The Remnant Rings Are Homeomorphic

In [6] the remnant rings $\mathbf{R}_k$ (where $k \geq 2$ is in $\mathbb{N}$) were shown to be compactifications of the integers $\mathbf{Z}$ with certain non-discrete topologies, which vary with k. It will first be shown here that, for relatively prime a, b, $\mathbf{R}_a$ and $\mathbf{R}_b$ are not equivalent compactifications in the generalized sense of [5].

The main result is that the topological space $\mathbf{R}_2$ is homeomorphic to $\mathbf{R}_k$ for all $k \geq 2$ in $\mathbb{N}$. Since, for p prime, $\mathbf{R}_p$ is topologically isomorphic to the p-adic integers by R17.1.19, this is a slight generalization of the known result that, for p, q prime, the p-adic integers are homeomorphic to the q-adic integers. The proof of the main result is based on the presentation of that fact in exercises 12 and 13, p. 65, of [1].

The remnant rings are also shown to be homeomorphic to the Cantor set and some results are derived on the question of when $\mathbb{N}_j$ and $\mathbb{N}_k$ are homeomorphic topological spaces.

The first proposition makes use of the following notation from [6]: Let $k \geq 2$ be in $\mathbb{N}$. For $n \in \mathbb{N}$ and $x \in \mathbf{Z}$, $D_n^x(k)$ is the equivalence class in $\mathbf{Z}$ of x mod k^n. In [6] it is shown that $\{D_n^x(k) : n \in \mathbb{N} \text{ and } x \in \mathbf{Z}\}$ is a clopen basis for a topology τ_k on $\mathbf{Z}$ and that $\mathbf{R}_k$ with a suitable embedding is a compactification of $(\mathbf{Z}, \tau_k)$.

Proposition R26.1 Let $a, b \geq 2$ be in $\mathbb{N}$ with $(a, b) = 1$. Then $\mathbf{R}_a \not\leq \mathbf{R}_b$.

Proof: By R13.1.5i it is sufficient to show that $\tau_a \not\subseteq \tau_b$, which will be done by showing $D_1^1(a)$ is not in τ_b. Deny. Then there are $n \in \mathbb{N}$ and $x \in \mathbf{Z}$ such that $1 \in D_n^x(b) \subseteq D_1^1(a)$. Since these are equivalence classes, $D_n^x(b) = D_n^1(b) \subseteq D_1^1(a)$. Then $1 + b^n \equiv 1 \bmod a$ so that $b^n \equiv 0 \bmod a$, i.e., a divides b^n, which contradicts the hypothesis that a and b are relatively prime.

Comment: In the last proposition the roles of a and b could be reversed thereby showing that the compactifications $\mathbf{R}_a$ and $\mathbf{R}_b$ are not related in either direction and, of course, are not equivalent. In R10.3.5 it was shown that, for two distinct primes p and q, $\mathbb{N}_p \not\leq \mathbb{N}_q$, a fact which could easily be generalized to a relatively prime pair. The proof of R10.3.5 could be modified to verify R26.1, but the approach used in R26.1 can't be used to prove R10.3.5, since $\mathbb{N}_p$ and $\mathbb{N}_q$ are both compactifications of $\mathbb{N}$ with the discrete topology.

Before presenting the main result, some needed properties of the derived sequence of an element in $\mathbf{R}_k$ for $k \geq 2$ will be developed. Recall that each element of $\mathbf{R}_k$ determines two related but distinct sequences: the derived sequence (defined in R20.1) of the form $\{a_n\}_{n=0}^{\infty}$, where $a_n \in \{0, \ldots, k-1\}$ for all n, and the associated sequence (defined in R10.2.3) of the form $\{x_n\}_{n=1}^{\infty}$, where $x_n \in \{1, \ldots, k^n\}$ for all n.

Lemma R26.2 Let $\{a_n\}_{n=0}^{\infty}$ be a sequence with each a_n in $\{0, \ldots, k-1\}$, where $k \geq 2$ is in $\mathbb{N}$. Then there is $\mathcal{F}(\{a_n\}_{n=0}^{\infty})$ in $\mathbf{R}_k$ such that the derived sequence of $\mathcal{F}(\{a_n\}_{n=0}^{\infty})$ is $\{a_n\}_{n=0}^{\infty}$.

Proof: First assume $a_n = 0$ for every n. Let $\mathcal{F}(\{a_n\}_{n=0}^{\infty})$ be the additive identity of $\mathbf{R}_k$, which has associated sequence $\{k^n\}_{n=1}^{\infty}$. By definition the derived sequence of $\mathcal{F}(\{a_n\}_{n=0}^{\infty})$ is $\{a_n\}_{n=0}^{\infty}$.

Now assume there is n with $a_n \neq 0$ and let m be the smallest of $\{n : a_n \neq 0\}$. If $m = 0$, let $x_1 = a_0$ and let $x_{n+1} = x_n + a_n k^n$ for $n \geq 1$. By R10.2.6 there is a unique $\mathcal{F}(\{a_n\}_{n=0}^{\infty})$ in $\mathbf{R}_k$ which is associated with $\{x_n\}_{n=1}^{\infty}$. Let $\{b_n\}_{n=0}^{\infty}$ be the derived sequence of $\mathcal{F}(\{a_n\}_{n=0}^{\infty})$. Since $x_1 = a_0 \neq k$, by definition $b_0 = x_1 = a_0$ and, for $n \geq 1$, $b_n = (x_{n+1} - x_n)/k^n = a_n$. Thus the derived sequence of $\mathcal{F}(\{a_n\}_{n=0}^{\infty})$ is $\{a_n\}_{n=0}^{\infty}$.

As a final case assume $m \geq 1$. Let $x_i = k^i$ for $1 \leq i \leq m$, let $x_{m+1} = k^m a_m$, and let $x_{i+1} = x_i + a_i k^i$ for $i \geq m+1$. Note that, for $1 \leq i \leq m-1$, $x_{i+1} = k^i + (k-1)k^i = x_i + (k-1)k^i$. Also, since $a_m \in \{1, \ldots, k-1\}$, $x_{m+1} = k^m + (a_m - 1)k^m = x_m + (a_m - 1)k^m$. Thus R10.2.6 applies: there is $\mathcal{F}(\{a_n\}_{n=0}^{\infty})$ in $\mathbf{R}_k$ such that $\mathcal{F}(\{a_n\}_{n=0}^{\infty})$ is associated with $\{x_n\}_{n=1}^{\infty}$. Let $\{b_n\}_{n=0}^{\infty}$ be the derived sequence of $\mathcal{F}(\{a_n\}_{n=0}^{\infty})$. Note that the smallest of $\{n : x_n \neq k^n\}$ is $m+1$ and so $b_i = 0 = a_i$ for $0 \leq i \leq m-1$. Also by definition, $b_m = x_{m+1}/k^m = a_m$. Finally, for $i \geq m+1$, by definition $b_i = (x_{i+1} - x_i)/k^i = a_i$. Thus $b_n = a_n$ for all n, i.e., the derived sequence of $\mathcal{F}(\{a_n\}_{n=0}^{\infty})$ is $\{a_n\}_{n=0}^{\infty}$.

Lemma R26.3 Let $k \geq 2$ in $\mathbb{N}$, let $\mathcal{F}$ be in $\mathbf{R}_k$, and let $\{a_n\}_{n=0}^{\infty}$ be the derived sequence of $\mathcal{F}$. Then $\mathcal{F}(\{a_n\}_{n=0}^{\infty}) = \mathcal{F}$.

Proof: Let $\{y_n\}_{n=1}^{\infty}$ be the associated sequence of $\mathcal{F}$. If $y_n = k^n$ for every n, then $\mathcal{F}$ is the additive identity of $\mathbf{R}_k$ as shown in [4], especially R12.5.9. By definition $a_n = 0$ for every n and by the construction in the previous lemma, $\mathcal{F}(\{a_n\}_{n=0}^{\infty}) = \mathcal{F}$. Now assume that $y_n \neq k^n$ for some n and let l be the smallest of $\{n : y_n \neq k^n\}$. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence associated with $\mathcal{F}(\{a_n\}_{n=0}^{\infty})$, as constructed in the proof of R26.2. If $l = 1$, by R20.1 $a_0 = y_1$ and $a_n = (y_{n+1} - y_n)/k^n$ for $n \geq 1$. Since $y_1 \neq 0$, by construction, $x_1 = a_0 = y_1$ and $x_{n+1} = x_n + a_n k^n$ for all $n \geq 1$. Induction and routine algebra show that $x_{n+1} = y_{n+1}$ for $n \geq 1$. By R10.2.4 $\mathcal{F}(\{a_n\}_{n=0}^{\infty}) = \mathcal{F}$. Finally, suppose $l > 1$. By R20.1 $a_0 = a_1 = \cdots = a_{l-2} = 0$, $a_{l-1} = y_l/k^{l-1}$, and, for $n \geq l$, $a_n = (y_{n+1} - y_n)/k^n$. The smallest of $\{n : a_n \neq 0\}$ is $l-1$. By construction, $x_n = k^n$ for $1 \leq n \leq l-1$, $x_l = k^{l-1}a_{l-1}$, and, for $n \geq l$, $x_{n+1} = x_n + a_n k^n$. By the choice of l, $y_n = k^n = x_n$ for $1 \leq n \leq l-1$. Also $x_l = k^{l-1}(y_l/k^{l-1}) = y_l$. For $n \geq l$, induction and routine algebra show $x_{n+1} = y_{n+1}$. Again by R10.2.4 $\mathcal{F}(\{a_n\}_{n=0}^{\infty}) = \mathcal{F}$.

Corollary R26.4 Let $k \geq 2$ in $\mathbb{N}$ and let $\mathcal{F}$ and $\mathcal{G}$ be in $\mathbf{R}_k$. Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be the derived sequences of $\mathcal{F}$ and $\mathcal{G}$ respectively. Then $\mathcal{F} = \mathcal{G}$ if and only if $a_n = b_n$ for all $n \geq 0$.

Proof: If $\mathcal{F} = \mathcal{G}$, each has the same associated sequence and so the same derived sequence. If $a_n = b_n$ for every n, then $\mathcal{F}(\{a_n\}_{n=0}^{\infty}) = \mathcal{F}(\{b_n\}_{n=0}^{\infty})$ and so By R26.3 $\mathcal{F} = \mathcal{G}$.

Next some terminology and notation used in Robert's hint will be presented: A word is a finite string with characters from $\{0, 1\}$ or, formally, a function from an initial segment of $\mathbb{N}$ into $\{0, 1\}$. Note that the function from $\emptyset$ is allowed. It will be called the null word or η. Each word has a length, which is its number of characters, i.e., the number of natural numbers in its domain. The length of η is 0. With the operation of concatenation, M_2, the set of all words, is a non-commutative semi-group with unit η, i.e., a monoid. An infinite word is a sequence in $\{0, 1\}$. If w denotes the infinite word $\{t_i\}_{i=1}^{\infty}$, $w_0 = \eta$ and w_n is the word $\{t_i\}_{i=1}^{n}$. For an infinite word w, $w \notin M_2$ but $w_n \in M_2$ for all n and the length of w_n is n. Usually string notation will be used instead of the more formal functional representation, e.g., 101 will denote the word $\{t_i\}_{i=1}^{3}$ with $t_1 = 1$, $t_2 = 0$, and $t_3 = 1$.

Definition R26.5 Define d from $\mathbf{R}_2$ to the set of infinite words by $d(\mathcal{F}) = \{a_n\}_{n=0}^{\infty}$, where $\{a_n\}_{n=0}^{\infty}$ is the derived sequence of $\mathcal{F}$.

Note: If one uses string notation, because the derived sequence indexing begins at 0, $d(\mathcal{F})_0 = \eta$, $d(\mathcal{F})_1 = a_0$, and $d(\mathcal{F})_{n+1} = d(\mathcal{F})_n a_n$.

Lemma R26.6 The map d is a bijection.

Proof: Each $\mathcal{F}$ in $\mathbf{R}_2$ has a unique associated sequence and so a unique derived sequence. Thus d is indeed a function. R26.2 shows that d is onto and R26.4 shows that d is one-to-one.

Lemma R26.7 Let $k \geq 2$ and $\mathcal{F}, \mathcal{G}$ be in $\mathbf{R}_k$ with associated sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ respectively. If $x_m = y_m$ for some m, then $x_j = y_j$ for $1 \leq j \leq m$.

Proof: By induction: The conclusion clearly holds for $m = 1$. Assume it holds for m and suppose $x_{m+1} = y_{m+1}$. By R10.2.5i $x_{m+1} = x_m + sk^m$ and $y_{m+1} = y_m + tk^m$, where $s, t \in \{0, 1, \ldots, k-1\}$. Then $x_m - y_m = (t-s)k^m$, i.e., if $s \neq t$, k^m divides $x_m - y_m$. Since $1 \leq x_m, y_m \leq k^m$, $|x_m - y_m| \leq k^m - 1$ and so $s = t$, i.e., $x_m = y_m$. By the induction hypothesis, the conclusion holds for $m+1$.

Lemma R26.8 Let $\mathcal{F}, \mathcal{G}$ be in $\mathbf{R}_2$ with associated sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ respectively. If $x_m = y_m$ for some m, then $d(\mathcal{F})_j = d(\mathcal{G})_j$ for $0 \leq j \leq m$.

