

## The Remnant Rings Are Homeomorphic

In [6] the remnant rings  $\mathbf{R}_k$  (where  $k \geq 2$  is in  $\mathbf{N}$ ) were shown to be compactifications of the integers  $\mathbf{Z}$  with certain non-discrete topologies, which vary with  $k$ . It will first be shown here that, for relatively prime  $a, b$ ,  $\mathbf{R}_a$  and  $\mathbf{R}_b$  are not equivalent compactifications in the generalized sense of [5].

The main result is that the topological space  $\mathbf{R}_2$  is homeomorphic to  $\mathbf{R}_k$  for all  $k \geq 2$  in  $\mathbf{N}$ . Since, for  $p$  prime,  $\mathbf{R}_p$  is topologically isomorphic to the  $p$ -adic integers by R17.1.19, this is a slight generalization of the known result that, for  $p, q$  prime, the  $p$ -adic integers are homeomorphic to the  $q$ -adic integers. The proof of the main result is based on the presentation of that fact in exercises 12 and 13, p. 65, of [1].

The remnant rings are also shown to be homeomorphic to the Cantor set and some results are derived on the question of when  $\mathbf{N}_j$  and  $\mathbf{N}_k$  are homeomorphic topological spaces.

The first proposition makes use of the following notation from [6]: Let  $k \geq 2$  be in  $\mathbf{N}$ . For  $n \in \mathbf{N}$  and  $x \in \mathbf{Z}$ ,  $D_n^x(k)$  is the equivalence class in  $\mathbf{Z}$  of  $x \bmod k^n$ . In [6] it is shown that  $\{D_n^x(k) : n \in \mathbf{N} \text{ and } x \in \mathbf{Z}\}$  is a clopen basis for a topology  $\tau_k$  on  $\mathbf{Z}$  and that  $\mathbf{R}_k$  with a suitable embedding is a compactification of  $(\mathbf{Z}, \tau_k)$ .

**Proposition R26.1** Let  $a, b \geq 2$  be in  $\mathbf{N}$  with  $(a, b) = 1$ . Then  $\mathbf{R}_a \not\cong \mathbf{R}_b$ .

Proof: By R13.1.5i it is sufficient to show that  $\tau_a \not\subseteq \tau_b$ , which will be done by showing  $D_1^1(a)$  is not in  $\tau_b$ . Deny. Then there are  $n \in \mathbf{N}$  and  $x \in \mathbf{Z}$  such that  $1 \in D_n^x(b) \subseteq D_1^1(a)$ . Since these are equivalence classes,  $D_n^x(b) = D_n^1(b) \subseteq D_1^1(a)$ . Then  $1 + b^n \equiv 1 \pmod{a}$  so that  $b^n \equiv 0 \pmod{a}$ , i.e.,  $a$  divides  $b^n$ , which contradicts the hypothesis that  $a$  and  $b$  are relatively prime.

Comment: In the last proposition the roles of  $a$  and  $b$  could be reversed thereby showing that the compactifications  $\mathbf{R}_a$  and  $\mathbf{R}_b$  are not related in either direction and, of course, are not equivalent. In R10.3.5 it was shown that, for two distinct primes  $p$  and  $q$ ,  $\mathbf{N}_p \not\cong \mathbf{N}_q$ , a fact which could easily be generalized to a relatively prime pair. The proof of R10.3.5 could be modified to verify R26.1, but the approach used in R26.1 can't be used to prove R10.3.5, since  $\mathbf{N}_p$  and  $\mathbf{N}_q$  are both compactifications of  $\mathbf{N}$  with the discrete topology.

Before presenting the main result, some needed properties of the derived sequence of an element in  $\mathbf{R}_k$  for  $k \geq 2$  will be developed. Recall that each element of  $\mathbf{R}_k$  determines two related but distinct sequences: the derived sequence (defined in R20.1) of the form  $\{a_n\}_{n=0}^\infty$ , where  $a_n \in \{0, \dots, k-1\}$  for all  $n$ , and the associated sequence (defined in R10.2.3) of the form  $\{x_n\}_{n=1}^\infty$ , where  $x_n \in \{1, \dots, k^n\}$  for all  $n$ .

**Lemma R26.2** Let  $\{a_n\}_{n=0}^\infty$  be a sequence with each  $a_n$  in  $\{0, \dots, k-1\}$ , where  $k \geq 2$  is in  $\mathbf{N}$ . Then there is  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$  in  $\mathbf{R}_k$  such that the derived sequence of  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$  is  $\{a_n\}_{n=0}^\infty$ .

Proof: First assume  $a_n = 0$  for every  $n$ . Let  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$  be the additive identity of  $\mathbf{R}_k$ , which has associated sequence  $\{k^n\}_{n=1}^\infty$ . By definition the derived sequence of  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$  is  $\{a_n\}_{n=0}^\infty$ .

Now assume there is  $n$  with  $a_n \neq 0$  and let  $m$  be the smallest of  $\{n : a_n \neq 0\}$ . If  $m = 0$ , let  $x_1 = a_0$  and let  $x_{n+1} = x_n + a_n k^n$  for  $n \geq 1$ . By R10.2.6 there is a unique  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$  in  $\mathbf{R}_k$  which is associated with  $\{x_n\}_{n=1}^\infty$ . Let  $\{b_n\}_{n=0}^\infty$  be the derived

sequence of  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$ . Since  $x_1 = a_0 \neq k$ , by definition  $b_0 = x_1 = a_0$  and, for  $n \geq 1$ ,  $b_n = (x_{n+1} - x_n)/k^n = a_n$ . Thus the derived sequence of  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$  is  $\{a_n\}_{n=0}^\infty$ .

As a final case assume  $m \geq 1$ . Let  $x_i = k^i$  for  $1 \leq i \leq m$ , let  $x_{m+1} = k^m a_m$ , and let  $x_{i+1} = x_i + a_i k^i$  for  $i \geq m+1$ . Note that, for  $1 \leq i \leq m-1$ ,  $x_{i+1} = k^i + (k-1)k^i = x_i + (k-1)k^i$ . Also, since  $a_m \in \{1, \dots, k-1\}$ ,  $x_{m+1} = k^m + (a_m - 1)k^m = x_m + (a_m - 1)k^m$ . Thus R10.2.6 applies: there is  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$  in  $\mathbf{R}_k$  such that  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$  is associated with  $\{x_n\}_{n=1}^\infty$ . Let  $\{b_n\}_{n=0}^\infty$  be the derived sequence of  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$ . Note that the smallest of  $\{n : x_n \neq k^n\}$  is  $m+1$  and so  $b_i = 0 = a_i$  for  $0 \leq i \leq m-1$ . Also by definition,  $b_m = x_{m+1}/k^m = a_m$ . Finally, for  $i \geq m+1$ , by definition  $b_i = (x_{i+1} - x_i)/k^i = a_i$ . Thus  $b_n = a_n$  for all  $n$ , i.e., the derived sequence of  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$  is  $\{a_n\}_{n=0}^\infty$ .

**Lemma R26.3** Let  $k \geq 2$  in  $\mathbf{N}$ , let  $\mathcal{F}$  be in  $\mathbf{R}_k$ , and let  $\{a_n\}_{n=0}^\infty$  be the derived sequence of  $\mathcal{F}$ . Then  $\mathcal{F}(\{a_n\}_{n=0}^\infty) = \mathcal{F}$ .

Proof: Let  $\{y_n\}_{n=1}^\infty$  be the associated sequence of  $\mathcal{F}$ . If  $y_n = k^n$  for every  $n$ , then  $\mathcal{F}$  is the additive identity of  $\mathbf{R}_k$  as shown in [4], especially R12.5.9. By definition  $a_n = 0$  for every  $n$  and by the construction in the previous lemma,  $\mathcal{F}(\{a_n\}_{n=0}^\infty) = \mathcal{F}$ . Now assume that  $y_n \neq k^n$  for some  $n$  and let  $l$  be the smallest of  $\{n : y_n \neq k^n\}$ . Let  $\{x_n\}_{n=1}^\infty$  be the sequence associated with  $\mathcal{F}(\{a_n\}_{n=0}^\infty)$ , as constructed in the proof of R26.2. If  $l = 1$ , by R20.1  $a_0 = y_1$  and  $a_n = (y_{n+1} - y_n)/k^n$  for  $n \geq 1$ . Since  $y_1 \neq 0$ , by construction,  $x_1 = a_0 = y_1$  and  $x_{n+1} = x_n + a_n k^n$  for all  $n \geq 1$ . Induction and routine algebra show that  $x_{n+1} = y_{n+1}$  for  $n \geq 1$ . By R10.2.4  $\mathcal{F}(\{a_n\}_{n=0}^\infty) = \mathcal{F}$ . Finally, suppose  $l > 1$ . By R20.1  $a_0 = a_1 = \dots = a_{l-2} = 0$ ,  $a_{l-1} = y_l/k^{l-1}$ , and, for  $n \geq l$ ,  $a_n = (y_{n+1} - y_n)/k^n$ . The smallest of  $\{n : a_n \neq 0\}$  is  $l-1$ . By construction,  $x_n = k^n$  for  $1 \leq n \leq l-1$ ,  $x_l = k^{l-1} a_{l-1}$ , and, for  $n \geq l$ ,  $x_{n+1} = x_n + a_n k^n$ . By the choice of  $l$ ,  $y_n = k^n = x_n$  for  $1 \leq n \leq l-1$ . Also  $x_l = k^{l-1}(y_l/k^{l-1}) = y_l$ . For  $n \geq l$ , induction and routine algebra show  $x_{n+1} = y_{n+1}$ . Again by R10.2.4  $\mathcal{F}(\{a_n\}_{n=0}^\infty) = \mathcal{F}$ .

**Corollary R26.4** Let  $k \geq 2$  in  $\mathbf{N}$  and let  $\mathcal{F}$  and  $\mathcal{G}$  be in  $\mathbf{R}_k$ . Let  $\{a_n\}_{n=0}^\infty$  and  $\{b_n\}_{n=0}^\infty$  be the derived sequences of  $\mathcal{F}$  and  $\mathcal{G}$  respectively. Then  $\mathcal{F} = \mathcal{G}$  if and only if  $a_n = b_n$  for all  $n \geq 0$ .

