_n) be in the compactification class corresponding to $\mathcal{U}_n$. Since $\mathcal{U}_2 \vee \mathcal{U}_3 = \mathcal{U}_6$, by R13.1.7 (Y_6, f_6) is in the compactification class corresponding to the the supremum of the classes of (Y_2, f_2) and (Y_3, f_3). Let (Z_n, g_n) be in the compactification class corresponding to $\widehat{\mathcal{U}_n}$. R13.1.7 the supremum of $[(Z_2, g_2)]$ and $[(Z_3, g_3)]$ corresponds to $\widehat{\mathcal{U}_2} \vee \widehat{\mathcal{U}_3}$. Since the latter is not $\widehat{\mathcal{U}_6}$, by R13.1.3 the class of (Z_6, g_6) is not the supremum of $[(Z_2, g_2)]$ and $[(Z_3, g_3)]$.

Next is an example showing that finite-point compactification need not be preserved when passing to a hyperspace.

Example R25.3.14 Again let X be $\mathbb{N}$ with the discrete topology and let $\mathcal{U}_{min}$ be the uniformity on X corresponding the class of the one-point compactification. As before, $(\hat{X}, \widehat{\mathcal{U}_{min}})$ is separated. Let (Y, f) be a one-point compactification of X with $Y - f[X] = \{\alpha\}$. For each $n \in \mathbb{N}$ let $B_n = \{f(n), \alpha\}$. Clearly $B_n \in 2^Y$, $n \neq j$ implies $B_n \neq B_j$, and $B_n \notin \hat{f}[\hat{X}]$. To see that $B_n \in c_{\hat{Y}}(\hat{f}[\hat{X}])$, let V be in the unique uniformity for Y. There is O open in Y with $\alpha \in O$ and $O \times O \subseteq V$. Since Y is the one-point compactification of a discrete space, $Y - O$ is a compact, i.e. finite, subset of $f[X]$, which is open in $Y$and discrete. Thus $W = \{(y, y) : y \in Y - O\} \cup (O \times O)$ is in the unique uniformity for Y. Clearly $W \subseteq V$ and $H(W) \subseteq H(V)$. Since f is one-to-one, $(O \cap f[X]) \cup \{f(n)\} = f[A] = \hat{f}(A)$, where A is an infinite subset of X. Note that $\hat{f}(A) \subseteq O \cup B_n = W[B_n]$ and $B_n \subseteq O \cup \{f(n)\} = W[\hat{f}(A)]$. Thus $H(V)[B_n] \cap \hat{f}[\hat{X}] \neq \emptyset$

and so $B_n \in c_{\hat{Y}}(\hat{f}[\hat{X}])$ as claimed. Finally by R25.3.11 $g : \hat{X} \to c_{\hat{Y}}(\hat{f}[\hat{X}]) \cap 2^Y$, where $g(S) = c_Y(f[S])$, is the embedding for the compactification of interest, which is in the class corresponding to $\widehat{\mathcal{U}_{min}}$. B_n is not the image of g since α is not in the Y-closure of any finite subset of $f[X]$. Thus $|c_{\hat{Y}}(\hat{f}[\hat{X}]) \cap 2^Y - g[\hat{X}]| \geq \aleph_0$, i.e., the compactification is not a finite-point compactification.

Before presenting an example related to Stone-Čech compactification, a few lemmas are necessary. These are versions of well-known facts.

Lemma R25.3.15 Let $(X, \mathcal{U})$ be a uniform space and let $O \in \tau(\mathcal{U})$. Then O^- is in $\tau(\hat{\mathcal{U}})$.

Proof: Let $A \in O^-$ and let $x \in A \cap O$. There is $U = U^{-1}$ in $\mathcal{U}$ with $U[x] \subseteq O$. It will be shown that $H(U)[A] \subseteq O^-$. Let $B \in H(U)[A]$. Since $A \subseteq U[B]$, there is $b \in B$ such that $(b, x) \in U$. Then $(x, b) \in U$ and so $b \in U[x]$, i.e., $b \in B \cap O$, i.e., $B \in O^-$. By definition of the topology generated by a uniformity, $O^- \in \tau(\hat{\mathcal{U}})$.