Proof: Let $\{a_n\}_{n=0}^{\infty}$ be the derived sequence of $\mathcal{F}$ and let $\{b_n\}_{n=0}^{\infty}$ be the derived sequence of $\mathcal{G}$. By the definition of d, it is sufficient to show $a_n = b_n$ for $0 \leq n \leq m-1$. As a first case, suppose $x_m = k^m$. Since the additive identity for $\mathbf{R}_2$ has associated sequence $\{k^n\}_{n=1}^{\infty}$, by R26.7 $x_n = k^n = y_n$ for $1 \leq n \leq m$. By definition of the derived sequence $a_n = b_n = 0$ for $0 \leq n \leq m-1$. Now suppose $x_m \neq k^m$ and let l be the smallest of $\{n : x_n \neq k^n\}$. Clearly $l \leq m$. Since $x_m = y_m$, by R26.7 l is also the smallest of $\{n : y_n \neq k^n\}$. If $l = 1$, by definition $a_0 = x_1 = y_1 = b_0$ and, for $n \geq 1$, $a_n = (x_{n+1} - x_n)/k^n$ and $b_n = (y_{n+1} - y_n)/k^n$. Since $x_n = y_n$ for $1 \leq n \leq m$, $a_n = b_n$ for $0 \leq n \leq m-1$. If $l > 1$, by definition $a_n = 0 = b_n$ for $0 \leq n \leq l-2$, $a_{l-1} = x_l/k^{l-1} = y_l/k^{l-1} = b_{l-1}$, and, for $n \geq l$, $a_n = (x_{n+1} - x_n)/k^n$ and $b_n = (y_{n+1} - y_n)/k^n$. For $0 \leq n \leq m-1$, it follows that $a_n = b_n$.

Lemma R26.9 Let $\{\mathcal{F}_i\}$ be a sequence in $\mathbf{R}_2$ and let $\mathcal{F}$ be in $\mathbf{R}_2$. Assume $\mathcal{F}_i \to \mathcal{F}$. Then for every $m \in \mathbb{N}$ there is $N(m)$ in $\mathbb{N}$ such that $i \geq N(m)$ implies $d(\mathcal{F}_i)_j = d(\mathcal{F})_j$ for $0 \leq j \leq m$.

Proof: Fix m in $\mathbb{N}$, let $\{{}^i x_n\}_{n=1}^{\infty}$ be the associated sequence of $\mathcal{F}_i$, and let $\{x_n\}_{n=1}^{\infty}$ be the associated sequence of $\mathcal{F}$. By R17.2.16 There is $N(m)$ such that $i \geq N(m)$ implies ${}^i x_m = x_m$. When $j \geq N(m)$, R26.8 implies $d(\mathcal{F}_i)_j = d(\mathcal{F})_j$ for $0 \leq j \leq m$.

Definition R26.10 An auxiliary function for a topological space (X, τ) is a map $\phi : M_2 \to \mathcal{P}(X)$ such that, for every $w \in M_2$, $\phi(w)$ is closed, $\phi(\eta) = X$ and, for every $w \in M_2$, $\phi(w) = \phi(w0) \cup \phi(w1)$.

Lemma R26.11 Let ϕ be an auxiliary function for the topological space (X, τ) and let w be an infinite word. Then $\phi(w_{n+1}) \subseteq \phi(w_n)$ for every $n \geq 0$.

Proof; Since w_{n+1} is either $w_n 0$ or $w_n 1$ and $\phi(w_n) = \phi(w_n 0) \cup \phi(w_n 1)$, the conclusion clearly holds.

Recall that the diameter of a non-empty set in a metric space is the supremum of distances of points in in the set. Of course, even for equivalent metrics, the diameter of a given set is not invariant. As an extreme example, in the interval $(0, 1)$, the subinterval $(0, 1/n)$ has diameter $1/n$ with the usual absolute value metric but, with the equivalent metric $\rho(x, y) = |1/x - 1/y|$, it has infinite diameter. The next lemma (undoubtedly known) could be omitted but shows that the subsequent proposition does not depend on a specific metric.

Lemma R26.12 Let ρ and σ be equivalent metrics on a set X and let $\{A_n\}$ be a sequence of non-empty subsets of X. Assume $\cap_{n=1}^{\infty} A_n \neq \emptyset$. If $\mathrm{diam}_\rho(A_n) \to 0$ as $n \to \infty$, then $\mathrm{diam}_\sigma(A_n) \to 0$ as $n \to \infty$.

Proof: Assume $\mathrm{diam}_\rho(A_n) \to 0$ as $n \to \infty$ and let $a \in \cap_{n=1}^{\infty} A_n$. Let $\epsilon > 0$. By definition equivalent metrics generate the same topology and so $B^\sigma_\epsilon(a)$ (the open σ-ball of radius ϵ centered at a) must contain $B^\rho_\gamma(a)$ for some $\gamma > 0$. By hypothesis there is M such that, for $n \geq M$, $\mathrm{diam}_\rho(A_n) < \gamma$ and so $A_n \subseteq B^\rho_\gamma(a)$. Thus $n \geq M$ implies $\mathrm{diam}_\sigma(A_n) \leq 2\epsilon$.

The next proposition is a version of problem 12 in [1: p.65].

Proposition R26.13 Let ϕ be an auxiliary function for a compact, metrizable space (E, τ). Assume that, for every word ν, $\phi(\nu) \neq \emptyset$ and that, for every infinite word w, $\mathrm{diam}(\phi(w_n)) \to 0$ as $n \to \infty$. Then there is a continuous surjection $f : \mathbf{R}_2 \to E$.

Proof: For any infinite word w, the collection of closed sets $\{\phi(w_n) : n \geq 0\}$ has the finite intersection property by hypothesis and R26.11 and so, since E is compact, $\cap_{n=0}^{\infty}\phi(w_n) \neq \emptyset$. Suppose $a \neq b$ are both in $\cap_{n=0}^{\infty}\phi(w_n)$ and let ϵ be smaller than the distance from a to b. Since $\mathrm{diam}(\phi(w_n)) \to 0$ (for any metric generating the topology of E by R26.12), there is n such that $\mathrm{diam}(\phi(w_n)) < \epsilon$, which contradicts $a, b \in \phi(w_n)$. Thus $\cap_{n=0}^{\infty}\phi(w_n)$ is a singleton and the map f can be defined by letting $f(\mathcal{F})$ be the unique element of $\cap_{n=0}^{\infty}\phi(d(\mathcal{F})_n)$ for each $\mathcal{F}$ in $\mathbf{R}_2$. To see that f is onto, let $e \in E$ and construct a sequence in $\{0,1\}$ as follows: Since $e \in E = \phi(\eta) = \phi(0) \cup \phi(1)$, pick $t_1 \in \{0,1\}$ such $e \in \phi(t_1)$. Now assume $t_1, \ldots, t_n$ chosen with $e \in \phi(\{t_i\}_{i=1}^{j})$ for all $1 \leq j \leq n$. Since $e \in \phi(\{t_i\}_{i=1}^{n}) = \phi(\{t_i\}_{i=1}^{n}0) \cup \phi(\{t_i\}_{i=1}^{n}1)$, pick t_{n+1} in $\{0,1\}$ such that $e \in \phi(\{t_i\}_{i=1}^{n+1})$. By induction we have an infinite word $w = \{t_i\}_{i=1}^{\infty}$ such that $e \in \phi(w_n)$ for every n. By R26.6 there is $\mathcal{F}$ in $\mathbf{R}_2$ such that $d(\mathcal{F}) = w$. By definition $f(\mathcal{F}) = e$. Lastly, for continuity, let $\{\mathcal{F}_i\}$ be a sequence in $\mathbf{R}_2$ and let $\mathcal{F}$ be in $\mathbf{R}_2$ with $\mathcal{F}_i \to \mathcal{F}$. It will be shown that $f(\mathcal{F}_i) \to f(\mathcal{F})$. Let $\epsilon > 0$. Since $\mathrm{diam}(\phi(d(\mathcal{F})_n)) \to 0$ as $n \to \infty$ and $f(\mathcal{F}) \in \phi(d(\mathcal{F})_n)$ for all n, there is $m \in \mathbb{N}$ such that $\phi(d(\mathcal{F})_m)$ is contained in $B_\epsilon(f(\mathcal{F}))$, the open ϵ-ball centered at $f(\mathcal{F})$. By R26.9 there is $N(m)$ in $\mathbb{N}$ such that $i \geq N(m)$ implies $d(\mathcal{F}_i)_j = d(\mathcal{F})_j$ for $0 \leq j \leq m$. Then $i \geq N(m)$ implies $f(\mathcal{F}_i) \in \phi(d(\mathcal{F}_i)_m) = \phi(d(\mathcal{F})_m) \subseteq B_\epsilon(f(\mathcal{F}))$ and the claim is verified. Thus f is continuous.

The next lemma and R26.16 provide a way to construct auxiliary functions.

Lemma R26.14 Let (X, τ) be a topological space and let $X = \cup_{i=1}^{t} F_i$, where each F_i is a non-empty closed subset and $t \in \mathbb{N}$. Then there is a finite set $D \subseteq M_2$ and ψ from D to the non-empty closed subsets of X such that

i) $\psi(\eta) = X$.

ii) If $w \in D$ and $v \in M_2$ with $\mathrm{length}(v) < \mathrm{length}(w)$, then $v \in D$.

iii) If $wv \in D$, then $\psi(wv) \subseteq \psi(w)$.

iv) There is $j \geq 1$ such that $\{w \in M_2 : \mathrm{length}(w) \leq j\} \subseteq D$.

v) $\psi(0) \cup \psi(1) = X$.

vi) If $w \in D$, then $w0 \in D$ if and only if $w1 \in D$.

vii) If w and $w0$ are in D, then $\psi(w) = \psi(w0) \cup \psi(w1)$.

viii) For every i there is $w \in D$ such that $\psi(w) = F_i$ and $w0 \notin D$.

ix) If $w \in D$ and $w0 \notin D$, then $\psi(w) = F_i$ for some i.

Proof: If $t = 1$, let $\psi(\eta) = \psi(0) = \psi(1) = F_1 = X$. Now assume $t \geq 2$. Define

$\psi(\eta) = X$ and let j be the unique positive integer such that $2^j \leq t < 2^{j+1}$. Let S_j be the set $\{w \in M_2 : \text{length}(w) = j\}$. By induction $|S_j| = 2^j$ and so there is a bijection f from S_j to $\{1, \ldots, 2^j\}$. For $w \in S_j$, define $\psi(w) = F_{f(w)}$ if $f(w) + 2^j > t$ and $\psi(w) = F_{f(w)} \cup F_{f(w)+2^j}$ if $f(w) + 2^j \leq t$. Also, if $f(w) + 2^j \leq t$, define $\psi(w0) = F_{f(w)}$ and $\psi(w1) = F_{f(w)+2^j}$. So far ψ is defined on $\{\eta\} \cup S_j \cup \{w0, w1 : w \in S_j \text{ and } f(w) + 2^j \leq t\}$. Also, if $w \in S_j$ and $f(w) + 2^j \leq t$, $\psi(w) = \psi(w0) \cup \psi(w1)$. Now extend the definition of ψ by a finite 'downward induction.' Let $S_s = \{w \in M_2 : \text{length}(w) = s\}$ and assume ψ has been defined on all of S_s, where $2 \leq s \leq j$. For $w \in S_{s-1}$, define $\psi(w) = \psi(w0) \cup \psi(w1)$. After a finite number of steps, i), iv) and ix) hold by definition. Clearly, for each $w \in D$ (the domain of ψ), $\psi(w)$ is a union of some non-empty subcollection of $\{F_1, \ldots, F_t\}$ and so is non-empty and closed. By definition D may include some words of length $j+1$, and these are the longest words in D. Thus iv) implies ii). The fact that words of length $j+1$ are included in D in pairs ($w0$ <u>and</u> $w1$) along with iv) imply vi). That and the definition yield vii). Let $1 \leq k \leq t$. If $k \leq 2^j$, there is $v \in S_j$ with $f(v) = k$. For $k + 2^j > t$, the initial step of the definition shows that $\psi(v) = F_k$, and, for $k + 2^j \leq t$, it says $\psi(v0) = F_k$. If $k > 2^j$, there is $v \in S_j$ with $f(v) = k - 2^j$ and by definition $\psi(v1) = F_k$. Thus viii) holds. For iii), suppose $wv \in D$. Proceed by induction on the length of v. If $v = \eta$, then $\psi(wv) = \psi(w)$ and the conclusion holds. Now assume the conclusion holds for any v of length s and suppose length$(v) = s+1$. Then $v = u0$ or $v = u1$, where length$(u) = s$. By ii) $wu \in D$ and by vii) and the induction hypothesis, $\psi(w) \supseteq \psi(wu) = \psi(wu0) \cup \psi(wu1) \supseteq \psi(wv)$. Lastly, for v), let $x \in X$. There is k with $x \in F_k$. By the initial step of ψ's definition there is $w \in D$ of length j or $j+1$ with $\psi(w) = F_k$. Since $w \neq \eta$, $w = 0u$ or $w = 1u$ and by iii) $x \in \psi(w) \subseteq \psi(0) \cup \psi(1)$. Conversely, since each value of ψ is a union of some subcollection of $\{F_1, \ldots, F_t\}$, clearly $\psi(0) \cup \psi(1) \subseteq X$.

Lemma R26.15 Let (X, τ) be a topological space and let $X = \cup_{i=1}^t F_i$, where each F_i is a non-empty closed subset and $t \geq 2$ is in $\mathbb{N}$. Assume $\{F_i : 1 \leq i \leq t\}$ is a pairwise disjoint collection. Let ψ be constructed as in the proof of R26.14. Assume $u \neq v$ are in dom(ψ) with length(u) = length(v). Then $\psi(u) \cap \psi(v) = \emptyset$. In addition, if $x \neq y$ are in dom(ψ) with $x0, y0 \notin$ dom(ψ), then $\psi(x) \cap \psi(y) = \emptyset$.