Proof: If  $\mathcal{F} = \mathcal{G}$ , each has the same associated sequence and so the same derived sequence. If  $a_n = b_n$  for every  $n$ , then  $\mathcal{F}(\{a_n\}_{n=0}^\infty) = \mathcal{F}(\{b_n\}_{n=0}^\infty)$  and so By R26.3  $\mathcal{F} = \mathcal{G}$ .

Next some terminology and notation used in Robert's hint will be presented: A word is a finite string with characters from  $\{0, 1\}$  or, formally, a function from an initial segment of  $\mathbf{N}$  into  $\{0, 1\}$ . Note that the function from  $\emptyset$  is allowed. It will be called the null word or  $\eta$ . Each word has a length, which is its number of characters, i.e., the number of natural numbers in its domain. The length of  $\eta$  is 0. With the operation of concatenation,  $M_2$ , the set of all words, is a non-commutative semi-group with unit  $\eta$ , i.e., a monoid. An infinite word is a sequence in  $\{0, 1\}$ . If  $w$  denotes the infinite word  $\{t_i\}_{i=1}^\infty$ ,  $w_0 = \eta$  and  $w_n$  is the word  $\{t_i\}_{i=1}^n$ . For an infinite word  $w$ ,  $w \notin M_2$  but  $w_n \in M_2$  for all  $n$  and the length of  $w_n$  is  $n$ . Usually string notation will be used instead of the more formal functional representation, e.g., 101 will denote the word  $\{t_i\}_{i=1}^3$  with  $t_1 = 1$ ,  $t_2 = 0$ , and  $t_3 = 1$ .

**Definition R26.5** Define  $d$  from  $\mathbf{R}_2$  to the set of infinite words by  $d(\mathcal{F}) = \{a_n\}_{n=0}^\infty$ , where  $\{a_n\}_{n=0}^\infty$  is the derived sequence of  $\mathcal{F}$ .

Note: If one uses string notation, because the derived sequence indexing begins at 0,  $d(\mathcal{F})_0 = \eta$ ,  $d(\mathcal{F})_1 = a_0$ , and  $d(\mathcal{F})_{n+1} = d(\mathcal{F})_n a_n$ .

**Lemma R26.6** The map  $d$  is a bijection.

Proof: Each  $\mathcal{F}$  in  $\mathbf{R}_2$  has a unique associated sequence and so a unique derived sequence. Thus  $d$  is indeed a function. R26.2 shows that  $d$  is onto and R26.4 shows that  $d$  is one-to-one.

**Lemma R26.7** Let  $k \geq 2$  and  $\mathcal{F}, \mathcal{G}$  be in  $\mathbf{R}_k$  with associated sequences  $\{x_n\}_{n=1}^{\infty}$  and  $\{y_n\}_{n=1}^{\infty}$  respectively. If  $x_m = y_m$  for some  $m$ , then  $x_j = y_j$  for  $1 \leq j \leq m$ .

Proof: By induction: The conclusion clearly holds for  $m = 1$ . Assume it holds for  $m$  and suppose  $x_{m+1} = y_{m+1}$ . By R10.2.5i  $x_{m+1} = x_m + sk^m$  and  $y_{m+1} = y_m + tk^m$ , where  $s, t \in \{0, 1, \dots, k-1\}$ . Then  $x_m - y_m = (t-s)k^m$ , i.e., if  $s \neq t$ ,  $k^m$  divides  $x_m - y_m$ . Since  $1 \leq x_m, y_m \leq k^m$ ,  $|x_m - y_m| \leq k^m - 1$  and so  $s = t$ , i.e.,  $x_m = y_m$ . By the induction hypothesis, the conclusion holds for  $m+1$ .

**Lemma R26.8** Let  $\mathcal{F}, \mathcal{G}$  be in  $\mathbf{R}_2$  with associated sequences  $\{x_n\}_{n=1}^{\infty}$  and  $\{y_n\}_{n=1}^{\infty}$  respectively. If  $x_m = y_m$  for some  $m$ , then  $d(\mathcal{F})_j = d(\mathcal{G})_j$  for  $0 \leq j \leq m$ .

Proof: Let  $\{a_n\}_{n=0}^{\infty}$  be the derived sequence of  $\mathcal{F}$  and let  $\{b_n\}_{n=0}^{\infty}$  be the derived sequence of  $\mathcal{G}$ . By the definition of  $d$ , it is sufficient to show  $a_n = b_n$  for  $0 \leq n \leq m-1$ . As a first case, suppose  $x_m = k^m$ . Since the additive identity for  $\mathbf{R}_2$  has associated sequence  $\{k^n\}_{n=1}^{\infty}$ , by R26.7  $x_n = k^n = y_n$  for  $1 \leq n \leq m$ . By definition of the derived sequence  $a_n = b_n = 0$  for  $0 \leq n \leq m-1$ . Now suppose  $x_m \neq k^m$  and let  $l$  be the smallest of  $\{n : x_n \neq k^n\}$ . Clearly  $l \leq m$ . Since  $x_m = y_m$ , by R26.7  $l$  is also the smallest of  $\{n : y_n \neq k^n\}$ . If  $l = 1$ , by definition  $a_0 = x_1 = y_1 = b_0$  and, for  $n \geq 1$ ,  $a_n = (x_{n+1} - x_n)/k^n$  and  $b_n = (y_{n+1} - y_n)/k^n$ . Since  $x_n = y_n$  for  $1 \leq n \leq m$ ,  $a_n = b_n$  for  $0 \leq n \leq m-1$ . If  $l > 1$ , by definition  $a_n = 0 = b_n$  for  $0 \leq n \leq l-2$ ,  $a_{l-1} = x_l/k^{l-1} = y_l/k^{l-1} = b_{l-1}$ , and, for  $n \geq l$ ,  $a_n = (x_{n+1} - x_n)/k^n$  and  $b_n = (y_{n+1} - y_n)/k^n$ . For  $0 \leq n \leq m-1$ , it follows that  $a_n = b_n$ .

**Lemma R26.9** Let  $\{\mathcal{F}_i\}$  be a sequence in  $\mathbf{R}_2$  and let  $\mathcal{F}$  be in  $\mathbf{R}_2$ . Assume  $\mathcal{F}_i \rightarrow \mathcal{F}$ . Then for every  $m \in \mathbf{N}$  there is  $N(m)$  in  $\mathbf{N}$  such that  $i \geq N(m)$  implies  $d(\mathcal{F}_i)_j = d(\mathcal{F})_j$  for  $0 \leq j \leq m$ .

Proof: Fix  $m$  in  $\mathbf{N}$ , let  $\{x_n\}_{n=1}^{\infty}$  be the associated sequence of  $\mathcal{F}_i$ , and let  $\{x_n\}_{n=1}^{\infty}$  be the associated sequence of  $\mathcal{F}$ . By R17.2.16 There is  $N(m)$  such that  $i \geq N(m)$  implies  $x_m = x_m$ . When  $j \geq N(m)$ , R26.8 implies  $d(\mathcal{F}_i)_j = d(\mathcal{F})_j$  for  $0 \leq j \leq m$ .

**Definition R26.10** An auxiliary function for a topological space  $(X, \tau)$  is a map  $\phi : M_2 \rightarrow \mathcal{P}(X)$  such that, for every  $w \in M_2$ ,  $\phi(w)$  is closed,  $\phi(\eta) = X$  and, for every  $w \in M_2$ ,  $\phi(w) = \phi(w0) \cup \phi(w1)$ .

**Lemma R26.11** Let  $\phi$  be an auxiliary function for the topological space  $(X, \tau)$  and let  $w$  be an infinite word. Then  $\phi(w_{n+1}) \subseteq \phi(w_n)$  for every  $n \geq 0$ .

Proof; Since  $w_{n+1}$  is either  $w_n0$  or  $w_n1$  and  $\phi(w_n) = \phi(w_n0) \cup \phi(w_n1)$ , the conclusion clearly holds.

Recall that the diameter of a non-empty set in a metric space is the supremum of distances of points in the set. Of course, even for equivalent metrics, the diameter of a given set is not invariant. As an extreme example, in the interval  $(0, 1)$ , the subinterval  $(0, 1/n)$  has diameter  $1/n$  with the usual absolute value metric but, with the equivalent metric  $\rho(x, y) = |1/x - 1/y|$ , it has infinite diameter. The next lemma (undoubtedly known) could be omitted but shows that the subsequent proposition does not depend on a specific metric.

**Lemma R26.12** Let  $\rho$  and  $\sigma$  be equivalent metrics on a set  $X$  and let  $\{A_n\}$  be a sequence of non-empty subsets of  $X$ . Assume  $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$ . If  $\text{diam}_{\rho}(A_n) \rightarrow 0$  as  $n \rightarrow \infty$ , then  $\text{diam}_{\sigma}(A_n) \rightarrow 0$  as  $n \rightarrow \infty$ .

Proof: Assume  $\text{diam}_{\rho}(A_n) \rightarrow 0$  as  $n \rightarrow \infty$  and let  $a \in \bigcap_{n=1}^{\infty} A_n$ . Let  $\epsilon > 0$ . By definition equivalent metrics generate the same topology and so  $B_{\epsilon}^{\sigma}(a)$  (the open  $\sigma$ -ball of radius  $\epsilon$  centered at  $a$ ) must contain  $B_{\gamma}^{\rho}(a)$  for some  $\gamma > 0$ . By hypothesis there is  $M$  such that, for  $n \geq M$ ,  $\text{diam}_{\rho}(A_n) < \gamma$  and so  $A_n \subseteq B_{\gamma}^{\rho}(a)$ . Thus  $n \geq M$  implies  $\text{diam}_{\sigma}(A_n) \leq 2\epsilon$ .

The next proposition is a version of problem 12 in [1: p.65].