The next definition is related to Kuratowski's topology of upper semi-continuity.

Definition R25.3.16 Let X be a set and let $A \subseteq X$. $A^+ = \{B : B \subseteq A\}$.

Lemma R25.3.17 Let X be a set and let $\mathcal{U}_M$ be the largest totally bounded uniformity for the discrete topology on X. Let $A \subseteq X$. Then $A^+ \in \tau(\widehat{\mathcal{U}_M})$.

Proof: As noted in R6.3.4, $\mathcal{U}_M$ is the collection of all supersets of unions of the form $\cup\{O_i \times O_i : i \in F\}$, where F is finite, each O_i is open, and $\cup\{O_i : i \in F\} = X$. Since A is clopen, $U = (A \times A) \cup ((X - A) \times (X - A))$ is in $\mathcal{U}_M$. Let $B \in A^+$. If $B = \emptyset$, $H(U)[B] = \{\emptyset\} \subseteq A^+$. If $B \neq \emptyset$, since $B \subseteq A$, $U[B] = A$ and so $H(U)[B] \subseteq A^+$. Thus the conclusion follows.

The notation in the next definition appears in [5].

Definition R25.3.18 Let (X, τ) be a topological space and let $O_1, \ldots, O_j$ be in τ. $[O_1, \ldots, O_j]$ is the set $(\cap_{i=1}^j O_i^-) \cap (\cup_{i=1}^j O_i)^+$.

Lemma R25.3.19 Let (X, τ) be a topological space. The collection containing $\{\emptyset\}$ and $[O_1, \ldots, O_j]$ for all finite collections of open sets is a basis for a topology on $\hat{X}$.

Proof: Since $\{\emptyset\}$ is in the collection and $[X] = \hat{X} - \{\emptyset\}$, the union of the collection is $\hat{X}$. Given $[O_1, \ldots, O_j]$ and $[G_1, \ldots, G_k]$, let $O = \cup_{i=1}^j O_i$ and $G = \cup_{i=1}^k G_i$. It is easy to check that $[O_1, \ldots, O_j] \cap [G_1, \ldots, G_k] = [O_1 \cap G, \ldots, O_j \cap G, O \cap G_1, \ldots, O \cap G_k]$. The conclusion now follows.

Let the topology generated on $\hat{X}$ by the basis of the last lemma be denoted τ_v. It is a version of the Vietoris topology.

Lemma R25.3.20 Let X be a set and let $\mathcal{U}$ be a totally bounded uniformity on X. Then $\tau(\hat{\mathcal{U}}) \subseteq (\tau(\mathcal{U}))_v$.

Proof: Let $A \in O \in \tau(\hat{\mathcal{U}})$. There is $U \in \mathcal{U}$ with $H(U)[A] \subseteq O$. If A is empty, $H(U)[A] = \{\emptyset\}$, which in the basis for $(\tau(\mathcal{U}))_v$. Otherwise, pick $V = V^{-1}$ in $\mathcal{U}$ such that $V \circ V \subseteq U$ and V is open in $X \times X$. By total boundedness, there are $x_1, \ldots, x_n$ in X such that $X = \cup_{i=1}^n V[x_i]$. That $\cup_{i-1}^n (V[x_i] \times V[x_i]) \subseteq U$ can be easily checked. Let $O_i = V[x_i]$, which is in $\tau(\mathcal{U})$ since V is open. Let $\Delta_A = \{i : A \cap O_i \neq \emptyset\}$. It is claimed that $A \in [\{O_i : i \in \Delta_A\}] \subseteq H(U)[A]$. Clearly $A \in O_i^-$ for each i in Δ_A. Since $X = \cup_{i=1}^n O_i$, $A \in (\cup\{O_i : i \in \Delta_A\})^+$. Now let $B \in [\{O_i : i \in \Delta_A\}]$. Then $U[A] \supseteq \cup\{O_i : i \in \Delta_A\} \supseteq B$. Also, $B \cap O_i \neq \emptyset$ for every $i \in \Delta_A$ implies $U[B] \supseteq \cup\{O_i : i \in \Delta_A\} \supseteq A$. Thus the claim is verified and the conclusion follows.