Proof: Let j be the unique positive integer such that $2^j \leq t < 2^{j+1}$ and use the notation defining ψ from the proof of R26.14. Since η is the only word of length 0, length$(u) \geq 1$. First assume u and v are in the initial part of the domain of ψ, i.e., $\{\eta\} \cup B_j \cup \{w0, w1 : w \in B_j \text{ and } f(w) + 2^j \leq t\}$. If length$(u) = j+1$, since f is one-to-one, it is easy to check that $\psi(u) = F_i$ and $\psi(v) = F_l$ where $i \neq l$. By the pairwise disjointness assumption $\psi(u) \cap \psi(v) = \emptyset$. Now assume length$(u) = j$. If $f(u) + 2^j > t$ and $f(v) + 2^j > t$, $\psi(u) = F_{f(u)}$ and $\psi(v) = F_{f(v)}$, which are disjoint since f is one-to-one. If $f(u) + 2^j \leq t$ and $f(v) + 2^j \leq t$, $\psi(u) = F_{f(u)} \cup F_{f(u)+2^j}$ and $\psi(v) = F_{f(v)} \cup F_{f(v)+2^j}$. Since f is one-to-one and has values 1 or larger, by pairwise disjointness $\psi(u) \cap \psi(v) = \emptyset$. If $f(u) + 2^j \leq t$ and $f(v) + 2^j > t$, $\psi(u) = F_{f(u)} \cup F_{f(u)+2^j}$ and $\psi(v) = F_{f(v)}$. For the same reasons, $\psi(u) \cap \psi(v) = \emptyset$. The fourth case is similar to the third. Now assume the conclusion holds for any two words of length s, where $j \geq s \geq 2$, and suppose length$(u) = s-1$. By R26.14 $\psi(u) = \psi(u0) \cup \psi(u1)$ and $\psi(v) = \psi(v0) \cup \psi(v1)$. Since $u \neq v$, $u0, u1, v0, v1$ are four distinct words of length s. By the inductive hypothesis, the first conclusion holds. Now assume $x \neq y$ are in dom(ψ) with $x0, y0 \notin$ dom(ψ). By R26.14iv neither is η and both are in the initial part of $\mathrm{dom}(\psi)$. If both have the same length, the first conclusion applies. If one, say x, has length j and the other has length $j+1$, there is z of length j such that y is either $z0$ or $z1$. By the first conclusion $\psi(x) \cap \psi(z) = \emptyset$ and by R26.14vi $\psi(y) \subseteq \psi(z)$ so that $\psi(x) \cap \psi(y) = \emptyset$.

Lemma R26.16 Let E be a compact metric space. Then there is an auxiliary function ϕ for E such that $\phi(v) \neq \emptyset$ for every $v \in M_2$ and, if w is an infinite word, $\mathrm{diam}(\phi(w_n)) \to 0$ as $n \to \infty$.

Proof: First, a sequence of maps will be defined by induction. Let ρ be a metric for E and, for $e \in E$ and $\epsilon > 0$, let $\overline{B_\epsilon(e)}$ denote the closure of the open ϵ-ball centered at e. Pick $m_1 \geq 1$. By compactness applied to the open balls, there are $e_1, \ldots, e_t$ such that $E = \cup_{i=1}^{t} \overline{B_{1/m_1}(e_i)}$. Apply R26.14 to obtain ϕ_1 on domain D_1 to the collection of unions from $\{\overline{B_{1/m_1}(e_1)}, \ldots, \overline{B_{1/m_1}(e_t)}\}$, where ϕ_1 has the lemma's properties, which easily imply the 8 properties listed below. Now assume $\phi_1, \ldots, \phi_n$ with domains $D_1 \subseteq D_2 \subseteq \cdots \subseteq D_n$ have been constructed with the following properties:

i) $D_n \subseteq M_2$ is finite and $\{w \in M_2 : \mathrm{length}(w) \leq n\} \subseteq D_n$.
ii) For $2 \leq i \leq n$, $\phi_{i-1} = \phi_i|_{D_{i-1}}$.
iii) $\phi_n(\eta) = E$.
iv) If $wv \in D_n$, then $w \in D_n$.
v) For $w \in D_n$, $w0 \in D_n$ if and only if $w1 \in D_n$.
vi) For $w \in D_n$ with $w0 \in D_n$, $\phi_n(w) = \phi_n(w0) \cup \phi_n(w1)$.
vii) If $w \in D_n$ and $w0 \notin D_n$, $\mathrm{diam}(\phi_n(w)) \leq 2/n$.
viii) For every $w \in D_n$, $\phi_n(w)$ is a non-empty closed set.

Now pick $m_{n+1} \geq n+1$ and call $w \in M_2$ a terminal word (of D_n) if $w \in D_n$ and $w0 \notin D_n$. There are finitely many terminal words. Note that no proper initial subword of a terminal word is a terminal word by properties iv) and v). For each terminal word w, by compactness $\phi_n(w) = \cup_{i=1}^{t(w)} (\overline{B_{1/m_{n+1}}(x_i(w))} \cap \phi_n(w))$, where $t(w)$ in $\mathbb{N}$ and $x_1(w), \ldots, x_{t(w)}(w)$ in $\phi_n(w)$. By R26.14 there is ψ_w from a finite domain A_w to the collection of unions from the family $\{\overline{B_{1/m_{n+1}}(x_i(w))} \cap \phi_n(w) : 1 \leq i \leq t(w)\}$, where ψ_w has the properties listed in R26.14. The finite set $\{wv : w$ is a terminal word and $v \in A_w, v \neq \eta\}$ will be denoted $C_{n+1}(w)$. Let C_{n+1} be the union of the $C_{n+1}(w)$ over all terminal words. Note that if $wv = us$ where w, u are terminal words, then $w = u$, since neither can be a proper initial subword of the other, and so also $v = s$. Thus each element of C_{n+1} has a unique representation in the specified form. Finally note that the requirement $v \neq \eta$ implies $D_n \cap C_{n+1} = \emptyset$. Let $D_{n+1} = D_n \cup C_{n+1}$. Define ϕ_{n+1} on D_{n+1} by $\phi_{n+1}(u) = \phi_n(u)$ if $u \in D_n$ and, if $u \in C_{n+1}$ with $u = wv$, $\phi(u) = \psi_w(v)$. Then ϕ_{n+1} is a function; clearly $D_n \subseteq D_{n+1}$ and ii), iii), and viii) hold for ϕ_{n+1}. For i), finiteness holds since each A_w is finite. Let w be a word of length $n+1$. Then $w = ux$, where $\mathrm{length}(u) = n$ and $x \in \{0,1\}$. Then $u \in D_n$. If $u0 \in D_n$, $w \in D_n \subseteq D_{n+1}$. Otherwise u is a terminal word of D_n and, since $\{0,1\} \subseteq A_u$, $w = ux \in C_{n+1} \subseteq D_{n+1}$. For iv), let $wv \in D_{n+1}$. Since iv) holds for D_n, assume $wv \in C_{n+1}$. Then $wv = us$, where u is a terminal word of D_n and $s \neq \eta$ is in A_u. If w is a subword of u, $u = wr$ and so $w \in D_n$ by the induction hypothesis. Otherwise u must be a proper subword of w, i.e., $w = ur$ where $r \neq \eta$. Then $wv = urv$ and so $rv = s \in A_u$. By R26.14ii $r \in A_u$ and so $w = ur \in C_{n+1} \subseteq D_{n+1}$. For v) let $w \in D_{n+1}$ with $w0 \in D_{n+1}$. If $w0 \in D_n$, by iv) and v) of the induction hypothesis, $w1 \in D_n \subseteq D_{n+1}$. If $w0 \in C_{n+1}$, $w0 = us$, where u is a terminal word of D_n and $s \neq \eta$ is in A_u. Now $s = r0$ and by R26.14vi $r1 \in A_u$. Thus $w1 = ur1$ is in $C_{n+1} \subseteq D_{n+1}$. The converse is similar. For vi) let $w0$ (and so w) be in D_{n+1}. If $w0 \in D_n$, since $\phi_n = \phi_{n+1}|_{D_n}$, apply vi) of the induction hypothesis for D_n. If $w0 \in C_{n+1}$, $w0 = us0$, where u is a terminal word of D_n and $s0 \in A_u$. By R26.14ii $s \in A_u$ and so $w = us \in C_{n+1}$. Applying the definition and R26.14vii, $\phi_{n+1}(w) = \psi_u(s) = \psi_u(s0) \cup \psi_u(s1) = \phi_{n+1}(w0) \cup \phi_{n+1}(w1)$. For vii) let $w \in D_{n+1}$ with $w0 \notin D_{n+1}$. Note that $w \notin D_n$ for, if so, w is a terminal word of D_n and $0 \in A_w$ so that $w0$ would be in $C_{n+1} \subseteq D_{n+1}$. Thus $w = us$, where u is a terminal word of D_n and $s \neq \eta$ is in A_u. Since $w0 = us0$ is not in C_{n+1}, $s0 \notin A_u$. By R26.14ix and the choice of ψ_u, $\phi_{n+1}(w) = \psi_u(s) \subseteq \overline{B_{1/m_{n+1}}(x)}$ for some x. Thus $\operatorname{diam}(\phi_{n+1}(w)) \leq 2/m_{n+1} \leq 2/(n+1)$. By induction there is an infinite sequence $\{\phi_n\}_{n=1}^{\infty}$ with the properties listed above. Since the domains are nested and ii) holds, $\phi = \cup_{n=1}^{\infty}\phi_n$ is a function and $\phi|_{D_n} = \phi_n$ for every n. Since i) holds, the domain of ϕ is M_2. For $w \in M_2$, pick n with $w0 \in D_n$. By vi) $\phi(w) = \phi_n(w) = \phi_n(w0) \cup \phi_n(w1) = \phi(w0) \cup \phi(w1)$. Thus ϕ is an auxiliary function for E and by viii) $\phi(w) \neq \emptyset$ for every w. Lastly, let w be an infinite word and let $\epsilon > 0$. Pick N such that $2/N < \epsilon$. Since D_N is finite, the set $\{k : w_k \notin D_N\}$ is non-empty and so has a smallest element M. Then $M \geq N + 1$ by i) and $w_{M-1} \in D_N$. By v) and the choice of M, $w_{M-1}0 \notin D_N$. By vii) $\operatorname{diam}(\phi(w_{M-1})) = \operatorname{diam}(\phi_N(w_{M-1})) \leq 2/N < \epsilon$. For $n \geq M$, by R26.11 $\phi(w_n) \subseteq \phi(w_{M-1})$ and so $\operatorname{diam}(\phi(w_n)) < \epsilon$. Thus the limit claim holds.

Corollary R26.17 Let E be a compact metric space. Then there is a continuous surjection $f : \mathbf{R}_2 \to E$.

Proof: This is immediate from R26.16 and R26.13.

In the next lemma, notation and terminology from the proof of R26.16 will continue to be used.

Lemma R26.18 Let E be a compact metric space. Assume for every n in $\mathbb{N}$, for every $\epsilon > 0$, and for every $e \in E$, there exist $m > n$ and $x_1, \ldots, x_t$ in $B_\epsilon(e)$ such that $t \geq 2$ and $\overline{B_\epsilon(e)} = \cup_{i=1}^{t}\overline{B_{1/m}(x_i)}$ where $\{\overline{B_{1/m}(x_i)} : 1 \leq i \leq t\}$ is a pairwise disjoint collection. Then there is an auxiliary function ϕ for E constructed as in R26.16 with the property that, if $u, v \in M_2$ with $u \neq v$ and $\operatorname{length}(u) = \operatorname{length}(v)$, then $\phi(u) \cap \phi(v) = \emptyset$.

Proof: It is possible to construct a sequence $\{\phi_n\}_{n=1}^{\infty}$ as in the proof of R26.16, using at each stage the pairwise disjoint collections of closed balls guaranteed by the assumption. To see this, it is sufficient to check (by induction) that each ϕ_n has the following property: If $w \in D_n$ and $w0 \notin D_n$, then there is $e \in E$ and $\epsilon > 0$ such that $\phi_n(w) = \overline{B_\epsilon(e)}$. This holds for ϕ_1 by R26.14ix. Assume it holds for ϕ_n and let $u \in D_{n+1}$ with $u0 \notin D_{n+1}$. By the construction, a terminal word of D_n is not terminal in D_{n+1} and so $u \in C_{n+1}$, i.e., $u = wv$ where w is a terminal word of D_n, $v \neq \eta$, and v is in the domain of ψ_w. By definition $\phi_{n+1}(w) = \psi_w(v)$. By the induction hypothesis $\phi_n(w)$ is some $\overline{B_\epsilon(e)}$, which is a union of closed balls by this lemma's assumption. Since $u0 \notin D_{n+1}$, $v0$ is not in the domain of ψ_w and so, by R26.14ix, $\psi_w(v)$ is one of those closed balls. With the sequence thus constructed, as in R26.16, the auxiliary function is $\phi = \cup_{n=1}^{\infty}\phi_n$. Since $D_n \subseteq D_{n+1}$ for all n, $\phi|_{D_n} = \phi_n$ for every n, and $M_2 = \cup_{n=1}^{\infty}D_n$, it is sufficient to show the following by induction: for every n, if $u \neq v$ are in D_n with $\operatorname{length}(u) = \operatorname{length}(v)$ or u, v both terminal words of D_n, then $\phi_n(u) \cap \phi_n(v) = \emptyset$. Since ϕ_1 is a map constructed as in R26.14