**Proposition R26.13** Let  $\phi$  be an auxiliary function for a compact, metrizable space  $(E, \tau)$ . Assume that, for every word  $\nu$ ,  $\phi(\nu) \neq \emptyset$  and that, for every infinite word  $w$ ,  $\text{diam}(\phi(w_n)) \rightarrow 0$  as  $n \rightarrow \infty$ . Then there is a continuous surjection  $f : \mathbf{R}_2 \rightarrow E$ .

Proof: For any infinite word  $w$ , the collection of closed sets  $\{\phi(w_n) : n \geq 0\}$  has the finite intersection property by hypothesis and R26.11 and so, since  $E$  is compact,  $\bigcap_{n=0}^{\infty} \phi(w_n) \neq \emptyset$ . Suppose  $a \neq b$  are both in  $\bigcap_{n=0}^{\infty} \phi(w_n)$  and let  $\epsilon$  be smaller than the distance from  $a$  to  $b$ . Since  $\text{diam}(\phi(w_n)) \rightarrow 0$  (for any metric generating the topology of  $E$  by R26.12), there is  $n$  such that  $\text{diam}(\phi(w_n)) < \epsilon$ , which contradicts  $a, b \in \phi(w_n)$ . Thus  $\bigcap_{n=0}^{\infty} \phi(w_n)$  is a singleton and the map  $f$  can be defined by letting  $f(\mathcal{F})$  be the unique element of  $\bigcap_{n=0}^{\infty} \phi(d(\mathcal{F})_n)$  for each  $\mathcal{F}$  in  $\mathbf{R}_2$ . To see that  $f$  is onto, let  $e \in E$  and construct a sequence in  $\{0, 1\}$  as follows: Since  $e \in E = \phi(\eta) = \phi(0) \cup \phi(1)$ , pick  $t_1 \in \{0, 1\}$  such  $e \in \phi(t_1)$ . Now assume  $t_1, \dots, t_n$  chosen with  $e \in \phi(\{t_i\}_{i=1}^j)$  for all  $1 \leq j \leq n$ . Since  $e \in \phi(\{t_i\}_{i=1}^n) = \phi(\{t_i\}_{i=1}^n 0) \cup \phi(\{t_i\}_{i=1}^n 1)$ , pick  $t_{n+1}$  in  $\{0, 1\}$  such that  $e \in \phi(\{t_i\}_{i=1}^{n+1})$ . By induction we have an infinite word  $w = \{t_i\}_{i=1}^{\infty}$  such that  $e \in \phi(w_n)$  for every  $n$ . By R26.6 there is  $\mathcal{F}$  in  $\mathbf{R}_2$  such that  $d(\mathcal{F}) = w$ . By definition  $f(\mathcal{F}) = e$ . Lastly, for continuity, let  $\{\mathcal{F}_i\}$  be a sequence in  $\mathbf{R}_2$  and let  $\mathcal{F}$  be in  $\mathbf{R}_2$  with  $\mathcal{F}_i \rightarrow \mathcal{F}$ . It will be shown that  $f(\mathcal{F}_i) \rightarrow f(\mathcal{F})$ . Let  $\epsilon > 0$ . Since  $\text{diam}(\phi(d(\mathcal{F})_n)) \rightarrow 0$  as  $n \rightarrow \infty$  and  $f(\mathcal{F}) \in \phi(d(\mathcal{F})_n)$  for all  $n$ , there is  $m \in \mathbf{N}$  such that  $\phi(d(\mathcal{F})_m)$  is contained in  $B_{\epsilon}(f(\mathcal{F}))$ , the open  $\epsilon$ -ball centered at  $f(\mathcal{F})$ . By R26.9 there is  $N(m)$  in  $\mathbf{N}$  such that  $i \geq N(m)$  implies  $d(\mathcal{F}_i)_j = d(\mathcal{F})_j$  for  $0 \leq j \leq m$ . Then  $i \geq N(m)$  implies  $f(\mathcal{F}_i) \in \phi(d(\mathcal{F}_i)_m) = \phi(d(\mathcal{F})_m) \subseteq B_{\epsilon}(f(\mathcal{F}))$  and the claim is verified. Thus  $f$  is continuous.

The next lemma and R26.16 provide a way to construct auxiliary functions.

**Lemma R26.14** Let  $(X, \tau)$  be a topological space and let  $X = \bigcup_{i=1}^t F_i$ , where each  $F_i$  is a non-empty closed subset and  $t \in \mathbf{N}$ . Then there is a finite set  $D \subseteq M_2$  and  $\psi$  from  $D$  to the non-empty closed subsets of  $X$  such that

- i)  $\psi(\eta) = X$ .
- ii) If  $w \in D$  and  $v \in M_2$  with  $\text{length}(v) < \text{length}(w)$ , then  $v \in D$ .
- iii) If  $wv \in D$ , then  $\psi(wv) \subseteq \psi(w)$ .
- iv) There is  $j \geq 1$  such that  $\{w \in M_2 : \text{length}(w) \leq j\} \subseteq D$ .
- v)  $\psi(0) \cup \psi(1) = X$ .
- vi) If  $w \in D$ , then  $w0 \in D$  if and only if  $w1 \in D$ .
- vii) If  $w$  and  $w0$  are in  $D$ , then  $\psi(w) = \psi(w0) \cup \psi(w1)$ .
- viii) For every  $i$  there is  $w \in D$  such that  $\psi(w) = F_i$  and  $w0 \notin D$ .
- ix) If  $w \in D$  and  $w0 \notin D$ , then  $\psi(w) = F_i$  for some  $i$ .

Proof: If  $t = 1$ , let  $\psi(\eta) = \psi(0) = \psi(1) = F_1 = X$ . Now assume  $t \geq 2$ . Define

$\psi(\eta) = X$  and let  $j$  be the unique positive integer such that  $2^j \leq t < 2^{j+1}$ . Let  $B_j$  be the set  $\{w \in M_2 : \text{length}(w) = j\}$ . By induction  $|B_j| = 2^j$  and so there is a bijection  $f$  from  $B_j$  to  $\{1, \dots, 2^j\}$ . For  $w \in B_j$ , define  $\psi(w) = F_{f(w)}$  if  $f(w) + 2^j > t$  and  $\psi(w) = F_{f(w)} \cup F_{f(w)+2^j}$  if  $f(w) + 2^j \leq t$ . Also, if  $f(w) + 2^j \leq t$ , define  $\psi(w0) = F_{f(w)}$  and  $\psi(w1) = F_{f(w)+2^j}$ . So far  $\psi$  is defined on  $\{\eta\} \cup B_j \cup \{w0, w1 : w \in B_j \text{ and } f(w) + 2^j \leq t\}$ . Also, if  $w \in B_j$  and  $f(w) + 2^j \leq t$ ,  $\psi(w) = \psi(w0) \cup \psi(w1)$ . Now extend the definition of  $\psi$  by a finite ‘downward induction.’ Let  $B_s = \{w \in M_2 : \text{length}(w) = s\}$  and assume  $\psi$  has been defined on all of  $B_s$ , where  $2 \leq s \leq j$ . For  $w \in B_{s-1}$ , define  $\psi(w) = \psi(w0) \cup \psi(w1)$ . After a finite number of steps, i), iv) and ix) hold by definition. Clearly, for each  $w \in D$  (the domain of  $\psi$ ),  $\psi(w)$  is a union of some non-empty subcollection of  $\{F_1, \dots, F_t\}$  and so is non-empty and closed. By definition  $D$  may include some words of length  $j + 1$ , and these are the longest words in  $D$ . Thus iv) implies ii). The fact that words of length  $j + 1$  are included in  $D$  in pairs ( $w0$  and  $w1$ ) along with iv) imply vi). That and the definition yield vii). Let  $1 \leq k \leq t$ . If  $k \leq 2^j$ , there is  $v \in B_j$  with  $f(v) = k$ . For  $k + 2^j > t$ , the initial step of the definition shows that  $\psi(v) = F_k$ , and, for  $k + 2^j \leq t$ , it says  $\psi(v0) = F_k$ . If  $k > 2^j$ , there is  $v \in B_j$  with  $f(v) = k - 2^j$  and by definition  $\psi(v1) = F_k$ . Thus viii) holds. For iii), suppose  $wv \in D$ . Proceed by induction on the length of  $v$ . If  $v = \eta$ , then  $\psi(wv) = \psi(w)$  and the conclusion holds. Now assume the conclusion holds for any  $v$  of length  $s$  and suppose  $\text{length}(v) = s + 1$ . Then  $v = u0$  or  $v = u1$ , where  $\text{length}(u) = s$ . By ii)  $wu \in D$  and by vii) and the induction hypothesis,  $\psi(w) \supseteq \psi(wu) = \psi(wu0) \cup \psi(wu1) \supseteq \psi(wv)$ . Lastly, for v), let  $x \in X$ . There is  $k$  with  $x \in F_k$ . By the initial step of  $\psi$ ’s definition there is  $w \in D$  of length  $j$  or  $j + 1$  with  $\psi(w) = F_k$ . Since  $w \neq \eta$ ,  $w = 0u$  or  $w = 1u$  and by iii)  $x \in \psi(w) \subseteq \psi(0) \cup \psi(1)$ . Conversely, since each value of  $\psi$  is a union of some subcollection of  $\{F_1, \dots, F_t\}$ , clearly  $\psi(0) \cup \psi(1) \subseteq X$ .

**Lemma R26.15** Let  $(X, \tau)$  be a topological space and let  $X = \cup_{i=1}^t F_i$ , where each  $F_i$  is a non-empty closed subset and  $t \geq 2$  is in  $\mathbf{N}$ . Assume  $\{F_i : 1 \leq i \leq t\}$  is a pairwise disjoint collection. Let  $\psi$  be constructed as in the proof of R26.14. Assume  $u \neq v$  are in  $\text{dom}(\psi)$  with  $\text{length}(u) = \text{length}(v)$ . Then  $\psi(u) \cap \psi(v) = \emptyset$ . In addition, if  $x \neq y$  are in  $\text{dom}(\psi)$  with  $x0, y0 \notin \text{dom}(\psi)$ , then  $\psi(x) \cap \psi(y) = \emptyset$ .