Lemma R25.3.21 Let X be a set and let $\mathcal{U}_M$ be the largest totally bounded uniformity for the discrete topology on X. Then $(\tau(\mathcal{U}_M))_v = \tau(\widehat{\mathcal{U}_M})$.

Proof: By R25.3.15 and R25.3.17, given $O_1, \ldots, O_j$ in $\tau(\mathcal{U}_M)$, $[O_1, \ldots, O_j]$ is in $\tau(\widehat{\mathcal{U}_M})$, as is $\{\emptyset\} = H(X \times X)[\emptyset]$. Thus $(\tau(\mathcal{U}_M))_v \subseteq \tau(\widehat{\mathcal{U}_M})$. The reverse containment follows from R25.3.20.

Now an example can be given showing that a uniformity corresponding to the Stone-Čech compactification may generate a hyper-uniformity which does not correspond to the Stone-Čech compactification.

Example R25.3.22 Let X be an infinite set and let $\mathcal{U}_M$ be the largest totally bounded uniformity for the discrete topology on X. As noted in [8], $\mathcal{U}_M$ corresponds to the Stone-Čech compactification of $(X, \tau(\mathcal{U}_M))$. Let $O_1 = \cup\{[\{x\}] : x \in X\}$, an open set in $(\tau(\mathcal{U}_M))_v = \tau(\widehat{\mathcal{U}_M})$. Let $O_2 = \hat{X} - O_1$. First, it will be shown that O_2 is also open. Let B be in O_2, and note that, for $x \in X$, $[\{x\}] = \{x\}^- \cap \{x\}^+ = \{\{x\}\}$. Thus, if $B \neq \emptyset$, $|B| \geq 2$ so that there exist $a, b \in B$ with $a \neq b$. Clearly, B is in the open set $[X - \{a\}, \{a, b\}, X - \{b\}]$. Now let A be in $[X - \{a\}, \{a, b\}, X - \{b\}]$. Since $A \in \{a, b\}^-$, at least one of a, b is in A. If $a \in A$, since $A \in (X - \{a\})^-$, $|A| \geq 2$ and $A \notin O_1$, i.e., $A \in O_2$. Similarly, if $b \in A$, $A \in O_2$ and so O_2 is a $(\tau(\mathcal{U}_M))_v$-neighborhood of B. If $B = \emptyset$, since $\{\emptyset\}$ is in the basis, again O_2 is a $(\tau(\mathcal{U}_M))_v$-neighborhood of B. Since O_2 is a neighborhood of all its points, O_2 is in $(\tau(\mathcal{U}_M))_v = \tau(\widehat{\mathcal{U}_M})$. Now let $E = (O_1 \times O_1) \cup (O_2 \times O_2)$, an equivalence relation on $\hat{X}$. For the uniformity $\mathcal{U}_E$ generated by E, (as in [9]), $\tau(\mathcal{U}_E) = \{\emptyset, O_1, O_2, \hat{X}\}$ and so the topology of the totally bounded uniformity $\widehat{\mathcal{U}_M} \vee \mathcal{U}_E$ equals $\tau(\widehat{\mathcal{U}_M})$. Thus E is in the largest totally bounded uniformity for $\hat{X}$ generating $\tau(\widehat{\mathcal{U}_M})$. Next it will be shown that $E \notin \widehat{\mathcal{U}_M}$. Deny this. Then there is $U \in \mathcal{U}_M$ with $H(U) \subseteq E$. Without loss of generality assume $U = \cup_{i=1}^n G_i \times G_i$, where $G_1, \ldots, G_n$ is an irreducible cover of X and G_1 is infinite. Then there is $x \in G_1$ with $x \notin \cup_{i=2}^n G_i$. It is easy to check that $(\{x\}, G_1) \in H(U)$. But, since $\{x\} \in O_1$ and $G_1 \in O_2$, $(\{x\}, G_1) \notin E$, a contradiction. Thus $\widehat{\mathcal{U}_M}$ is not the largest totally bounded uniformity generating $\tau(\widehat{\mathcal{U}_M})$, i.e., $\widehat{\mathcal{U}_M}$ does not correspond to the Stone-Čech compactification of $(\hat{X}, \tau(\widehat{\mathcal{U}_M}))$.

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