using a pairwise disjoint union with at least two sets, by R26.15 the claim holds for $n = 1$. Now assume it holds for n and let $u \neq v$ be in D_{n+1}. First assume length$(u) =$ length(v). If both are in D_n, apply the induction hypothesis. If both are in C_{n+1}, $u = xs$ and $v = yt$ where x, y are terminal words of D_n, neither of s, t is η, $s \in \text{dom}(\psi_x)$, and t is in $\text{dom}(\psi_y)$. By definition $\phi_{n+1}(u) = \psi_x(s) \subseteq \phi_n(x)$ and $\phi_{n+1}(v) = \psi_y(t) \subseteq \phi_n(y)$. If $x \neq y$, by the induction hypothesis $\phi_n(x) \cap \phi_n(y) = \emptyset$. If $x = y$, $s \neq t$ and length$(s) =$ length(t). By R26.15 $\psi_x(s) \cap \psi_x(t) = \emptyset$. As a third case, suppose one, say u, is in C_{n+1} and the other is in D_n. Again $u = xs$ where x is a terminal word of D_n and $s \neq \eta$ is in $\text{dom}(\psi_x)$, $\phi_{n+1}(u) = \psi_x(s) \subseteq \phi_n(x)$, and $\phi_{n+1}(v) = \phi_n(v)$. Since length$(u) =$ length(v) and length$(s) \geq 1$, $v = v_1 z$ where length$(v_1) =$ length(x). Note that $v_1 \in D_n$ by property iv) in the proof of R26.16 and, since length$(z) =$ length$(s) \geq 1$, $v_1 0 \in D_n$ by the same property. Thus v_1 is not a terminal word of D_n and so $v_1 \neq x$. By the induction hypothesis $\phi_n(v_1) \cap \phi_n(x) = \emptyset$ and so $\phi_n(v_1) \cap \phi_{n+1}(u) = \emptyset$. It follows from vi) in the proof of R26.16 that $\phi_n(v) \subseteq \phi_n(v_1)$ and so $\phi_{n+1}(u) \cap \phi_{n+1}(v) = \emptyset$. Lastly assume u, v are both terminal words of D_{n+1}. By the construction of ϕ_{n+1}, if w is a terminal word of D_n, $w0$ is in D_{n+1}, i.e., w is not a terminal word of D_{n+1}. Thus u, v are both in C_{n+1}. As before, $u = xs$ and $v = yt$ where x, y are terminal words of D_n, neither of s, t is η, $s \in \text{dom}(\psi_x)$, and t is in $\text{dom}(\psi_y)$. By definition $\phi_{n+1}(u) = \psi_x(s) \subseteq \phi_n(x)$ and $\phi_{n+1}(v) = \psi_y(t) \subseteq \phi_n(y)$. If $x \neq y$, by the induction hypothesis $\phi_n(x) \cap \phi_n(y) = \emptyset$. If $x = y$, $s \neq t$ and length$(s) =$ length(t). By R26.15 $\psi_x(s) \cap \psi_x(t) = \emptyset$.

Corollary R26.19 Let E be a compact metric space. Assume for every n in $\mathbb{N}$, for every $\epsilon > 0$, and for every $e \in E$, there exist $m > n$ and $x_1, \ldots, x_t$ in $B_\epsilon(e)$ such that $t \geq 2$ and $\overline{B_\epsilon(e)} = \cup_{i=1}^{t} \overline{B_{1/m}(x_i)}$ where $\{\overline{B_{1/m}(x_i)} : 1 \leq i \leq t\}$ is a pairwise disjoint collection. Then $\mathbf{R}_2$ is homeomorphic to E.

Proof: Use the auxiliary function ϕ from R26.18 to construct a continuous surjection $f : \mathbf{R}_2 \to E$ as in R26.13. Since both spaces are compact and Hausdorff, it is sufficient to show that f is one-to-one. Let $\mathcal{F} \neq \mathcal{G}$ be in $\mathbf{R}_2$. The infinite words $d(\mathcal{F})$ and $d(\mathcal{G})$ are different and so there is n with $d(\mathcal{F})_n \neq d(\mathcal{G})_n$. By R26.18 $\phi(d(\mathcal{F})_n) \cap \phi(d(\mathcal{G})_n) = \emptyset$. Since $f(\mathcal{F}) \in \phi(d(\mathcal{F})_n)$ and $f(\mathcal{G}) \in \phi(d(\mathcal{G})_n)$, $f(\mathcal{F}) \neq f(\mathcal{G})$.

In [8], for $k \geq 2$, the metric d_k was described and shown to generate the topology of $\mathbf{R}_k$. In the next two results, the open balls are assumed to be generated with d_k as the underlying metric for $\mathbf{R}_k$.

The next lemma is implicit in [8]. Recall the following facts and notation: For Z in a normal basis $\mathcal{Z}$, Z^ω is the set of $\mathcal{Z}$-filters containing Z. For $k \geq 2$ and $x, n \in \mathbb{N}$, $C_n^x(k) = \{m \in \mathbb{N} : m \equiv x \bmod k^n\}$, which is in the normal basis generating $\mathbb{N}_k$.

Lemma R26.20 Let $k \geq 2$ be in $\mathbb{N}$ and let $\mathcal{F}$ in $\mathbf{R}_k$ .Let $\mathcal{F}$ have associated sequence $\{x_n\}_{n=1}^\infty$. Let $1 \geq \epsilon > 0$ and let j be the unique positive integer with $1/k^j < \epsilon \leq 1/k^{j-1}$. Then

i) $\overline{B_\epsilon(\mathcal{F})}$ is clopen.
ii) $\overline{B_\epsilon(\mathcal{F})} = B_\epsilon(\mathcal{F}) = Z^\omega \cap \mathbf{R}_k$, where $Z = C_j^{x_j}(k)$.
iii) If $\mathcal{G}$ is in $B_\epsilon(\mathcal{F})$, then $B_\epsilon(\mathcal{G}) = B_\epsilon(\mathcal{F})$.

Proof: Part i) is an application of R20.23ii. Part ii) is verified in the proof of R20.23. For iii) let $\mathcal{G}$ be in $B_\epsilon(\mathcal{F})$ and suppose $\mathcal{G}$ has associated sequence $\{y_n\}_{n=1}^\infty$. By ii) $C_j^{x_j}(k) \in \mathcal{G}$ and so by R20.21 $y_j = x_j$. Apply part ii) to $\mathcal{G}$ in place of $\mathcal{F}$ to obtain the conclusion.

Lemma R26.21 Let $k \geq 2$ be in $\mathbb{N}$ and let $\mathcal{F}$ in $\mathbf{R}_k$. Let $\epsilon > 0$ and let $n \in \mathbb{N}$. Then there exist $m > n$ and $\mathcal{G}_1, \ldots, \mathcal{G}_t$ in $B_\epsilon(\mathcal{F})$ such that $t \geq 2$ and $\overline{B_\epsilon(\mathcal{F})} = \cup_{i=1}^t \overline{B_{1/m}(\mathcal{G}_i)}$ where $\{\overline{B_{1/m}(\mathcal{G}_i)} : 1 \leq i \leq t\}$ is a pairwise disjoint collection.

Proof: Let $\mathcal{F}$ have associated sequence $\{x_i\}_{i=1}^\infty$. Pick $j \geq 2$ such that $1/k^{j-1} < \epsilon$ and $k^{j-1} > n$. Since $C_j^1(k), \ldots, C_j^{k^j}(k)$ are the distinct classes of equivalence mod k^j, the collection $\{(C_j^i(k))^\omega \cap \mathbf{R}_k : 1 \leq i \leq k^j\}$ is a finite cover of $\mathbf{R}_k$ by pairwise disjoint clopen sets. By R26.20 each of those sets is a $1/k^{j-1}$-open ball, any element of which can be taken as its center. Thus the elements of this cover which have a non-empty intersection with $B_\epsilon(\mathcal{F})$ can be written as $\{B_{1/k^{j-1}}(\mathcal{G}_i) : 1 \leq i \leq t\}$, where each $\mathcal{G}_i \in B_\epsilon(\mathcal{F})$. By choice of the $\mathcal{G}_i$, $B_\epsilon(\mathcal{F}) \subseteq \cup\{B_{1/k^{j-1}}(\mathcal{G}_i) : 1 \leq i \leq t\}$. Let $\mathcal{H}_i \in B_\epsilon(\mathcal{F}) \cap B_{1/k^{j-1}}(\mathcal{G}_i)$. Since $1/k^{j-1} < \epsilon$, by R26.20iii (twice), $B_{1/k^{j-1}}(\mathcal{G}_i) = B_{1/k^{j-1}}(\mathcal{H}_i) \subseteq B_\epsilon(\mathcal{H}_i) = B_\epsilon(\mathcal{F})$. Thus $\cup\{B_{1/k^{j-1}}(\mathcal{G}_i) : 1 \leq i \leq t\} \subseteq B_\epsilon(\mathcal{F})$. Since the balls are clopen, the conclusion follows with $m = k^{j-1}$ provided $t \geq 2$. That can be shown as follows: By R10.2.5i $x_j = x_{j-1} + sk^{j-1}$, where $s \in \{0, 1, \ldots, k-1\}$. Pick $r \in \{0, 1, \ldots, k-1\}$ with $r \neq s$. Define a sequence $\{y_i\}_{i=1}^\infty$ by $y_i = x_i$ for $i \leq j-1$, $y_j = x_{j-1} + rk^{j-1}$, and $y_{i+1} = y_i + k^i$ for $i \geq j$. By R10.2.6 there is $\mathcal{G}$ in $\mathbf{R}_k$ with $\{y_i\}_{i=1}^\infty$ as its associated sequence. Since $y_j \neq x_j$, the classes $C_j^{x_j}(k)$ and $C_j^{y_j}(k)$ are distinct. By definition of the associated sequence, $C_j^{y_j}(k) \in \mathcal{G}$ and $C_j^{x_j}(k) \in \mathcal{F}$ so that $\mathcal{G}$ and $\mathcal{F}$ are in different elements of the cover $\{(C_j^i(k))^\omega \cap \mathbf{R}_k : 1 \leq i \leq k^j\}$. To verify that $t \geq 2$, it is now sufficient to show that $\mathcal{G} \in B_\epsilon(\mathcal{F})$. If $\epsilon > 1$, this clear since $d_k \leq 1$ so that $B_\epsilon(\mathcal{F}) = \mathbf{R}_k$. If $\epsilon \leq 1$, there is a unique l such that $1/k^l < \epsilon \leq 1/k^{l-1}$. Since $1/k^{j-1} < \epsilon \leq 1/k^{l-1}$, $j > l$ and so by definition $y_l = x_l$. Thus $\mathcal{G}$ is in the set $(C_l^{x_l}(k))^\omega \cap \mathbf{R}_k$, which is $B_\epsilon(\mathcal{F})$ by R26.20.

Corollary R26.22 Let $k \geq 2$ be in $\mathbb{N}$. Then the topological spaces $\mathbf{R}_2$ and $\mathbf{R}_k$ are homeomorphic.

Proof: This follows from R26.19 and R26.21.

Corollary R26.23 Let $k, j \geq 2$ be in $\mathbb{N}$. Then the topological spaces $\mathbf{R}_k$ and $\mathbf{R}_j$ are homeomorphic.

Proof: Transitivity and symmetry of homeomorphism: each is homeomorphic to $\mathbf{R}_2$.

In [3] the the following question was left unanswered: For distinct primes p, q, are the topological spaces $\mathbb{N}_p$ and $\mathbb{N}_q$ homeomorphic? The next few results do not answer the question but are related.

Lemma R26.24 Let X be an infinite discrete space. Let (Y, f) and (Z, g) be T_2-compactifications of X. Assume $h : Y \to Z$ is a homeomorphism. Then $h[f[X]] = g[X]$.

Proof: Let $x \in X$. Since $f[X]$ is an open, discrete subspace of Y, $\{f(x)\}$ is open in Y and so $\{h(f(x))\}$ is open in Z. By density $\{h(f(x))\} \cap g[X]$ is non-empty, i.e. $h \circ f(x) \in g[X]$. Thus $h[f[X]] \subseteq g[X]$. Similarly, using h^{-1}, $h[f[X]] \supseteq g[X]$.

Corollary R26.25 Let $k, j \geq 2$ be in $\mathbb{N}$. If the topological spaces $\mathbb{N}_k$ and $\mathbb{N}_j$ are homeomorphic, then the topological spaces $\mathbf{R}_k$ and $\mathbf{R}_j$ are homeomorphic.

Proof: Let $h : \mathbb{N}_k \to \mathbb{N}_j$ be a homeomorphism and let ι_k and ι_j be the embeddings of $\mathbb{N}$ with the discrete topology into $\mathbb{N}_k$, $\mathbb{N}_j$ respectively. By R26.24 $h[\iota_k[\mathbb{N}]] = \iota_j[\mathbb{N}]$. Since $\mathbf{R}_k = \mathbb{N}_k - \iota_k[\mathbb{N}]$ and $\mathbf{R}_j = \mathbb{N}_j - \iota_j[\mathbb{N}]$, h restricted to $\mathbf{R}_k$ maps onto $\mathbf{R}_j$ and so is a homeomorphism from $\mathbf{R}_k$ to $\mathbf{R}_j$.

In other words, R26.23 is a necessary condition for topological spaces $\mathbb{N}_k$ and $\mathbb{N}_j$ to be homeomorphic. The next proposition is a partial answer as to whether it is sufficient.

Lemma R26.26 Let $k \geq 2$ be in $\mathbb{N}$ and let ι_k, f_k be the embeddings of $\mathbb{N}$ into the compactifications $\mathbb{N}_k$, $\mathbf{R}_k$ respectively. Let $\mathcal{F}$ be in $\mathbf{R}_k$ and let $\{x_i\}$ be a sequence in $\mathbb{N}$ such that $\{\iota_k(x_i)\}$ converges to $\mathcal{F}$ in $\mathbb{N}_k$. Then $\{f_k(x_i)\}$ converges to $\mathcal{F}$ in $\mathbf{R}_k$.