Proof: Let  $j$  be the unique positive integer such that  $2^j \leq t < 2^{j+1}$  and use the notation defining  $\psi$  from the proof of R26.14. Since  $\eta$  is the only word of length 0,  $\text{length}(u) \geq 1$ . First assume  $u$  and  $v$  are in the initial part of the domain of  $\psi$ , i.e.,  $\{\eta\} \cup B_j \cup \{w0, w1 : w \in B_j \text{ and } f(w) + 2^j \leq t\}$ . If  $\text{length}(u) = j + 1$ , since  $f$  is one-to-one, it is easy to check that  $\psi(u) = F_i$  and  $\psi(v) = F_l$  where  $i \neq l$ . By the pairwise disjointness assumption  $\psi(u) \cap \psi(v) = \emptyset$ . Now assume  $\text{length}(u) = j$ . If  $f(u) + 2^j > t$  and  $f(v) + 2^j > t$ ,  $\psi(u) = F_{f(u)}$  and  $\psi(v) = F_{f(v)}$ , which are disjoint since  $f$  is one-to-one. If  $f(u) + 2^j \leq t$  and  $f(v) + 2^j \leq t$ ,  $\psi(u) = F_{f(u)} \cup F_{f(u)+2^j}$  and  $\psi(v) = F_{f(v)} \cup F_{f(v)+2^j}$ . Since  $f$  is one-to-one and has values 1 or larger, by pairwise disjointness  $\psi(u) \cap \psi(v) = \emptyset$ . If  $f(u) + 2^j \leq t$  and  $f(v) + 2^j > t$ ,  $\psi(u) = F_{f(u)} \cup F_{f(u)+2^j}$  and  $\psi(v) = F_{f(v)}$ . For the same reasons,  $\psi(u) \cap \psi(v) = \emptyset$ . The fourth case is similar to the third. Now assume the conclusion holds for any two words of length  $s$ , where  $j \geq s \geq 2$ , and suppose  $\text{length}(u) = s - 1$ . By R26.14  $\psi(u) = \psi(u0) \cup \psi(u1)$  and  $\psi(v) = \psi(v0) \cup \psi(v1)$ . Since  $u \neq v$ ,  $u0, u1, v0, v1$  are four distinct words of length  $s$ . By the inductive hypothesis, the first conclusion holds. Now assume  $x \neq y$  are in  $\text{dom}(\psi)$  with  $x0, y0 \notin \text{dom}(\psi)$ . By R26.14iv neither is  $\eta$  and both are

in the initial part of  $\text{dom}(\psi)$ . If both have the same length, the first conclusion applies. If one, say  $x$ , has length  $j$  and the other has length  $j + 1$ , there is  $z$  of length  $j$  such that  $y$  is either  $z0$  or  $z1$ . By the first conclusion  $\psi(x) \cap \psi(z) = \emptyset$  and by R26.14vi  $\psi(y) \subseteq \psi(z)$  so that  $\psi(x) \cap \psi(y) = \emptyset$ .

**Lemma R26.16** Let  $E$  be a compact metric space. Then there is an auxiliary function  $\phi$  for  $E$  such that  $\phi(v) \neq \emptyset$  for every  $v \in M_2$  and, if  $w$  is an infinite word,  $\text{diam}(\phi(w_n)) \rightarrow 0$  as  $n \rightarrow \infty$ .

Proof: First, a sequence of maps will be defined by induction. Let  $\rho$  be a metric for  $E$  and, for  $e \in E$  and  $\epsilon > 0$ , let  $\overline{B_\epsilon(e)}$  denote the closure of the open  $\epsilon$ -ball centered at  $e$ . Pick  $m_1 \geq 1$ . By compactness applied to the open balls, there are  $e_1, \dots, e_t$  such that  $E = \cup_{i=1}^t \overline{B_{1/m_1}(e_i)}$ . Apply R26.14 to obtain  $\phi_1$  on domain  $D_1$  to the collection of unions from  $\{\overline{B_{1/m_1}(e_1)}, \dots, \overline{B_{1/m_1}(e_t)}\}$ , where  $\phi_1$  has the lemma's properties, which easily imply the 8 properties listed below. Now assume  $\phi_1, \dots, \phi_n$  with domains  $D_1 \subseteq D_2 \subseteq \dots \subseteq D_n$  have been constructed with the following properties:

- i)  $D_n \subseteq M_2$  is finite and  $\{w \in M_2 : \text{length}(w) \leq n\} \subseteq D_n$ .
- ii) For  $2 \leq i \leq n$ ,  $\phi_{i-1} = \phi_i|_{D_{i-1}}$ .
- iii)  $\phi_n(\eta) = E$ .
- iv) If  $wv \in D_n$ , then  $w \in D_n$ .
- v) For  $w \in D_n$ ,  $w0 \in D_n$  if and only if  $w1 \in D_n$ .
- vi) For  $w \in D_n$  with  $w0 \in D_n$ ,  $\phi_n(w) = \phi_n(w0) \cup \phi_n(w1)$ .
- vii) If  $w \in D_n$  and  $w0 \notin D_n$ ,  $\text{diam}(\phi_n(w)) \leq 2/n$ .
- viii) For every  $w \in D_n$ ,  $\phi_n(w)$  is a non-empty closed set.

Now pick  $m_{n+1} \geq n + 1$  and call  $w \in M_2$  a terminal word (of  $D_n$ ) if  $w \in D_n$  and  $w0 \notin D_n$ . There are finitely many terminal words. Note that no proper initial subword of a terminal word is a terminal word by properties iv) and v). For each terminal word  $w$ , by compactness  $\phi_n(w) = \cup_{i=1}^{t(w)} (\overline{B_{1/m_{n+1}}(x_i(w))} \cap \phi_n(w))$ , where  $t(w)$  in  $\mathbf{N}$  and  $x_1(w), \dots, x_{t(w)}(w)$  in  $\phi_n(w)$ . By R26.14 there is  $\psi_w$  from a finite domain  $A_w$  to the collection of unions from the family  $\{\overline{B_{1/m_{n+1}}(x_i(w))} \cap \phi_n(w) : 1 \leq i \leq t(w)\}$ , where  $\psi_w$  has the properties listed in R26.14. The finite set  $\{wv : w \text{ is a terminal word and } v \in A_w, v \neq \eta\}$  will be denoted  $C_{n+1}$ . Note that if  $wv = us$  where  $w, u$  are terminal words, then  $w = u$ , since neither can be a proper initial subword of the other, and so also  $v = s$ . Thus each element of  $C_{n+1}$  has a unique representation in the specified form. Finally note that the requirement  $v \neq \eta$  implies  $D_n \cap C_{n+1} = \emptyset$ . Let  $D_{n+1} = D_n \cup C_{n+1}$ . Define  $\phi_{n+1}$  on  $D_{n+1}$  by  $\phi_{n+1}(u) = \phi_n(u)$  if  $u \in D_n$  and, if  $u \in C_{n+1}$  with  $u = wv$ ,  $\phi(u) = \psi_w(v)$ . Then  $\phi_{n+1}$  is a function; clearly  $D_n \subseteq D_{n+1}$  and ii), iii), and viii) hold for  $\phi_{n+1}$ . For i), finiteness holds since each  $A_w$  is finite. Let  $w$  be a word of length  $n + 1$ . Then  $w = ux$ , where  $\text{length}(u) = n$  and  $x \in \{0, 1\}$ . Then  $u \in D_n$ . If  $u0 \in D_n$ ,  $w \in D_n \subseteq D_{n+1}$ . Otherwise  $u$  is a terminal word of  $D_n$  and, since  $\{0, 1\} \subseteq A_u$ ,  $w = ux \in C_{n+1} \subseteq D_{n+1}$ . For iv), let  $wv \in D_{n+1}$ . Since iv) holds for  $D_n$ , assume  $wv \in C_{n+1}$ . Then  $wv = us$ , where  $u$  is a terminal word of  $D_n$  and  $s \neq \eta$  is in  $A_u$ . If  $w$  is a subword of  $u$ ,  $u = wr$  and so  $w \in D_n$  by the induction hypothesis. Otherwise  $u$  must be a proper subword of  $w$ , i.e.,  $w = ur$  where  $r \neq \eta$ . Then  $wv = urv$  and so  $rv = s \in A_u$ . By R26.14ii  $r \in A_u$  and so  $w = ur \in C_{n+1} \subseteq D_{n+1}$ . For v) let  $w \in D_{n+1}$  with  $w0 \in D_{n+1}$ . If  $w0 \in D_n$ , by iv) and v) of the induction hypothesis,  $w1 \in D_n \subseteq D_{n+1}$ . If  $w0 \in C_{n+1}$ ,  $w0 = us$ , where