Proof: Since $\mathcal{F}$ is in $\mathbf{R}_k$, it is a non-point ultrafilter in $\mathbb{N}_k$ and so has an associated sequence $\{y_n\}_{n=1}^{\infty}$. Recall that for each i the associated sequence of $f_k(x_i)$ is $\{{}^i z_n\}_{n=1}^{\infty}$ where ${}^i z_n \equiv x_i \bmod k^n$ for each n. By R10.2.11 $\{x_i\}$ is unbounded and for every m in $\mathbb{N}$ there is i_0 such that $i \geq i_0$ implies $x_i \equiv y_m \bmod k^m$. Then by transitivity $i \geq i_0$ implies ${}^i z_m \equiv y_m \bmod k^m$, i.e., since ${}^i z_m$ and y_m are both in $\{1, 2, \ldots, k^m\}$, ${}^i z_m = y_m$. By R17.2.16 $\{f_k(x_i)\}$ converges to $\mathcal{F}$ in $\mathbf{R}_k$.

The previous lemma can also be proven by using the continuity of addition and the fact that $f_k(x) = \iota_k(x) + \mathcal{O}_k$, where $\mathcal{O}_k$ is the additive identity in $\mathbf{R}_k$. The next lemma is a partial converse of R26.26.

Lemma R26.27 Let $k \geq 2$ be in $\mathbb{N}$ and let ι_k, f_k be the embeddings of $\mathbb{N}$ into the compactifications $\mathbb{N}_k$, $\mathbf{R}_k$ respectively. Let $\mathcal{F}$ be in $\mathbf{R}_k$ and let $\{x_i\}$ be an unbounded sequence in $\mathbb{N}$ such that $\{f_k(x_i)\}$ converges to $\mathcal{F}$ in $\mathbf{R}_k$. Then $\{\iota_k(x_i)\}$ converges to $\mathcal{F}$ in $\mathbb{N}_k$.

Proof: As above, since $\mathcal{F}$ is in $\mathbf{R}_k$, it is a non-point ultrafilter in $\mathbb{N}_k$ and so has an associated sequence $\{y_n\}_{n=1}^{\infty}$. Recall that for each i the associated sequence of $f_k(x_i)$ is $\{{}^i z_n\}_{n=1}^{\infty}$ where ${}^i z_n \equiv x_i \bmod k^n$ for each n. Let m be in $\mathbb{N}$. By R17.2.16 there is i_0 such that $i \geq i_0$ implies ${}^i z_m = y_m$. Then $i \geq i_0$ implies $x_i \equiv y_m \bmod k^m$. Since $\{x_i\}$ is given to be unbounded, by R10.2.11 $\{\iota_k(x_i)\}$ converges to $\mathcal{F}$ in $\mathbb{N}_k$.

Proposition R26.28 Let $k, j \geq 2$ be in $\mathbb{N}$ and let f_k, f_j be the embeddings of $\mathbb{N}$ into the compactifications $\mathbf{R}_k$, $\mathbf{R}_j$ respectively. Let $h : \mathbf{R}_k \to \mathbf{R}_j$ be a homeomorphism such that $h[f_k[\mathbb{N}\,]] = f_j[\mathbb{N}\,]$. Then the topoloical spaces $\mathbb{N}_k$ and $\mathbb{N}_j$ are homeomorphic.

Proof: Let ι_k and ι_j be the embeddings of $\mathbb{N}$ into compactifications $\mathbb{N}_k$, $\mathbb{N}_j$ respectively. Define $H : \mathbb{N}_k \to \mathbb{N}_j$ as follows: For $\mathcal{F}$ in $\mathbf{R}_k$, let $H(\mathcal{F}) = h(\mathcal{F})$. For $\mathcal{F}$ in $\mathbb{N}_k - \mathbf{R}_k$, $\mathcal{F} = \iota_k(n)$ for some n. Let $H(\mathcal{F}) = \iota_j(m)$, where $h(f_k(n)) = f_j(m)$. Since h is bijective, $h[f_k[\mathbb{N}\,]] = f_j[\mathbb{N}\,]$, and the embeddings are one-to-one, it is easy to check that H is a one-to-one, onto function. Since the spaces are compact and Hausdorff, the continuity of H would imply that it is a homeorphism. Let C be a closed subset of $\mathbb{N}_j$ and decompose C into disjoint subsets C_1 and C_2, where $C_1 = \iota_j[\mathbb{N}\,] \cap C$ and $C_2 = \mathbf{R}_j \cap C$. Note that $H^{-1}[C_2] = h^{-1}[C_2]$ is closed in $\mathbf{R}_k$ by continuity of h. It is also closed in $\mathbb{N}_k$ since $\mathbf{R}_k$ is the complement of the open $\iota_k[\mathbb{N}\,]$. Thus to show that $H^{-1}[C] = H^{-1}[C_1] \cup H^{-1}[C_2]$ is closed in $\mathbb{N}_k$, it is sufficient to show that the closure of $H^{-1}[C_1]$ is a subset of $H^{-1}[C]$. Let $\mathcal{F}$ be in the closure of $H^{-1}[C_1]$. Since $\mathbb{N}_k$ is metrizable and $H^{-1}[C_1] \subseteq \iota_k[\mathbb{N}\,]$, there is a sequence $\{x_i\}$ in $\mathbb{N}$ with $\iota_k(x_i) \in H^{-1}[C_1]$ for every i such that $\{\iota_k(x_i)\}$ converges to $\mathcal{F}$ in $\mathbb{N}_k$. If $\mathcal{F}$ is in $f_k[\mathbb{N}\,]$, since $\{\mathcal{F}\}$ is open, eventually $\iota_k(x_i) = \mathcal{F}$, i.e., $\mathcal{F} \in H^{-1}[C_1] \subseteq H^{-1}[C]$. Thus assume $\mathcal{F}$ is in $\mathbf{R}_k$ so that $H(\mathcal{F}) = h(\mathcal{F})$. For each i let y_i be the positive integer with $h(f_k(x_i)) = f_j(y_i)$, i.e., $H(\iota_k(x_i)) = \iota_j(y_i)$ by definition. By R10.2.11 the sequence $\{x_i\}$ is unbounded and so, since h, f_k, and f_j are one-to-one, $\{y_i\}$ is also unbounded. By R26.26 $f_k(x_i) \to \mathcal{F}$ and so by continuity $h(f_k(x_i)) = f_j(y_i) \to h(\mathcal{F})$. By R26.27 $\iota_j(y_i) \to h(\mathcal{F})$, i.e., $H(\iota_k(x_i)) \to H(\mathcal{F})$. Since C is closed, $H(\mathcal{F})$ is in C, i.e., $\mathcal{F}$ is in $H^{-1}[C]$ as required.

Not every homeomorphism h from $\mathbf{R}_k$ to $\mathbf{R}_j$ has the property assumed in R26.28: Let $\mathcal{F}_0$ be in $\mathbf{R}_j$ but not in $f_j[\mathbf{Z}\,]$. Since $\mathbf{R}_j$ with addition is a topological group, the map t

defined by $t(\mathcal{F}) = \mathcal{F} + \mathcal{F}_0$ is a homeomorphism of $\mathbf{R}_j$ onto itself. Since f_j is an additive homomorphism, at least one of h and $t \circ h$ does not have the assumed property.

The uniformities $\mathcal{U}_k, \mathcal{U}_j$ in the next proposition are described in detail in R10.3.8.

Proposition R26.29 Let $k, j \geq 2$ be in $\mathbb{N}$ and let $\mathcal{U}_k, \mathcal{U}_j$ be the totally bounded uniformities corresponding to $\mathbb{N}_k$ and $\mathbb{N}_j$ respectively. Then $\mathbb{N}_k$ and $\mathbb{N}_j$ are homeomorphic topological spaces if and only if there exists a permutation σ of the positive integers such that $\sigma : (\mathbb{N}, \mathcal{U}_k) \to (\mathbb{N}, \mathcal{U}_j)$ is a unimorphism.

Proof: Let ι_k and ι_j be the embeddings of $\mathbb{N}$ into compactifications $\mathbb{N}_k$, $\mathbb{N}_j$ respectively. First assume h is a homeomorphism from $\mathbb{N}_k$ to $\mathbb{N}_j$. By R26.24 h maps $\iota_k[\mathbb{N}]$ onto $\iota_j[\mathbb{N}]$. Both ι_k and ι_j are unimorphisms onto their images and, by compactness, h is also a unimorphism. Thus $\sigma = \iota_j^{-1} \circ h \circ \iota_k$ is the required permutation and unimorphism. Conversely let a unimorphism σ be given. By R7.1.3 there are unique continuous maps $P : \mathbb{N}_k \to \mathbb{N}_j$ and $Q : \mathbb{N}_j \to \mathbb{N}_k$ such that $\sigma = \iota_j^{-1} \circ P \circ \iota_k$ and $\sigma^{-1} = \iota_k^{-1} \circ Q \circ \iota_j$. Letting id be the identity on $\mathbb{N}$, $id = \iota_j^{-1} \circ P \circ Q \circ \iota_j = \iota_k^{-1} \circ Q \circ P \circ \iota_k$, i.e., $P \circ Q$ is the unique continuous map from $\mathbb{N}_j$ to $\mathbb{N}_j$ corresponding to id and $Q \circ P$ is the unique continuous map from $\mathbb{N}_k$ to $\mathbb{N}_k$ corresponding to id. By uniqueness $P \circ Q$ is the identity on $\mathbb{N}_j$ and $Q \circ P$ is the identity on $\mathbb{N}_k$. Thus P ia a homeomorphism.

This section will conclude by showing that the remnant rings are homeomorphic to the Cantor set by using an argument based on auxiliary functions. In [1] Robert derives this result for the p-adic numbers, i.e. the remnant ring $\mathbf{R}_p$ where p is prime, by a different method and also describes several linear and Euclidean models which illustrate the fractal nature of the spaces.

To begin, let's review the construction of the Cantor set and establish notation for what follows: C_1 is obtained by removing the middle third from $[0, 1]$, i.e., C_1 is the union $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Suppose C_n has been defined and is the union of 2^n pairwise disjoint closed subintervals of $[0, 1]$. Define C_{n+1} the union of the 2^{n+1} pairwise disjoint closed subintervals obtained by removing the the middle third from each of the subintervals making up C_n. Clearly $C_{n+1} \subseteq C_n$ and by induction C_n is defined for every n. The Cantor set C is defined by $C = \cap_{n=1}^{\infty} C_n$. C is a non-empty, closed, compact subset of $[0, 1]$.

Lemma R26.30 Let $n \in \mathbb{N}$ and let $[a, b]$ be one of the subintervals in the union making up C_n. Then $|a - b| = 1/3^n$ and $a, b \in C$.

Proof: Removing the middle third of an interval leaves two closed intervals, each one-third the length of the original. This and a routine induction show that $|a - b| = 1/3^n$. For the second claim, note first that, since $C_{j+1} \subseteq C_j$ for all j, $a, b \in C_i$ for all $i \leq n$. Moreover, when the middle third is removed from an interval, the endpoints are still in the result and are again endpoints of different intervals in the union making up the result. Thus $a, b \in C_i$ for all $i \geq n$. Therefore a and b are in C.

Note that removing the middle third from an interval leaves a union of two subintervals, which can naturally be described as the left and right subintervals of the result.

Definition R26.31 The map $\phi_C : M_2 \to \mathcal{P}(C)$ is defined inductively: $\phi_C(\eta) = C$. Assume ϕ_C has been defined for all words of length less than n and $w \in M_2$ has length n. Let $w = vx$ where length$(v) = n - 1$ and $x \in \{0, 1\}$. Let $\phi_C(v0)$ be $\phi_C(v)$ intersected with the union of the left subintervals making up C_n and $\phi_C(v1)$ be $\phi_C(v)$ intersected with the union of the right subintervals making up C_n.

The diameter in the next lemma refers of course to the absolute value metric on $[0,1]$.

Lemma R26.32 The map ϕ_C has the following properties.

i) For every $w \in M_2$, $\phi_C(w)$ is a closed subset of C.

ii) If $w \in M_2$ has length $n \geq 1$, $\phi_C(w) = [a,b] \cap C$, where $[a,b]$ is one of the subintervals in the union making up C_n.

iii) If $w \in M_2$ has length $n \geq 1$, $\phi_C(w)$ is non-empty.

iv) If $w \in M_2$ has length $n \geq 1$, $\mathrm{diam}(\phi_C(w)) = 1/3^n$.

v) For every $w \in M_2$, $\phi_C(w) = \phi_c(w0) \cup \phi_C(w1)$.

Proof: Part i) follows by induction since C is closed and there are 2^{n-1} left (or right) subintervals in the union making up C_n. For part ii), $\phi_C(0) = [0, \frac{1}{3}] \cap C$ and $\phi_C(1) = [\frac{2}{3}, 1] \cap C$. Since $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, the claim holds for words of length one. Assume it holds for all words of length n and let w be of length $n+1$. Then $w = vx$ where v is of length n and $x \in \{0,1\}$. By the induction hypothesis $\phi_C(v) = [a,b] \cap C$, where $[a,b]$ is one of the subintervals making up C_n. Among the intervals making up C_{n+1}, only the left and right thirds of $[a,b]$ have a possibly non-empty intersection with $\phi_C(v)$. By definition of ϕ_C and the induction hypothesis, $\phi_C(w)$ equals C intersected with either the left or right subinterval of $[a,b]$, both of which are subintervals in the union making up C_{n+1}. Part iii) is immediate from R26.30 and part ii). For part iv), let w have length $n \geq 1$ and let $\phi_C(w) = [a,b] \cap C$ where $[a,b]$ is one of the subintervals in the union making up C_n. By R26.30 $a, b \in \phi_C(w)$ and so $\mathrm{diam}(\phi_C(w)) \geq |a-b| = 1/3^n$. Since $\phi_C(w) \subseteq [a,b]$, $\mathrm{diam}(\phi_C(w)) \leq 1/3^n$. For v), given $w \in M_2$ of length n, since C_{n+1} is the union of its left and right subintervals, by definition $\phi(w0) \cup \phi(w1) = C_{n+1} \cap \phi_C(w)$. Since $\phi_C(w) \subseteq C \subseteq C_{n+1}$, the desired equality holds.

Proposition R26.33 The topological space $\mathbf{R}_2$ is homeomorphic to the Cantor set.

Proof: By definition $\phi_C(\eta) = C$ and so by i) and v) of the previous lemma ϕ_C is an auxiliary function for C. Part iv) of the lemma shows that, if w is an infinite word, $\mathrm{diam}(w_n) \to 0$ as $n \to \infty$. That and part iii) of the allow an application of R26.13: There is a continuous, onto map $f : \mathbf{R}_2 \to C$. Since the spaces are compact and T_2, it is sufficient to show that the f constructed as in the proof of R26.13 is one-to-one. Let $\mathcal{F} \neq \mathcal{G}$ be in $\mathbf{R}_2$ with corresponding infinite words $d(\mathcal{F}) = w$ and $d(\mathcal{G}) = v$. By R26.6 $w \neq v$ and so there is n with $w_n \neq v_n$. Assume n is the least such positive integer. Without loss of generality assume $w_n = x0$ and $v_n = x1$, where x (possibly η) has length $n-1$. By definition $\phi_C(w_n)$ is contained in the union of the left subintervals making up C_n and $\phi_C(v_n)$ is contained in the union of the right subintervals making up C_n. Since these two unions are disjoint, $\phi_C(w_n) \cap \phi_C(v_n) = \emptyset$. By the construction of f in the proof of R26.13, $f(\mathcal{F}) \in \phi_C(w_n)$ and $f(\mathcal{G}) \in \phi_C(v_n)$. Thus $f(\mathcal{F}) \neq f(\mathcal{G})$.

Corollary R26.34 Let $k \geq 2$ be in $\mathbb{N}$. The topological space $\mathbf{R}_k$ is homeomorphic to the Cantor set.

Proof: This follows from R26.23 and the symmetry and transitivity of homeomorphism.

Comment: Robert [1] has an exercise (14 p.65), which asks the reader to show that a compact metric space E is path-connected if and only there is a continuous surjection $f : [0,1] \to E$. In outline, the difficult half of this can be done as follows: Use R26.17 and R26.33 to obtain a continuous surjection $f_0 : C \to E$. Extend f_0 inductively by using appropriate curves on the closures of the removed middle thirds. For example, obtain f_1 on $C \cup [\frac{1}{3}, \frac{2}{3}]$ by pasting together f_0 and a continuous curve $\gamma : [\frac{1}{3}, \frac{2}{3}] \to E$ such that $\gamma(\frac{1}{3}) = f_0(\frac{1}{3})$ and $\gamma(\frac{2}{3}) = f_0(\frac{2}{3})$. Given f_n, extend it to f_{n+1}in a similar way with finitely many curves. The desired map f is $\cup_{n=0}^{\infty} f_n$. This result shows the existence of a continuous surjection from $[0, 1]$ onto $[0, 1] \times [0, 1]$, i.e., a space-filling curve.

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http://www.susanjkleinart.com/compactification/

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Added 2019

The main result of this added subsection is that $(\mathbf{R}_k\,, <_k)$ and $(\mathbf{R}_l\,, <_l)$ are order isomorphic for $k, l \geq 2$ in $\mathbb{N}$. It follows that the topological spaces $\mathbb{N}_k$ and $\mathbb{N}_l$ are homeomorphic.

The argument for R26.14 uses an arbitrary bijection from S_j to $\{1, \ldots, 2^j\}$ in the initialization of the preliminary function ψ. This added subsection in effect makes a more specific choice in the case of the remnant rings, a choice which eventually leads to a homeomorphism also preserving order. The notation for ϵ-balls will be as above, with d_k as the underlying metric on $\mathbf{R}_k$. By R20.23.ii all d_k-balls are clopen.

The first lemma is a variation of R26.21.

Lemma R26.Add.1 Let $k, l \in \mathbb{N}$ with $k \geq 2$. Then

i) $\mathbf{R}_k = \cup_{t=1}^{k} B_1(f_k(t))$.

ii) If $1 \leq x \leq k^l$ with $x \in \mathbb{N}$, $B_{1/k^{l-1}}(f_k(x)) = \cup_{s=0}^{k-1} B_{1/k^l}(f_k(x + sk^l))$.

iii) The balls in either union form a pairwise disjoint collection.

Proof: For any integer t with $1 \leq t \leq k$, by R26.20.ii $B_1(f_k(t)) = (C_1^t(k))^{\omega} \cap \mathbf{R}_k$. Since $\mathbb{N} = \cup_{t=1}^{k} C_1^t(k)$, by definition of the associated sequence and R20.21, every element of $\mathbf{R}_k$ is in some $(C_1^t(k))^{\omega} \cap \mathbf{R}_k$ and so i) holds. For integers $t \neq r$ with $1 \leq t, r \leq k$ the equivalence classes are disjoint and so $\{B_1(f_k(t)) : 1 \leq t \leq k\}$ is a pairwise disjoint collection as claimed in iii). Now let x be an integer with $1 \leq x \leq k^l$. By R26.20ii again, $B_{1/k^{l-1}}(f_k(x)) = (C_l^x(k))^{\omega} \cap \mathbf{R}_k$ and, for $s \in \{0, \ldots, k-1\}$, $B_{1/k^l}(f_k(x + sk^l)) = (C_{l+1}^{x+sk^l}(k))^{\omega} \cap \mathbf{R}_k$. Since $C_l^x(k) = \cup_{s=0}^{k-1} C_{l+1}^{x+sk^l}(k)$, by definition of the associated sequence and R20.21, every element of $(C_l^x(k))^{\omega} \cap \mathbf{R}_k$ is in some $(C_{l+1}^{x+sk^l}(k))^{\omega} \cap \mathbf{R}_k$. Similarly $(C_{l+1}^{x+sk^l}(k))^{\omega} \cap \mathbf{R}_k$ is a subset of $(C_l^x(k))^\omega \cap \mathbf{R}_k$. Thus ii) holds. Lastly, because the equivalence classes are pairwise disjoint, so are the corresponding balls.

Lemma R26.Add.2 Let $k, l, a, b \in \mathbb{N}$ with $k \geq 2$ and $a \not\equiv b \bmod k^l$. Assume $f_k(a) <_k f_k(b)$. Let $\mathcal{F} \in B_{1/k^{l-1}}(f_k(a))$ and $\mathcal{G} \in B_{1/k^{l-1}}(f_k(b))$. Then $\mathcal{F} <_k \mathcal{G}$.

Proof: Let $\mathcal{F}$ be associated with $\{y_n\}_{n=1}^\infty$, $\mathcal{G}$ with $\{z_n\}_{n=1}^\infty$, $f_k(a)$ with $\{a_n\}_{n=1}^\infty$, and $f_k(b)$ with $\{b_n\}_{n=1}^\infty$. Since $a_l \equiv a \bmod k^l$, $C_l^{a_l}(k) = C_l^a(k)$. By R26.20ii $B_{1/k^{l-1}}(f_k(a)) = (C_l^a(k))^\omega \cap \mathbf{R}_k$. Since $\mathcal{F}$ is in the ball, $C_l^a(k) \in \mathcal{F}$ and, by the definition of the associated sequence, $C_l^{y_l}(k) \in \mathcal{F}$. Since two elements of a $\mathcal{Z}_k$-filter must have a non-empty intersection, $C_l^{y_l}(k) = C_l^a(k)$, i.e., $y_l \equiv a \equiv a_l \bmod k^l$. Since $1 \leq y_l, a_l \leq k^l$, $y_l = a_l$. By R26.7 $y_n = a_n$ for $1 \leq n \leq l$. Similarly, $z_l \equiv b \bmod k^l$ and $z_n = b_n$ for $1 \leq n \leq l$. Also note that $f_k(a) <_k f_k(b)$ implies $f_k(a) \neq f_k(b)$. Since f_k is one-to-one, $a \neq b$. Now let M be the smallest of $\{n : a_n \neq b_n\}$. Since $a \not\equiv b \bmod k^l$, $C_l^a(k) \cap C_l^b(k) = \emptyset$ and so $a_l \neq b_l$. Thus $M \leq l$ so that M is also the smallest of $\{n : y_n \neq z_n\}$. By hypothesis and R19.Add.2, $a_M < b_M$. Thus $y_M < z_M$. By R19.Add.2 again $\mathcal{F} <_k \mathcal{G}$.

For convenience the usage of '$<_k$' will be extended as follows: for $S, T \subseteq \mathbf{R}_k$ such that $\mathcal{F} \in S$ and $\mathcal{G} \in T$ imply $\mathcal{F} <_k \mathcal{G}$, $S <_k T$ will be written. Of course, a pair of subsets may not be related, but related subsets must be disjoint. With this convention, the conclusion of R26.Add.2 could be written $B_{1/k^{l-1}}(f_k(a)) <_k B_{1/k^{l-1}}(f_k(b))$.

It will also be convenient to write $\mathcal{M}$ for the set of all infinite words with characters from $\{0,1\}$. With the natural order on $\{0,1\}$, the dictionary order $<_D$ is a linear order on $\mathcal{M}$. As in the proof of R26.14, for $t \in \mathbb{N}$, $S_t = \{w \in M_2 : \text{length}(w) = t\}$. Dictionary order applies to the pairs from S_t and the same notation will be used. The $<_D$-ordered indexing of S_t is $\{u_i : 1 \leq i \leq 2^t\}$, where $u_i \in S_t$ for all i and $u_i <_D u_{i+1}$ for $1 \leq i \leq 2^t - 1$.

Similarly, the k-ordered indexing of a non-empty, finite subset S of $\mathbb{N}$ will be written $\{x_i : 1 \leq i \leq |S|\}$, where $x_i \in S$ for all i and $f_k(x_i) <_k f_k(x_{i+1})$ for $1 \leq i \leq |S| - 1$. By R26.Add.2, if the elements of S represent distinct equivalence classes mod k^l, the k-ordered indexing of S induces the k-ordered indexing of the corresponding $1/k^{l-1}$-balls.

The next lemma is used repeatedly, sometimes implicitly, in the following definition and related results.

Lemma R26.Add.3 Let $k, j \in \mathbb{N}$ with $2^j \leq k < 2^{j+1}$. Let $c = 2^{j+1} - k$. Then $c \leq 2^j$ and $k - c$ is even. Moreover, $\{i : c < i \leq 2^j\} = \{c + t : 1 \leq t \leq (k-c)/2\}$. Lastly, the map $t \mapsto \{c + 2t - 1, c + 2t\}$ from the domain $\{n \in \mathbb{N} : 1 \leq n \leq (k-c)/2\}$ has disjoint images for $s \neq t$ and the union of the images is $\{n \in \mathbb{N} : c + 1 \leq n \leq k\}$.

Proof: Because $-k \leq -2^j$, $c = 2^{j+1} - k \leq 2^{j+1} - 2^j = 2^j$. Next $k - c$ equals $k - (2^{j+1} - k) = 2(k - 2^j)$, which is even. The second claim follows easily from the observation that $2^j - c = k - 2^j$, which is $(k-c)/2$. Next, for the map with $s \neq t$, the odd-determined elements of the images ($c + 2t - 1$ and $c + 2s - 1$) are distinct, as are the even-determined elements. It follows easily that the images of s an t are disjoint. Lastly let $n \in \mathbb{N}$ with $c + 1 \leq n \leq k$. If $n - c$ is even, the integer $t = (n-c)/2$ is in the specified domain and $n = c + 2t$ is in the image of t. If $n - c$ is odd, $t = (n - c + 1)/2$ is an integer between 1 and $(k-c)/2 + 1/2$, i.e., t is in the specified domain. Clearly, $n = c + 2t - 1$ is in the image of t.

Definition R26.Add.4 Let k, l, x be integers with $k \geq 2$, $x \geq 1$, and $l \geq 0$. Let X be the (clopen in $\mathbf{R}_k$) set $\cup_{s=0}^{k-1} B_{1/k^l}(f_k(x + sk^l))$. Assume $j \in \mathbb{N}$ is such that $2^j \leq k < 2^{j+1}$.

Let $\{x_i : 1 \le i \le k\}$ be the k-ordering of $S = \{x + sk^l : 0 \le s \le k-1\}$ and $\{u_i : 1 \le i \le 2^j\}$ be the $<_D$-indexed ordering of S_j. Define $g_{lx} : S_j \to \mathcal{P}(X)$ as follows: For $1 \le i \le 2^{j+1} - k$, let $g_{lx}(u_i) = B_{1/k^l}(f_k(x_i))$. For $c = 2^{j+1} - k$ and $t \in \{1, \ldots, (k-c)/2\}$, $g_{lx}(u_{c+t})$ is defined to be $B_{1/k^l}(f_k(x_{c+2t-1})) \cup B_{1/k^l}(f_k(x_{c+2t}))$.

Lemma R26.Add.5 Let k, l, x be integers with $k \ge 2$, $x \ge 1$, and $l \ge 0$. Let $S = \{x + sk^l : 0 \le s \le k-1\}$. Assume j is the integer with $2^j \le k < 2^{j+1}$. Let $\{x_i : 1 \le i \le k\}$ be the k-ordering of S and $\{u_i : 1 \le i \le 2^l\}$ be the $<_D$-indexed ordering of S_j. Then

i) If $x_i \in S$, then there is a unique $u_n \in S_j$ such that $B_{1/k^l}(f_k(x_i)) \subseteq g_{lx}(u_n)$.

ii) If $u_i, u_n \in S_j$ with $u_i <_D u_n$, then $g_{lx}(u_i) <_k g_{lx}(u_n)$.