$u$  is a terminal word of  $D_n$  and  $s \neq \eta$  is in  $A_u$ . Now  $s = r0$  and by R26.14vi  $r1 \in A_u$ . Thus  $w1 = ur1$  is in  $C_{n+1} \subseteq D_{n+1}$ . The converse is similar. For vi) let  $w0$  (and so  $w$ ) be in  $D_{n+1}$ . If  $w0 \in D_n$ , since  $\phi_n = \phi_{n+1}|_{D_n}$ , apply vi) of the induction hypothesis for  $D_n$ . If  $w0 \in C_{n+1}$ ,  $w0 = us0$ , where  $u$  is a terminal word of  $D_n$  and  $s0 \in A_u$ . By R26.14ii  $s \in A_u$  and so  $w = us \in C_{n+1}$ . Applying the definition and R26.14vii,  $\phi_{n+1}(w) = \psi_u(s) = \psi_u(s0) \cup \psi_u(s1) = \phi_{n+1}(w0) \cup \phi_{n+1}(w1)$ . For vii) let  $w \in D_{n+1}$  with  $w0 \notin D_{n+1}$ . Note that  $w \notin D_n$  for, if so,  $w$  is a terminal word of  $D_n$  and  $0 \in A_w$  so that  $w0$  would be in  $C_{n+1} \subseteq D_{n+1}$ . Thus  $w = us$ , where  $u$  is a terminal word of  $D_n$  and  $s \neq \eta$  is in  $A_u$ . Since  $w0 = us0$  is not in  $C_{n+1}$ ,  $s0 \notin A_u$ . By R26.14ix and the choice of  $\psi_u$ ,  $\phi_{n+1}(w) = \psi_u(s) \subseteq \overline{B_{1/m_{n+1}}(x)}$  for some  $x$ . Thus  $\text{diam}(\phi_{n+1}(w)) \leq 2/m_{n+1} \leq 2/(n+1)$ . By induction there is an infinite sequence  $\{\phi_n\}_{n=1}^\infty$  with the properties listed above. Since the domains are nested and ii) holds,  $\phi = \bigcup_{n=1}^\infty \phi_n$  is a function and  $\phi|_{D_n} = \phi_n$  for every  $n$ . Since i) holds, the domain of  $\phi$  is  $M_2$ . For  $w \in M_2$ , pick  $n$  with  $w0 \in D_n$ . By vi)  $\phi(w) = \phi_n(w) = \phi_n(w0) \cup \phi_n(w1) = \phi(w0) \cup \phi(w1)$ . Thus  $\phi$  is an auxiliary function for  $E$  and by viii)  $\phi(w) \neq \emptyset$  for every  $w$ . Lastly, let  $w$  be an infinite word and let  $\epsilon > 0$ . Pick  $N$  such that  $2/N < \epsilon$ . Since  $D_N$  is finite, the set  $\{k : w_k \notin D_N\}$  is non-empty and so has a smallest element  $M$ . Then  $M \geq N + 1$  by i) and  $w_{M-1} \in D_N$ . By v) and the choice of  $M$ ,  $w_{M-1}0 \notin D_N$ . By vii)  $\text{diam}(\phi(w_{M-1})) = \text{diam}(\phi_N(w_{M-1})) \leq 2/N < \epsilon$ . For  $n \geq M$ , by R26.11  $\phi(w_n) \subseteq \phi(w_{M-1})$  and so  $\text{diam}(\phi(w_n)) < \epsilon$ . Thus the limit claim holds.

**Corollary R26.17** Let  $E$  be a compact metric space. Then there is a continuous surjection  $f : \mathbf{R}_2 \rightarrow E$ .

Proof: This is immediate from R26.16 and R26.13.

In the next lemma, notation and terminology from the proof of R26.16 will continue to be used.

**Lemma R26.18** Let  $E$  be a compact metric space. Assume for every  $n$  in  $\mathbf{N}$ , for every  $\epsilon > 0$ , and for every  $e \in E$ , there exist  $m > n$  and  $x_1, \dots, x_t$  in  $B_\epsilon(e)$  such that  $t \geq 2$  and  $\overline{B_\epsilon(e)} = \bigcup_{i=1}^t \overline{B_{1/m}(x_i)}$  where  $\{\overline{B_{1/m}(x_i)} : 1 \leq i \leq t\}$  is a pairwise disjoint collection. Then there is an auxiliary function  $\phi$  for  $E$  constructed as in R26.18 with the property that, if  $u, v \in M_2$  with  $u \neq v$  and  $\text{length}(u) = \text{length}(v)$ , then  $\phi(u) \cap \phi(v) = \emptyset$ .

Proof: It is possible to construct a sequence  $\{\phi_n\}_{n=1}^\infty$  as in the proof of R26.16, using at each stage the pairwise disjoint collections of closed balls guaranteed by the assumption. To see this, it is sufficient to check (by induction) that each  $\phi_n$  has the following property: If  $w \in D_n$  and  $w0 \notin D_n$ , then there is  $e \in E$  and  $\epsilon > 0$  such that  $\phi_n(w) = \overline{B_\epsilon(e)}$ . This holds for  $\phi_1$  by R26.14ix. Assume it holds for  $\phi_n$  and let  $u \in D_{n+1}$  with  $u0 \notin D_{n+1}$ . By the construction, a terminal word of  $D_n$  is not terminal in  $D_{n+1}$  and so  $u \in C_{n+1}$ , i.e.,  $u = wv$  where  $w$  is a terminal word of  $D_n$ ,  $v \neq \eta$ , and  $v$  is in the domain of  $\psi_w$ . By definition  $\phi_{n+1}(u) = \psi_w(v)$ . By the induction hypothesis  $\phi_n(w)$  is some  $\overline{B_\epsilon(e)}$ , which is a union of closed balls by this lemma's assumption. Since  $u0 \notin D_{n+1}$ ,  $v0$  is not in the domain of  $\psi_w$  and so, by R26.14ix,  $\psi_w(v)$  is one of those closed balls. With the sequence thus constructed, as in R26.16, the auxiliary function is  $\phi = \bigcup_{n=1}^\infty \phi_n$ . Since  $D_n \subseteq D_{n+1}$  for all  $n$ ,  $\phi|_{D_n} = \phi_n$  for every  $n$ , and  $M_2 = \bigcup_{n=1}^\infty D_n$ , it is sufficient to show the following by induction: for every  $n$ , if  $u \neq v$  are in  $D_n$  with  $\text{length}(u) = \text{length}(v)$  or  $u, v$  both terminal words of  $D_n$ , then  $\phi_n(u) \cap \phi_n(v) = \emptyset$ . Since  $\phi_1$  is a map constructed as in R26.14 using a pairwise disjoint union with at least two sets, by R26.15 the claim holds for  $n = 1$ .

Now assume it holds for  $n$  and let  $u \neq v$  be in  $D_{n+1}$ . First assume  $\text{length}(u) = \text{length}(v)$ . If both are in  $D_n$ , apply the induction hypothesis. If both are in  $C_{n+1}$ ,  $u = xs$  and  $v = yt$  where  $x, y$  are terminal words of  $D_n$ , neither of  $s, t$  is  $\eta$ ,  $s \in \text{dom}(\psi_x)$ , and  $t$  is in  $\text{dom}(\psi_y)$ . By definition  $\phi_{n+1}(u) = \psi_x(s) \subseteq \phi_n(x)$  and  $\phi_{n+1}(v) = \psi_y(t) \subseteq \phi_n(y)$ . If  $x \neq y$ , by the induction hypothesis  $\phi_n(x) \cap \phi_n(y) = \emptyset$ . If  $x = y$ ,  $s \neq t$  and  $\text{length}(s) = \text{length}(t)$ . By R26.15  $\psi_x(s) \cap \psi_x(t) = \emptyset$ . As a third case, suppose one, say  $u$ , is in  $C_{n+1}$  and the other is in  $D_n$ . Again  $u = xs$  where  $x$  is a terminal word of  $D_n$  and  $s \neq \eta$  is in  $\text{dom}(\psi_x)$ ,  $\phi_{n+1}(u) = \psi_x(s) \subseteq \phi_n(x)$ , and  $\phi_{n+1}(v) = \phi_n(v)$ . Since  $\text{length}(u) = \text{length}(v)$  and  $\text{length}(s) \geq 1$ ,  $v = v_1z$  where  $\text{length}(v_1) = \text{length}(x)$ . Note that  $v_1 \in D_n$  by property iv) in the proof of R26.16 and, since  $\text{length}(z) = \text{length}(s) \geq 1$ ,  $v_10 \in D_n$  by the same property. Thus  $v_1$  is not a terminal word of  $D_n$  and so  $v_1 \neq x$ . By the induction hypothesis  $\phi_n(v_1) \cap \phi_n(x) = \emptyset$  and so  $\phi_n(v_1) \cap \phi_{n+1}(u) = \emptyset$ . It follows from vi) in the proof of R26.16 that  $\phi_n(v) \subseteq \phi_n(v_1)$  and so  $\phi_{n+1}(u) \cap \phi_{n+1}(v) = \emptyset$ . Lastly assume  $u, v$  are both terminal words of  $D_{n+1}$ . By the construction of  $\phi_{n+1}$ , if  $w$  is a terminal word of  $D_n$ ,  $w0$  is in  $D_{n+1}$ , i.e.,  $w$  is not a terminal word of  $D_{n+1}$ . Thus  $u, v$  are both in  $C_{n+1}$ . As before,  $u = xs$  and  $v = yt$  where  $x, y$  are terminal words of  $D_n$ , neither of  $s, t$  is  $\eta$ ,  $s \in \text{dom}(\psi_x)$ , and  $t$  is in  $\text{dom}(\psi_y)$ . By definition  $\phi_{n+1}(u) = \psi_x(s) \subseteq \phi_n(x)$  and  $\phi_{n+1}(v) = \psi_y(t) \subseteq \phi_n(y)$ . If  $x \neq y$ , by the induction hypothesis  $\phi_n(x) \cap \phi_n(y) = \emptyset$ . If  $x = y$ ,  $s \neq t$  and  $\text{length}(s) = \text{length}(t)$ . By R26.15  $\psi_x(s) \cap \psi_x(t) = \emptyset$ .

**Corollary R26.19** Let  $E$  be a compact metric space. Assume for every  $n$  in  $\mathbf{N}$ , for every  $\epsilon > 0$ , and for every  $e \in E$ , there exist  $m > n$  and  $x_1, \dots, x_t$  in  $B_\epsilon(e)$  such that  $t \geq 2$  and  $\overline{B_\epsilon(e)} = \cup_{i=1}^t \overline{B_{1/m}(x_i)}$  where  $\{B_{1/m}(x_i) : 1 \leq i \leq t\}$  is a pairwise disjoint collection. Then  $\mathbf{R}_2$  is homeomorphic to  $E$ .

Proof: Use the auxiliary function  $\phi$  from R26.18 to construct a continuous surjection  $f : \mathbf{R}_2 \rightarrow E$  as in R26.13. Since both spaces are compact and Hausdorff, it is sufficient to show that  $f$  is one-to-one. Let  $\mathcal{F} \neq \mathcal{G}$  be in  $\mathbf{R}_2$ . The infinite words  $d(\mathcal{F})$  and  $d(\mathcal{G})$  are different and so there is  $n$  with  $d(\mathcal{F})_n \neq d(\mathcal{G})_n$ . By R26.18  $\phi(d(\mathcal{F})_n) \cap \phi(d(\mathcal{G})_n) = \emptyset$ . Since  $f(\mathcal{F}) \in \phi(d(\mathcal{F})_n)$  and  $f(\mathcal{G}) \in \phi(d(\mathcal{G})_n)$ ,  $f(\mathcal{F}) \neq f(\mathcal{G})$ .