Proof: First note that for $a, b \in \mathbb{N}$ with $a \not\equiv b \bmod k^{l+1}$, the balls $B_{1/k^l}(f_k(a))$ and $B_{1/k^l}(f_k(b))$ are disjoint by R26.20.ii, because $C^a_{l+1}(k) \cap C^b_{l+1}(k) = \emptyset$. Thus, because no two elements of S are equivalent mod k^{l+1}, the balls used to assign the values of g_{lx} are pairwise disjoint. Now let $x_i \in S$. By the disjointness of the balls and R26.Add.3, there is at most one $u_n \in S_j$ such that $B_{1/k^l}(f_k(x_i)) \subseteq g_{lx}(u_n)$. If $1 \le i \le 2^{j+1} - k$, by definition $g_{lx}(u_i) = B_{1/k^l}(f_k(x_i))$. If $2^{j+1} - k < i \le k$, let $c = 2^{j+1} - k$. If $i - c$ is even, let $t = (i-c)/2$. Clearly, t is an integer and $i - c \ge 2$ so that $t \ge 1$. Also $i - c \le= k - c$ so that $t \le (k-c)/2$. Clearly, $i = c + 2t$ and so by definition $B_{1/k^l}(f_k(x_i)) \subseteq g_{lx}(u_{c+t})$. Next suppose suppose $i - c$ is odd. Let $t = (i - c + 1)/2$. Clearly t is a positive integer. Because $i = k$ would imply $i - c = k - c$ which is even, in this case $i \le k - 1$ so that $i - c + 1 \le k - c$, i.e., $t \le (k-c)/2$. Now $i = c + 2t - 1$ and so by definition $B_{1/k^l}(f_k(x_i)) \subseteq g_{lx}(u_{c+t})$. Thus the first claim holds. Now suppose $u_i, u_n \in S_j$ with $u_i <_D u_n$. Because a $<_D$-indexed ordering is assumed, $i < n$. Again let $c = 2^{j+1} - k$. If $i, n \le c$, since k-ordering is given, $f_k(x_i) <_k f_k(x_n)$ and so by R26Add.2 the balls are related in the same way, i.e., $g_{lx}(u_i) <_k g_{lx}(u_n)$. If $i \le c < n$, the image of u_n is the union of two balls determined by subscripts greater than $c > i$. The conclusion follows easily from R26Add.2. Lastly assume $c < i < n$. Write $i = c + s$ and $n = c + t$ with s, t in $\mathbb{N}$ and $s < t$ so that $2s < 2t$. Also $2s < 2t - 1$ because of the even/odd pairing. Then $g_{lx}(u_i) <_k g_{lx}(u_n)$ by the definition of g_{lx} and a routine application of R26Add.2.

The following lemma is a better version of a key step in the proof of R26.14.

Lemma R26.Add.6 Let the set $T = \cup_{i=1}^t T_i$ and let $j \in \mathbb{N}$. Assume $\psi_1 : S_j \to \mathcal{P}(T)$ is such that each word in S_j maps to a union of one or more elements of $\{T_i : 1 \le i \le t\}$ and $\cup\{\psi_1(w) : w \in S_j\} = T$. Let $E_j = \{w \in M_2 : \text{length}(w) \le j\}$. Then there is a unique $\Psi : E_j \to \mathcal{P}(T)$ such that

i) $\Psi(\eta) = T$.

ii) For each $w \in E_j$ $\Psi(w)$ is a union of one or more elements of $\{T_i : 1 \le i \le t\}$.

iii) For w with $w0, w1 \in E_j$ $\Psi(w) = \Psi(w0) \cup \Psi(w1)$.

iv) For $w \in S_j$ $\Psi(w) = \psi_1(w)$.

Proof: Proceed by induction on j. For $j = 1$, $E_1 = \{\eta, 0, 1\}$ and Ψ must be the extension of ψ_1 with $\Psi(\eta) = \psi_1(0) \cup \psi_1(1)$, which is T by hypothesis. Clearly Ψ works and is unique. Now assume the conclusion holds for j and suppose ψ_1 is defined on S_{j+1}. Let ψ_2 be defined on S_j by $\psi_2(w) = \psi_1(w0) \cup \psi_1(w1)$. The induction hypothesis applied to ψ_2 yields an extension σ of ψ_2. Define Ψ on E_{j+1} by $\Psi(w) = \sigma(w)$ if $\text{length}(w) \le j$ and $\Psi(w) = \psi_1(w)$ if $w \in S_{j+1}$. Clearly Ψ has the required properties. If $\overline{\Psi}$ is another such, $\overline{\Psi}$

restricted to the set of words of length less than or equal to j is a suitable extension of ψ_2 and so must be σ. It follows easily that $\overline{\Psi} = \Psi$.

Corollary R26.Add.7 Let k, j, l, x be integers with $k \geq 2$, $x \geq 1$, $l \geq 0$, and $2^j \leq k < 2^{j+1}$. Let $S = \{x + sk^l : 0 \leq s \leq k-1\}$ and $X = \cup\{B_{1/k^l}(f_k(x)) : x \in S\}$. Let $\{x_i : 1 \leq i \leq k\}$ be the k-ordering of S and $\{u_n : 1 \leq n \leq 2^j\}$ be the $<_D$-indexed ordering of S_j. Let $E_j = \{w \in M_2 : \text{length}(w) \leq j\}$. Let G be the extension of g_{lx} to E_j guaranteed by the previous lemma. If $u, v \in E_j$ are of equal length and $u <_D v$, then $G(u) <_k G(v)$.

Proof: Let $u, v \in E_j$ be of equal length with $u <_D v$. If their length is j, the conclusion follows from R26.Add.5ii, since G is an extension of g_{lx}. Let length$(u) = t$ where $t \leq j-1$ and the conclusion hold for words of length $t+1$. The words $u0 <_D u1 <_D v0 <_D v1$ are of length $t+1$ and so $G(u0) <_k G(u1) <_k G(v0) <_k G(v1)$. Since $G(u) = G(u0) \cup G(u1)$ and $G(v) = G(v0) \cup G(v1)$, it is routine to check that $G(u) <_k G(v)$ as required.

The next lemma is a version of R26.14 with an added property. In brief, the arbitrary bijection f used in the proof of R26.14 is replaced by the $<_D$-indexed ordering of S_j and a suitable g_{lx} is used as the initial map. The complete proof given here is somewhat repetitive but also uses a partly different presentation.

Lemma R26.Add.8 Let k, l, x be integers with $k \geq 2$, $x \geq 1$, and $l \geq 0$. Let $S = \{x + sk^l : 0 \leq s \leq k-1\}$ and $X = \cup\{B_{1/k^l}(f_k(x)) : x \in S\}$. Then there is a finite set $E \subseteq M_2$ and ψ from E to the non-empty closed subsets of X such that

i) $\psi(\eta) = X$.

ii) If $w \in E$ and $v \in M_2$ with length$(v) <$ length(w), then $v \in E$.

iii) If $wv \in E$, then $\psi(wv) \subseteq \psi(w)$.

iv) There is $j \geq 1$ such that $\{w \in M_2 : \text{length}(w) \leq j\} \subseteq E$.

v) $\psi(0) \cup \psi(1) = X$.

vi) If $w \in E$, then $w0 \in E$ if and only if $w1 \in E$.

vii) If w and $w0$ are in E, then $\psi(w) = \psi(w0) \cup \psi(w1)$.

viii) For $t \in S$ there is $w \in E$ such that $\psi(w) = B_{1/k^l}(f_k(t))$ and $w0 \notin E$.

ix) If $w \in E$ and $w0 \notin E$, then $\psi(w) = B_{1/k^l}(f_k(t))$ for some $t \in S$.

x) If u, w are equal-length elements of E with $u <_D w$, then $\psi(u) <_k \psi(w)$.

Proof: Assume j is the unique positive integer with $2^j \leq k < 2^{j+1}$ and let E_j be the set $\{w \in M_2 : \text{length}(w) \leq j\}$. Let $\{x_i : 1 \leq i \leq k\}$ be the k-ordering of S and $\{u_i : 1 \leq i \leq 2^j\}$ be the $<_D$-indexed ordering of S_j. Let G be the unique extenion of g_{lx} established in R26.Add.6. The domain of G is E_j. Next ψ will be defined. Let $c = 2^{j+1} - k$ and $E^* = \{u_i0, u_i1 : c < i \leq 2^j\}$. By R26.Add.3 E^* can also be represented as $\{u_{c+t}0, u_{c+t}1 : 1 \leq t \leq (k^m - c)/2\}$. Note that the elements of E^* have length $j+1$ and so ψ can be defined on $E_j \cup E^*$by $\psi = G$ on E_j, $\psi(u_{c+t}0) = B_{1/k^{m-1}}(f_k(x_{c+2t-1}))$ and $\psi(u_{c+t}1) = B_{1/k^{m-1}}(f_k(x_{c+2t}))$. Property x) will be verified first. Let u, w be equal-length elements of E with $u <_D w$. Either both are in E_j or both are in E^*, since elements of E^* are all of length $j+1$. If both are in E_j, the conclusion follows from R26.Add.7. Assume both are in E^*. As a first case, suppose both have the same initial string. Since $u <_D w$, it must be that $u = u_{c+t}0$ and $w = u_{c+t}1$. Then $\psi(u) = B_{1/k^l}(f_k(x_{c+2t-1}))$ and $\psi(w) = B_{1/k^l}(f_k(x_{c+2t}))$. Because the k-ordering of S is assumed, $\psi(u) <_k \psi(w)$. In the other case, the initial strings must be different, say $u = u_{c+s}a$ and $w = u_{c+t}b$, where $s \neq t$ and $a, b \in \{0, 1\}$. Since the initial strings are different, the dictionary order of u and w

is determined by their order, i.e., $u_{c+s} <_D u_{c+t}$. Because the $<_D$-indexed ordering of S_j is assumed, $c+s < c+t$. Moreover, $c+2s \leq c+2t-1$ since $c+2s < c+2t$. Equality cannot hold because it would imply $2s = 2t-1$. Thus $c+2s < c+2t-1$. Because the k-ordering of S is assumed, $\psi(u) <_k \psi(w)$. Property i) holds for G (and so for ψ) by R26.Add.6i. Property ii) holds since the shorter word must be in E_j. $E_j \subseteq E$ verifies iv). Property vi) holds for w of length less than or equal to $j-1$. For elements of S_j, say u_i with $1 \leq i \leq c$, neither u_i0 nor u_i1 is in E. For u_i with $c < i \leq 2^j$, $\{u_i0, u_i1\} \subseteq E^*$ by definition. Lastly $w \in E^*$ implies neither $w0$ nor $w1$ is in E. Thus vi) holds. Property vii) holds for any word w with length$(w) \leq j-1$ because it holds for G (and so for ψ) by R26.Add.6iii. It is vacuously true on E^* and $\{u_i : 1 \leq i \leq c\}$. For u_i with $c < i \leq 2^j$, vii) holds by the definitions of ψ and g_{lx}, which G extends. Properties vii) and i) imply v). For iii) assume $wv \in E$ and proceed by induction: If length$(v) = 0$, i.e. $v = \eta$, clearly $\psi(wv) \subseteq \psi(w)$. Now assume iii) holds for any word of length less than or equal t and suppose length$(v) = t+1$. Let $v = ux$ where $x \in \{0,1\}$. By ii) $wu \in E$ and by vi) $wu0$ and $wu1$, one of which is wv, are both in E. By vii) $\psi(wu) = \psi(wu0) \cup \psi(wu1)$ and by the induction hypothesis $\psi(wu) \subseteq \psi(w)$. It follows that $\psi(wv) \subseteq \psi(w)$ as required. For viii) let $x_i \in S$. If $1 \leq i \leq c$, by definition $\psi(u_i) = G(u_i) = g_{lx}(u_i) = B_{1/k^l}(f_k(x_i))$ and $u_i0 \notin E$. If $c+1 \leq i \leq k^m$, by R26.Add.3 there is t with $1 \leq t \leq (k^m - c)/2$ such that $i \in \{c+2t-1, c+2t\}$. For $w = u_{c+t}0$ if $i = c+2t-1$ and $w = u_{c+t}1$ if $i = c+2t$, $\psi(w) = B_{1/k^l}(f_k(x_i))$. In either case $w0$ has length $j+2$ and so $w0 \notin E$. Lastly, ix) holds: If $w \in E$ with $w0 \notin E$, either $w \in \{u_i : 1 \leq i \leq c\}$ or $w = u_{c+t}x$ where $x \in \{0,1\}$ and $1 \leq t \leq (k^m - c)/2\}$. By definition of ψ, those are the elements which map to the balls.

Lemma R26.Add.9 Let k, l, x be integers with $k \geq 2$, $x \geq 1$, and $l \geq 0$. Let $S = \{x + sk^l : 0 \leq s \leq k-1\}$ and $X = \cup\{B_{1/k^l}(f_k(x)) : x \in S\}$. Let ψ with domain E be constructed as in the previous lemma. Assume $u \neq v$ are in E with length$(u) =$ length(v). Then $\psi(u) \cap \psi(v) = \emptyset$. In addition, if $x \neq y$ are in E with $x0, y0 \notin E$, then $\psi(x) \cap \psi(y) = \emptyset$.

Proof: Let j be the unique positive integer such that $2^j \leq k < 2^{j+1}$ and use the notation defining ψ from the proof of R26.Add.8. Because no two elements of S are equivalent class mod k^{l+1}, the set of balls $\{B_{1/k^l}(f_k(x)) : x \in S\}$ is a pairwise disjoint collection. By R26.Add.3 the images of elements in E^* are distinct, and so disjoint, balls. In addition, the elements $e \in E$ with $e0 \notin E$ must come from $E^* \cup \{u_i : 1 \leq i \leq c\}$, each of which maps to a distinct ball. Thus the second claim holds. By R26.Add.8x, since distinct words of the same length must be $<_D$-related, the first claim holds.