In [8], for  $k \geq 2$ , the metric  $d_k$  was described and shown to generate the topology of  $\mathbf{R}_k$ . In the next two results, the open balls are assumed to be generated with  $d_k$  as the underlying metric for  $\mathbf{R}_k$ .

The next lemma is implicit in [8]. Recall the following facts and notation: For  $Z$  in a normal basis  $\mathcal{Z}$ ,  $Z^\omega$  is the set of  $\mathcal{Z}$ -filters containing  $Z$ . For  $k \geq 2$  and  $x, n \in \mathbf{N}$ ,  $C_n^x(k) = \{m \in \mathbf{N} : m \equiv x \pmod{k^n}\}$ , which is in the normal basis generating  $\mathbf{N}_k$ .

**Lemma R26.20** Let  $k \geq 2$  be in  $\mathbf{N}$  and let  $\mathcal{F}$  in  $\mathbf{R}_k$ . Let  $1 \geq \epsilon > 0$  and let  $j$  be the unique positive integer with  $1/k^j < \epsilon \leq 1/k^{j-1}$ . Then

- i)  $B_\epsilon(\mathcal{F})$  is clopen.
- ii)  $\overline{B_\epsilon(\mathcal{F})} = B_\epsilon(\mathcal{F}) = Z^\omega \cap \mathbf{R}_k$ , where  $Z = C_j^{x_j}(k)$ .
- iii) If  $\mathcal{G}$  is in  $B_\epsilon(\mathcal{F})$ , then  $B_\epsilon(\mathcal{G}) = B_\epsilon(\mathcal{F})$ .

Proof: Let  $\mathcal{F}$  have associated sequence  $\{x_n\}_{n=1}^\infty$ . Part i) is an application of R20.23ii. Part ii) is verified in the proof of R20.23. For iii) let  $\mathcal{G}$  be in  $B_\epsilon(\mathcal{F})$  and suppose  $\mathcal{G}$  has associated sequence  $\{y_n\}_{n=1}^\infty$ . By ii)  $C_j^{x_j}(k) \in \mathcal{G}$  and so by R20.21  $y_j = x_j$ . Apply part ii) to  $\mathcal{G}$  in place of  $\mathcal{F}$  to obtain the conclusion.

**Lemma R26.21** Let  $k \geq 2$  be in  $\mathbf{N}$  and let  $\mathcal{F}$  in  $\mathbf{R}_k$ . Let  $\epsilon > 0$  and let  $n \in \mathbf{N}$ . Then



there exist  $m > n$  and  $\mathcal{G}_1, \dots, \mathcal{G}_t$  in  $B_\epsilon(\mathcal{F})$  such that  $t \geq 2$  and  $\overline{B_\epsilon(\mathcal{F})} = \cup_{i=1}^t \overline{B_{1/m}(\mathcal{G}_i)}$  where  $\{\overline{B_{1/m}(\mathcal{G}_i)} : 1 \leq i \leq t\}$  is a pairwise disjoint collection.

Proof: Let  $\mathcal{F}$  have associated sequence  $\{x_n\}_{n=1}^\infty$ . Pick  $j \geq 2$  such that  $1/k^{j-1} < \epsilon$  and  $k^{j-1} > n$ . Since  $C_j^1(k), \dots, C_j^{k^j}(k)$  are the distinct classes of equivalence mod  $k^j$ , the collection  $\{(C_j^i(k))^\omega \cap \mathbf{R}_k : 1 \leq i \leq k^j\}$  is a finite cover of  $\mathbf{R}_k$  by pairwise disjoint clopen sets. By R26.20 each of those sets is a  $1/k^{j-1}$ -open ball, any element of which can be taken as its center. Thus the elements of this cover which have a non-empty intersection with  $B_\epsilon(\mathcal{F})$  can be written as  $\{B_{1/k^{j-1}}(\mathcal{G}_i) : 1 \leq i \leq t\}$ , where each  $\mathcal{G}_i \in B_\epsilon(\mathcal{F})$ . By choice of the  $\mathcal{G}_i$ ,  $B_\epsilon(\mathcal{F}) \subseteq \cup\{B_{1/k^{j-1}}(\mathcal{G}_i) : 1 \leq i \leq t\}$ . Let  $\mathcal{H}_i \in B_\epsilon(\mathcal{F}) \cap B_{1/k^{j-1}}(\mathcal{G}_i)$ . Since  $1/k^{j-1} < \epsilon$ , by R26.20iii (twice),  $B_{1/k^{j-1}}(\mathcal{G}_i) = B_{1/k^{j-1}}(\mathcal{H}_i) \subseteq B_\epsilon(\mathcal{H}_i) = B_\epsilon(\mathcal{F})$ . Thus  $\cup\{B_{1/k^{j-1}}(\mathcal{G}_i) : 1 \leq i \leq t\} \subseteq B_\epsilon(\mathcal{F})$ . Since the balls are clopen, the conclusion follows with  $m = k^{j-1}$  provided  $t \geq 2$ . That can be shown as follows: By R10.2.5i  $x_j = x_{j-1} + sk^{j-1}$ , where  $s \in \{0, 1, \dots, k-1\}$ . Pick  $r \in \{0, 1, \dots, k-1\}$  with  $r \neq s$ . Define a sequence  $\{y_n\}_{n=1}^\infty$  by  $y_n = x_n$  for  $n \leq j-1$ ,  $y_j = x_{j-1} + rk^{j-1}$ , and  $y_{n+1} = y_n + k^n$  for  $n \geq j$ . By R10.2.6 there is  $\mathcal{G}$  in  $\mathbf{R}_k$  with  $\{y_n\}_{n=1}^\infty$  as its associated sequence. Since  $y_j \neq x_j$ , the classes  $C_j^{x_j}(k)$  and  $C_j^{y_j}(k)$  are distinct. By definition of the associated sequence,  $C_j^{y_j}(k) \in \mathcal{G}$  and  $C_j^{x_j}(k) \in \mathcal{F}$  so that  $\mathcal{G}$  and  $\mathcal{F}$  are in different elements of the cover  $\{(C_j^i(k))^\omega \cap \mathbf{R}_k : 1 \leq i \leq k^j\}$ . To verify that  $t \geq 2$ , it is now sufficient to show that  $\mathcal{G} \in B_\epsilon(\mathcal{F})$ . If  $\epsilon > 1$ , this clear since  $d_k \leq 1$  so that  $B_\epsilon(\mathcal{F}) = \mathbf{R}_k$ . If  $\epsilon \leq 1$ , there is a unique  $l$  such that  $1/k^l < \epsilon \leq 1/k^{l-1}$ . Since  $1/k^{j-1} < \epsilon \leq 1/k^{l-1}$ ,  $j > l$  and so by definition  $y_l = x_l$ . Thus  $\mathcal{G}$  is in the set  $(C_l^{x_l}(k))^\omega \cap \mathbf{R}_k$ , which is  $B_\epsilon(\mathcal{F})$  by R26.20.

**Corollary R26.22** Let  $k \geq 2$  be in  $\mathbf{N}$ . Then the topological spaces  $\mathbf{R}_2$  and  $\mathbf{R}_k$  are homeomorphic.

Proof: This follows from R26.19 and R26.21.

**Corollary R26.23** Let  $k, j \geq 2$  be in  $\mathbf{N}$ . Then the topological spaces  $\mathbf{R}_k$  and  $\mathbf{R}_j$  are homeomorphic.

Proof: Transitivity and symmetry of homeomorphism: each is homeomorphic to  $\mathbf{R}_2$ .

In [3] the the following question was left unanswered: For distinct primes  $p, q$ , are the topological spaces  $\mathbf{N}_p$  and  $\mathbf{N}_q$  homeomorphic? The next few results do not answer the question but are related.

**Lemma R26.24** Let  $X$  be an infinite discrete space. Let  $(Y, f)$  and  $(Z, g)$  be  $T_2$ -compactifications of  $X$ . Assume  $h : Y \rightarrow Z$  is a homeomorphism. Then  $h[f[X]] = g[X]$ .

Proof: Let  $x \in X$ . Since  $f[X]$  is an open, discrete subspace of  $Y$ ,  $\{f(x)\}$  is open in  $Y$  and so  $\{h(f(x))\}$  is open in  $Z$ . By density  $\{h(f(x))\} \cap g[X]$  is non-empty, i.e.  $h \circ f(x) \in g[X]$ . Thus  $h[f[X]] \subseteq g[X]$ . Similarly, using  $h^{-1}$ ,  $h[f[X]] \supseteq g[X]$ .

**Corollary R26.25** Let  $k, j \geq 2$  be in  $\mathbf{N}$ . If the topological spaces  $\mathbf{N}_k$  and  $\mathbf{N}_j$  are homeomorphic, then the topological spaces  $\mathbf{R}_k$  and  $\mathbf{R}_j$  are homeomorphic.

Proof: Let  $h : \mathbf{N}_k \rightarrow \mathbf{N}_j$  be a homeomorphism and let  $\iota_k$  and  $\iota_j$  be the embeddings of  $\mathbf{N}$  with the discrete topology into  $\mathbf{N}_k, \mathbf{N}_j$  respectively. By R26.24  $h[\iota_k[\mathbf{N}]] = \iota_j[\mathbf{N}]$ . Since  $\mathbf{R}_k = \mathbf{N}_k - \iota_k[\mathbf{N}]$  and  $\mathbf{R}_j = \mathbf{N}_j - \iota_j[\mathbf{N}]$ ,  $h$  restricted to  $\mathbf{R}_k$  maps onto  $\mathbf{R}_j$  and so is a homeomorphism from  $\mathbf{R}_k$  to  $\mathbf{R}_j$ .

In other words, R26.23 is a necessary condition for topological spaces  $\mathbf{N}_k$  and  $\mathbf{N}_j$  to be homeomorphic. The next proposition is a partial answer as to whether it is sufficient.

**Lemma R26.26** Let  $k \geq 2$  be in  $\mathbf{N}$  and let  $\iota_k, f_k$  be the embeddings of  $\mathbf{N}$  into the

compactifications  $\mathbf{N}_k, \mathbf{R}_k$  respectively. Let  $\mathcal{F}$  be in  $\mathbf{R}_k$  and let  $\{x_i\}$  be a sequence in  $\mathbf{N}$  such that  $\{\iota_k(x_i)\}$  converges to  $\mathcal{F}$  in  $\mathbf{N}_k$ . Then  $\{f_k(x_i)\}$  converges to  $\mathcal{F}$  in  $\mathbf{R}_k$ .

Proof: Since  $\mathcal{F}$  is in  $\mathbf{R}_k$ , it is a non-point ultrafilter in  $\mathbf{N}_k$  and so has an associated sequence  $\{y_n\}_{n=1}^\infty$ . Recall that for each  $i$  the associated sequence of  $f_k(x_i)$  is  $\{z_n\}_{n=1}^\infty$  where  $z_n \equiv x_i \pmod{k^n}$  for each  $n$ . By R10.2.11  $\{x_i\}$  is unbounded and for every  $m$  in  $\mathbf{N}$  there is  $i_0$  such that  $i \geq i_0$  implies  $x_i \equiv y_m \pmod{k^m}$ . Then by transitivity  $i \geq i_0$  implies  $z_m \equiv y_m \pmod{k^m}$ , i.e., since  $z_m$  and  $y_m$  are both in  $\{1, 2, \dots, k^m\}$ ,  $z_m = y_m$ . By R17.2.16  $\{f_k(x_i)\}$  converges to  $\mathcal{F}$  in  $\mathbf{R}_k$ .

The previous lemma can also be proven by using the continuity of addition and the fact that  $f_k(x) = \iota_k(x) + \mathcal{O}_k$ , where  $\mathcal{O}_k$  is the additive identity in  $\mathbf{R}_k$ . The next lemma is a partial converse of R26.26.

**Lemma R26.27** Let  $k \geq 2$  be in  $\mathbf{N}$  and let  $\iota_k, f_k$  be the embeddings of  $\mathbf{N}$  into the compactifications  $\mathbf{N}_k, \mathbf{R}_k$  respectively. Let  $\mathcal{F}$  be in  $\mathbf{R}_k$  and let  $\{x_i\}$  be an unbounded sequence in  $\mathbf{N}$  such that  $\{f_k(x_i)\}$  converges to  $\mathcal{F}$  in  $\mathbf{R}_k$ . Then  $\{\iota_k(x_i)\}$  converges to  $\mathcal{F}$  in  $\mathbf{N}_k$ .

Proof: As above, since  $\mathcal{F}$  is in  $\mathbf{R}_k$ , it is a non-point ultrafilter in  $\mathbf{N}_k$  and so has an associated sequence  $\{y_n\}_{n=1}^\infty$ . Recall that for each  $i$  the associated sequence of  $f_k(x_i)$  is  $\{z_n\}_{n=1}^\infty$  where  $z_n \equiv x_i \pmod{k^n}$  for each  $n$ . Let  $m$  be in  $\mathbf{N}$ . By R17.2.16 there is  $i_0$  such that  $i \geq i_0$  implies  $z_m = y_m$ . Then  $i \geq i_0$  implies  $x_i \equiv y_m \pmod{k^m}$ . Since  $\{x_i\}$  is given to be unbounded, by R10.2.11  $\{\iota_k(x_i)\}$  converges to  $\mathcal{F}$  in  $\mathbf{N}_k$ .

**Proposition R26.28** Let  $k, j \geq 2$  be in  $\mathbf{N}$  and let  $f_k, f_j$  be the embeddings of  $\mathbf{N}$  into the compactifications  $\mathbf{R}_k, \mathbf{R}_j$  respectively. Let  $h : \mathbf{R}_k \rightarrow \mathbf{R}_j$  be a homeomorphism such that  $h[f_k[\mathbf{N}]] = f_j[\mathbf{N}]$ . Then the topological spaces  $\mathbf{N}_k$  and  $\mathbf{N}_j$  are homeomorphic.

Proof: Let  $\iota_k$  and  $\iota_j$  be the embeddings of  $\mathbf{N}$  into compactifications  $\mathbf{N}_k, \mathbf{N}_j$  respectively. Define  $H : \mathbf{N}_k \rightarrow \mathbf{N}_j$  as follows: For  $\mathcal{F}$  in  $\mathbf{R}_k$ , let  $H(\mathcal{F}) = h(\mathcal{F})$ . For  $\mathcal{F}$  in  $\mathbf{N}_k - \mathbf{R}_k$ ,  $\mathcal{F} = \iota_k(n)$  for some  $n$ . Let  $H(\mathcal{F}) = \iota_j(m)$ , where  $h(f_k(n)) = f_j(m)$ . Since  $h$  is bijective,  $h[f_k[\mathbf{N}]] = f_j[\mathbf{N}]$ , and the embeddings are one-to-one, it is easy to check that  $H$  is a one-to-one, onto function. Since the spaces are compact and Hausdorff, the continuity of  $H$  would imply that it is a homeomorphism. Let  $C$  be a closed subset of  $\mathbf{N}_j$  and decompose  $C$  into disjoint subsets  $C_1$  and  $C_2$ , where  $C_1 = \iota_j[\mathbf{N}] \cap C$  and  $C_2 = \mathbf{R}_j \cap C$ . Note that  $H^{-1}[C_2] = h^{-1}[C_2]$  is closed in  $\mathbf{R}_k$  by continuity. It is also a closed in  $\mathbf{N}_k$  since  $\mathbf{R}_k$  is the complement of the open  $\iota_k[\mathbf{N}]$ . Thus to show that  $H^{-1}[C] = H^{-1}[C_1] \cup H^{-1}[C_2]$  is closed in  $\mathbf{N}_k$ , it is sufficient to show that the closure of  $H^{-1}[C_1]$  is a subset of  $H^{-1}[C]$ . Let  $\mathcal{F}$  be in the closure of  $H^{-1}[C_1]$ . Since  $\mathbf{N}_k$  is metrizable and  $H^{-1}[C_1] \subseteq \iota_k[\mathbf{N}]$ , there is a sequence  $\{x_i\}$  in  $\mathbf{N}$  with  $\iota_k(x_i) \in H^{-1}[C_1]$  for every  $i$  such that  $\{\iota_k(x_i)\}$  converges to  $\mathcal{F}$  in  $\mathbf{N}_k$ . If  $\mathcal{F}$  is in  $f_k[\mathbf{N}]$ , since  $\{f_k(x_i)\}$  is open, eventually  $\iota_k(x_i) = \mathcal{F}$ , i.e.,  $\mathcal{F} \in H^{-1}[C_1] \subseteq H^{-1}[C]$ . Thus assume  $\mathcal{F}$  is in  $\mathbf{R}_k$  so that  $H(\mathcal{F}) = h(\mathcal{F})$ . For each  $i$  let  $y_i$  be the positive integer with  $h(f_k(x_i)) = f_j(y_i)$ , i.e.,  $H(\iota_k(x_i)) = \iota_j(y_i)$  by definition. By R10.2.11 the sequence  $\{x_i\}$  is unbounded and so, since  $h, f_k$ , and  $f_j$  are one-to-one,  $\{y_i\}$  is also unbounded. By R26.26  $f_k(x_i) \rightarrow \mathcal{F}$  and so by continuity  $h(f_k(x_i)) = f_j(y_i) \rightarrow h(\mathcal{F})$ . By R26.27  $\iota_j(y_i) \rightarrow h(\mathcal{F})$ , i.e.,  $H(\iota_k(x_i)) \rightarrow H(\mathcal{F})$ . Since  $C$  is closed,  $H(\mathcal{F})$  is in  $C$ , i.e.,  $\mathcal{F}$  is in  $H^{-1}[C]$  as required.

Not every homeomorphism  $h$  from  $\mathbf{R}_k$  to  $\mathbf{R}_j$  has the property assumed in R26.28: Let  $\mathcal{F}_0$  be in  $\mathbf{R}_j$  but not in  $f_j[\mathbf{Z}]$ . Since  $\mathbf{R}_j$  with addition is a topological group, the map  $t$  defined by  $t(\mathcal{F}) = \mathcal{F} + \mathcal{F}_0$  is a homeomorphism of  $\mathbf{R}_j$  onto itself. Since  $f_j$  is an additive

homomorphism, at least one of  $h$  and  $t \circ h$  does not have the assumed property.

The uniformities  $\mathcal{U}_k, \mathcal{U}_j$  in the next proposition are described in detail in R10.3.8.

**Proposition R26.29** Let  $k, j \geq 2$  be in  $\mathbf{N}$  and let  $\mathcal{U}_k, \mathcal{U}_j$  be the totally bounded uniformities corresponding to  $\mathbf{N}_k$  and  $\mathbf{N}_j$  respectively. Then  $\mathbf{N}_k$  and  $\mathbf{N}_j$  are homeomorphic topological spaces if and only if there exists a permutation  $\sigma$  of the positive integers such that  $\sigma : (\mathbf{N}, \mathcal{U}_k) \rightarrow (\mathbf{N}, \mathcal{U}_j)$  is a unimorphism.

Proof: Let  $\iota_k$  and  $\iota_j$  be the embeddings of  $\mathbf{N}$  into compactifications  $\mathbf{N}_k, \mathbf{N}_j$  respectively. First assume  $h$  is a homeomorphism from  $\mathbf{N}_k$  to  $\mathbf{N}_j$ . By R26.24  $h$  maps  $\iota_k[\mathbf{N}]$  onto  $\iota_j[\mathbf{N}]$ . Both  $\iota_k$  and  $\iota_j$  are unimorphisms onto their images and, by compactness,  $h$  is also a unimorphism. Thus  $\sigma = \iota_j^{-1} \circ h \circ \iota_k$  is the required permutation and unimorphism. Conversely let a unimorphism  $\sigma$  be given. By R7.1.3 there are unique continuous maps  $P : \mathbf{N}_k \rightarrow \mathbf{N}_j$  and  $Q : \mathbf{N}_j \rightarrow \mathbf{N}_k$  such that  $\sigma = \iota_j^{-1} \circ P \circ \iota_k$  and  $\sigma^{-1} = \iota_k^{-1} \circ Q \circ \iota_j$ . Letting  $id$  be the identity on  $\mathbf{N}$ ,  $id = \iota_j^{-1} \circ P \circ Q \circ \iota_j = \iota_k^{-1} \circ Q \circ P \circ \iota_k$ , i.e.,  $P \circ Q$  is the unique continuous map from  $\mathbf{N}_j$  to  $\mathbf{N}_j$  corresponding to  $id$  and  $Q \circ P$  is the unique continuous map from  $\mathbf{N}_k$  to  $\mathbf{N}_k$  corresponding to  $id$ . By uniqueness  $P \circ Q$  is the identity on  $\mathbf{N}_j$  and  $Q \circ P$  is the identity on  $\mathbf{N}_k$ . Thus  $P$  is a homeomorphism.

This section will conclude by showing that the remnant rings are homeomorphic to the Cantor set by using an argument based on auxiliary functions. In [1] Robert derives this result for the  $p$ -adic numbers, i.e. the remnant ring  $\mathbf{R}_p$  where  $p$  is prime, by a different method and also describes several linear and Euclidean models which illustrate the fractal nature of the spaces.

To begin, let's review the construction of the Cantor set and establish notation for what follows:  $C_1$  is obtained by removing the middle third from  $[0, 1]$ , i.e.,  $C_1$  is the union  $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ . Suppose  $C_n$  has been defined and is the union of  $2^n$  pairwise disjoint closed subintervals of  $[0, 1]$ . Define  $C_{n+1}$  the union of the  $2^{n+1}$  pairwise disjoint closed subintervals obtained by removing the the middle third from each of the subintervals making up  $C_n$ . Clearly  $C_{n+1} \subseteq C_n$  and by induction  $C_n$  is defined for every  $n$ . The Cantor set  $C$  is defined by  $C = \bigcap_{n=1}^{\infty} C_n$ .  $C$  is a non-empty, closed, compact subset of  $[0, 1]$ .

**Lemma R26.30** Let  $n \in \mathbf{N}$  and let  $[a, b]$  be one of the subintervals in the union making up  $C_n$ . Then  $|a - b| = 1/3^n$  and  $a, b \in C$ .

Proof: Removing the middle third of an interval leaves two closed intervals, each one-third the length of the original. This and a routine induction show that  $|a - b| = 1/3^n$ . For the second claim, note first that, since  $C_{j+1} \subseteq C_j$  for all  $j$ ,  $a, b \in C_i$  for all  $i \leq n$ . Moreover, when the middle third is removed from an interval, the endpoints are still in the result and are again endpoints of different intervals in the union making up the result. Thus  $a, b \in C_i$  for all  $i \geq n$ . Therefore  $a$  and  $b$  are in  $C$ .

Note that removing the middle third from an interval leaves a union of two subintervals, which can naturally be described as the left and right subintervals of the result.

**Definition R26.31** The map  $\phi_C : M_2 \rightarrow \mathcal{P}(C)$  is defined inductively:  $\phi_C(\eta) = C$ . Assume  $\phi_C$  has been defined for all words of length less than  $n$  and  $w \in M_2$  has length  $n$ . Let  $w = vx$  where  $\text{length}(v) = n - 1$  and  $x \in \{0, 1\}$ . Let  $\phi_C(v0)$  be  $\phi_C(v)$  intersected with the union of the left subintervals making up  $C_n$  and  $\phi_C(v1)$  be  $\phi_C(v)$  intersected with the union of the right subintervals making up  $C_n$ .

The diameter in the next lemma refers of course to the absolute value metric on  $[0, 1]$ .

**Lemma R26.32** The map  $\phi_C$  has the following properties.

- i) For every  $w \in M_2$ ,  $\phi_C(w)$  is a closed subset of  $C$ .
- ii) If  $w \in M_2$  has length  $n \geq 1$ ,  $\phi_C(w) = [a, b] \cap C$ , where  $[a, b]$  is one of the subintervals in the union making up  $C_n$ .
- iii) If  $w \in M_2$  has length  $n \geq 1$ ,  $\phi_C(w)$  is non-empty.
- iv) If  $w \in M_2$  has length  $n \geq 1$ ,  $\text{diam}(\phi_C(w)) = 1/3^n$ .
- v) For every  $w \in M_2$ ,  $\phi_C(w) = \phi_C(w0) \cup \phi_C(w1)$ .

Proof: Part i) follows by induction since  $C$  is closed and there are  $2^{n-1}$  left (or right) subintervals in the union making up  $C_n$ . For part ii),  $\phi_C(0) = [0, \frac{1}{3}] \cap C$  and  $\phi_C(1) = [\frac{2}{3}, 1] \cap C$ . Since  $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ , the claim holds for words of length one. Assume it holds for all words of length  $n$  and let  $w$  be of length  $n + 1$ . Then  $w = vx$  where  $v$  is of length  $n$  and  $x \in \{0, 1\}$ . By the induction hypothesis  $\phi_C(v) = [a, b] \cap C$ , where  $[a, b]$  is one of the subintervals making up  $C_n$ . Among the intervals making up  $C_{n+1}$ , only the left and right thirds of  $[a, b]$  have a possibly non-empty intersection with  $\phi_C(v)$ . By definition of  $\phi_C$  and the induction hypothesis,  $\phi_C(w)$  equals  $C$  intersected with either the left or right subinterval of  $[a, b]$ , both of which are subintervals in the union making up  $C_{n+1}$ . Part iii) is immediate from R26.30 and part ii). For part iv), let  $w$  have length  $n \geq 1$  and let  $\phi_C(w) = [a, b] \cap C$  where  $[a, b]$  is one of the subintervals in the union making up  $C_n$ . By R26.30  $a, b \in \phi_C(w)$  and so  $\text{diam}(\phi_C(w)) \leq |a - b| = 1/3^n$ . Since  $\phi_C(w) \subseteq [a, b]$ ,  $\text{diam}(\phi_C(w)) \leq 1/3^n$ . For v), given  $w \in M_2$  of length  $n$ , since  $C_{n+1}$  is the union of its left and right subintervals, by definition  $\phi_C(w0) \cup \phi_C(w1) = C_{n+1} \cap \phi_C(w)$ . Since  $\phi_C(w) \subseteq C \subseteq C_{n+1}$ , the desired equality holds.

**Proposition R26.33** The topological space  $\mathbf{R}_2$  is homeomorphic to the Cantor set.

Proof: By definition  $\phi_C(\eta) = C$  and so by i) and v) of the previous lemma  $\phi_C$  is an auxiliary function for  $C$ . Part iv) of the lemma shows that, if  $w$  is an infinite word,  $\text{diam}(w_n) \rightarrow 0$  as  $n \rightarrow \infty$ . That and part iii) of the allow an application of R26.13: There is a continuous, onto map  $f : \mathbf{R}_2 \rightarrow C$ . Since the spaces are compact and  $T_2$ , it is sufficient to show that the  $f$  constructed as in the proof of R26.13 is one-to-one. Let  $\mathcal{F} \neq \mathcal{G}$  be in  $\mathbf{R}_2$  with corresponding infinite words  $d(\mathcal{F}) = w$  and  $d(\mathcal{G}) = v$ . By R26.6  $w \neq v$  and so there is  $n$  with  $w_n \neq v_n$ . Assume  $n$  is the least such positive integer. Without loss of generality assume  $w_n = x0$  and  $v_n = x1$ , where  $x$  (possibly  $\eta$ ) has length  $n - 1$ . By definition  $\phi_C(w_n)$  is contained in the union of the left subintervals making up  $C_n$  and  $\phi_C(v_n)$  is contained in the union of the right subintervals making up  $C_n$ . Since these two unions are disjoint,  $\phi_C(w_n) \cap \phi_C(v_n) = \emptyset$ . By the construction of  $f$  in the proof of R26.13,  $f(\mathcal{F}) \in \phi_C(w_n)$  and  $f(\mathcal{G}) \in \phi_C(v_n)$ . Thus  $f(\mathcal{F}) \neq f(\mathcal{G})$ .

**Corollary R26.34** Let  $k \geq 2$  be in  $\mathbf{N}$ . The topological space  $\mathbf{R}_k$  is homeomorphic to the Cantor set.

Proof: This follows from R26.23 and the symmetry and transitivity of homeomorphism.

Comment: Robert [1] has an exercise (14 p.65), which asks the reader to show that a compact metric space  $E$  is path-connected if and only there is a continuous surjection  $f : [0, 1] \rightarrow E$ . In outline, the difficult half of this can be done as follows: Use R26.17 and R26.33 to obtain a continuous surjection  $f_0 : C \rightarrow E$ . Extend  $f_0$  inductively by using appropriate curves on the closures of the removed middle thirds. For example, obtain  $f_1$

on  $C \cup [\frac{1}{3}, \frac{2}{3}]$  by pasting together  $f_0$  and a continuous curve  $\gamma : [\frac{1}{3}, \frac{2}{3}] \rightarrow E$  such that  $\gamma(\frac{1}{3}) = f_0(\frac{1}{3})$  and  $\gamma(\frac{2}{3}) = f_0(\frac{2}{3})$ . Given  $f_n$ , extend it to  $f_{n+1}$  in a similar way with finitely many curves. The desired map  $f$  is  $\cup_{n=0}^{\infty} f_n$ . This result shows the existence of a continuous surjection from  $[0, 1]$  onto  $[0, 1] \times [0, 1]$ , i.e., a space-filling curve.

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### References

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