Proposition R26.Add.10 Let k be in $\mathbb{N}$ with $k \geq 2$. Then there is an auxiliary function ϕ for $\mathbf{R}_k$ such that $\phi(v) \neq \emptyset$ for every $v \in M_2$ and, if w is an infinite word, diam$(\phi(w_n)) \to 0$ as $n \to \infty$.

Proof: First, a sequence of maps will be defined inductively. By R26.Add.1i $\mathbf{R}_k$ can be written as $\cup_{i=1}^k B_1(f_k(i))$. Let j be the positive integer with $2^j \leq k < 2^{j+1}$. Apply R26.Add.8 with $x = 1$ and $l = 0$ to obtain ϕ_1 from D_1 to the collection of unions from $\{B_1(f_k(1)), \ldots, B_1(f_k(k)\}$, where ϕ_1 has the lemma's properties, which easily imply the 10 properties listed below. Now assume $\phi_1, \ldots, \phi_n$ with domains $D_1 \subseteq D_2 \subseteq \cdots \subseteq D_n$ have been constructed with the following properties:

i) $D_n \subseteq M_2$ is finite and $\{w \in M_2 : \text{length}(w) \leq n\} \subseteq D_n$.

ii) For $2 \le i \le n$, $\phi_{i-1} = \phi_i|_{D_{i-1}}$.
iii) $\phi_n(\eta) = \mathbf{R}_k$.
iv) If $wv \in D_n$, then $w \in D_n$.
v) For $w \in D_n$, $w0 \in D_n$ if and only if $w1 \in D_n$.
vi) For $w \in D_n$ with $w0 \in D_n$, $\phi_n(w) = \phi_n(w0) \cup \phi_n(w1)$.
vii) If $w \in D_n$ and $w0 \notin D_n$, $\mathrm{diam}(\phi_n(w)) \le 2/k^{n-1}$.
viii) For every $w \in D_n$, $\phi_n(w)$ is a non-empty closed set.
ix) If $w \in D_n$ and $w0 \notin D_n$, $\phi_n(w) = B_{1/k^{n-1}}(f_k(x(w)))$ for some $x(w) \in \mathbb{N}$.
x) If $w \in D_n$ and $w = ab$, then $\phi_n(w) \subseteq \phi_n(a)$.

Call $w \in M_2$ a terminal word (of D_n) if $w \in D_n$ and $w0 \notin D_n$. There are finitely many terminal words. Note that no proper initial subword of a terminal word is a terminal word by properties iv) and v). For each terminal word w, by ix) and R26.Add.1ii, $\phi_n(w) = \cup_{s=0}^{k-1} B_{1/k^n}(f_k(x(w) + sk^n))$ for some $x(w) \in \mathbb{N}$. By R26.Add.8 there is ψ_w from a finite domain $A_w \subseteq M_2$ to the collection of unions from the family $\{B_{1/k_n}(x(w) + sk^n)\}_{s=0}^{k-1}$, where ψ_w has the properties listed in R26.Add.8. Let C_{n+1} denote the (finite) union over all terminal words of the finite sets $\{wv : v \in A_w, v \neq \eta\}$. Note that if $wv = us$ where w, u are terminal words, then $w = u$, since neither can be a proper initial subword of the other, and so also $v = s$. Thus each element of C_{n+1} has a unique representation in the specified form. Finally note that the requirement $v \neq \eta$ implies $D_n \cap C_{n+1} = \emptyset$. Let $D_{n+1} = D_n \cup C_{n+1}$. Define ϕ_{n+1} on D_{n+1} by $\phi_{n+1}(u) = \phi_n(u)$ if $u \in D_n$ and, if $u \in C_{n+1}$ with $u = wv$, $\phi(u) = \psi_w(v)$. Then ϕ_{n+1} is a function; clearly $D_n \subseteq D_{n+1}$ and $\phi_n = \phi_{n+1}|_{D_n}$ so that ii) holds for $2 \le i \le n+1$. Likewise, iii) and viii) hold for ϕ_{n+1}. For i), finiteness holds since each A_w is finite. Let w be a word of length $n+1$. Then $w = ux$, where $\mathrm{length}(u) = n$ and $x \in \{0, 1\}$. Then $u \in D_n$. If $u0 \in D_n$, $w \in D_n \subseteq D_{n+1}$. Otherwise u is a terminal word of D_n and, since $\{0, 1\} \subseteq A_u$, $w = ux \in C_{n+1} \subseteq D_{n+1}$. For iv), let $wv \in D_{n+1}$. Since iv) holds for D_n, assume $wv \in C_{n+1}$. Then $wv = us$, where u is a terminal word of D_n and $s \neq \eta$ is in A_u. If w is a subword of u, $u = wr$ and so $w \in D_n$ by the induction hypothesis. Otherwise u must be a proper subword of w, i.e., $w = ut$ where $t \neq \eta$. Then $wv = utv$ and so $tv = s \in A_u$. By R26.Add.2ii $t \in A_u$ and so $w = ut \in C_{n+1} \subseteq D_{n+1}$. For v) let $w \in D_{n+1}$ with $w0 \in D_{n+1}$. If $w0 \in D_n$, by iv) and v) of the induction hypothesis, $w1 \in D_n \subseteq D_{n+1}$. If $w0 \in C_{n+1}$, $w0 = us$, where u is a terminal word of D_n and $s \neq \eta$ is in A_u. Now $s = r0$ and by R26.Add.8vi $r1 \in A_u$. Thus $w1 = ur1$ is in $C_{n+1} \subseteq D_{n+1}$. The converse is similar. For vi) let $w0$ (and so w) be in D_{n+1}. If $w0 \in D_n$, since $\phi_n = \phi_{n+1}|_{D_n}$, apply vi) of the induction hypothesis for D_n. If $w0 \in C_{n+1}$, $w0 = us0$, where u is a terminal word of D_n and $s0 \in A_u$. By R26.Add.8ii $s \in A_u$ and so $w = us \in C_{n+1}$. Applying the definition and R26.Add.8vii, $\phi_{n+1}(w) = \psi_u(s) = \psi_u(s0) \cup \psi_u(s1) = \phi_{n+1}(w0) \cup \phi_{n+1}(w1)$. For ix) let $w \in D_{n+1}$ with $w0 \notin D_{n+1}$. Note that $w \notin D_n$ for, if so, w is a terminal word of D_n and $0 \in A_w$ so that $w0$ would be in $C_{n+1} \subseteq D_{n+1}$. Thus $w = us$, where u is a terminal word of D_n and $s \neq \eta$ is in A_u. Since $w0 = us0$ is not in C_{n+1}, $s0 \notin A_u$. As above, $\phi_n(u)$ is a union of balls of radius $1/k^n$. By R26.Add.8ix $\psi_u(s) = \phi_{n+1}(w)$ is a ball of radius $1/k^n$ as required for ix). It follows that $\mathrm{diam}(\phi_{n+1}(w)) \le 2/k^n$ so that vii) also holds. For x) let $w = ab$ be in D_n. By iv) $a \in D_n$. The conclusion is trivial if $b = \eta$ and so assume $b \neq \eta$ and proceed by induction on the length of b. For the initial case, $b \in \{0, 1\}$. By vi)

$\phi_{n+1}(a) = \phi_{n+1}(a0) \cup \phi_{n+1}(a1)$ and the conclusion holds. Assume it holds for any word of length r and let length$(b) = r + 1$. Write b as cx where $x \in \{0, 1\}$ and length$(c) = r$. By the initial step $\phi_{n+1}(w) \subseteq \phi_{n+1}(ac)$ and by the induction hypothesis $\phi_{n+1}(ac) \subseteq \phi_{n+1}(a)$. By induction there is an infinite sequence $\{\phi_n\}_{n=1}^{\infty}$ with the properties listed above. Since the domains are nested and ii) holds, $\phi = \cup_{n=1}^{\infty}\phi_n$ is a function and $\phi|_{D_n} = \phi_n$ for every n. Since i) holds, the domain of ϕ is M_2. For $w \in M_2$, pick n with $w0 \in D_n$. By vi) $\phi(w) = \phi_n(w) = \phi_n(w0) \cup \phi_n(w1) = \phi(w0) \cup \phi(w1)$. Thus ϕ is an auxiliary function for E and by viii) $\phi(w) \neq \emptyset$ for every w. Lastly, let w be an infinite word and let $\epsilon > 0$. Pick N such that $2/k^{N-1} < \epsilon$. Since D_N is finite, the set $\{n : w_n \notin D_N\}$ is non-empty and so has a smallest element M. Then $M \geq N + 1$ by i) and $w_{M-1} \in D_N$. By v) and the choice of M, $w_{M-1}0 \notin D_N$. By vii) diam$(\phi(w_{M-1}))$ =diam$(\phi_N(w_{M-1})) \leq 2/k^{N-1} < \epsilon$. For $n \geq M$, by R26.11 $\phi(w_n) \subseteq \phi(w_{M-1})$ and so diam$(\phi(w_n)) < \epsilon$. Thus the limit claim holds.

Lemma R26.Add.11 Let k be in $\mathbb{N}$ with $k \geq 2$. Let $\{\phi_n\}_{n=1}^{\infty}$ be the sequence of maps constructed in the previous proof. For every n, if u, v are equal length elements of D_n with $u <_D v$, then $\phi_n(u) <_k \phi_n(v)$.

Proof: Continue with the notation and terminology from the previous proof and proceed by induction on n. For $n = 1$ the claim holds by R26.Add.8x and so assume it holds for n and let u, v be equal-length elements of D_{n+1} with $u <_D v$. There several cases to consider. If both u, v are in D_n, $\phi_{n+1}(u) = \phi_n(u) <_k \phi_n(v) = \phi_{n+1}(v)$. Next suppose $u \in D_n$ and $v \in C_{n+1}$. Let $v = ws$, where w is a terminal word of D_n. Since $s \neq \eta$, w is shorter than v and u. Let $u = u_1 t$ where length(u_1) = length(w) so that $t \neq \eta$. By property iv) of ϕ_n, u_1 and $u_1 x$ for some $x \in \{0, 1\}$ are both in D_n and so by v) u_1 is not a terminal word of D_n. Thus $u_1 \neq w$ and so the dictionary ordering of u_1 and w determines the ordering of u and v, i.e., $u_1 <_D w$. By the induction hypothesis $\phi_n(u_1) <_k \phi_n(w)$. By property x) for ϕ_{n+1}, $\phi_{n+1}(u) \subseteq \phi_{n+1}(u_1) = \phi_n(u_1)$ and $\phi_{n+1}(v) \subseteq \phi_{n+1}(w) = \phi_n(w)$. It follows easily that $\phi_{n+1}(u) <_k \phi_{n+1}(v)$. As a second case, assume $u \in C_{n+1}$ and $v \in D_n$. Let $u = yt$ where y is a terminal word of D_n and $t \neq \eta$ so that y is shorter than both u and v. Write $v = v_1 s$ where length(v_1) = length(y) and so $s \neq \eta$. As in the first case v_1 is in D_n but is not a terminal word so that $v_1 \neq y$, the ordering of v_1 and y determines the ordering of v and u, and $y <_D v_1$. Using the induction hypothesis and x) again, $\phi_n(y) <_k \phi_n(v_1)$, $\phi_{n+1}(u) \subseteq \phi_{n+1}(y) = \phi_n(y)$ and $\phi_{n+1}(v) \subseteq \phi_{n+1}(v_1) = \phi_n(v_1)$. It follows easily that $\phi_{n+1}(u) <_k \phi_{n+1}(v)$. The last case, both u and v are in C_{n+1}, has several subcases. Write $u = yt$ and $v = ws$, where y, w are both terminal elements of D_n and both s, t are non-null. If $y = w$, t and s are equal-length length elements of the domain of $\psi_y = \psi_w$. The dictionary ordering of u and v is determined by the ordering of t and s, i.e., $t <_D s$. By of R26.Add.8x and the definition of ϕ_{n+1}, $\phi_{n+1}(u) = \psi_y(t) <_k \psi_y(s) = \psi_w(s) = \phi_{n+1}(v)$. Next suppose $y \neq w$ and length(y) = length(w). Here the ordering of u and v is determined by the order of y and w, i.e., $y <_D w$. By the induction hypothesis and the definition of ϕ_{n+1}, $\phi_{n+1}(y) = \phi_n(y) <_k \phi_n(w) = \phi_{n+1}(w)$. By property x) for ϕ_{n+1}, $\phi_{n+1}(u) \subseteq \phi_{n+1}(y)$ and $\phi_{n+1}(v) \subseteq \phi_{n+1}(w)$. The required conclusion now follows easily. For the next case suppose $y \neq w$ and length$(y) <$ length(w). Write $w = w_1 q$, where length(w_1) = length(y) and $q \neq \eta$. Since $w \in D_n$ and $q \neq \eta$, $w_1 \in D_n$ but is not a terminal word. Thus $w_1 \neq y$ and so the ordering of u, v is determined by y and w_1, i.e., $y <_D w_1$. By the induction hypothesis, $\phi_{n+1}(y) = \phi_n(y) <_k \phi_n(w_1) = \phi_{n+1}(w_1)$. By property x) for ϕ_{n+1}, $\phi_{n+1}(u) \subseteq \phi_{n+1}(y)$ and $\phi_{n+1}(v) \subseteq \phi_{n+1}(w_1)$, and so $\phi_{n+1}(u) <_k \phi_{n+1}(v)$. Finally, suppose $y \neq w$ and length$(y) >$ length(w). Write $y = y_1 r$, where length$(y_1) =$ length(w) and $r \neq \eta$. As in the previous case, y_1 is in D_n but is not a terminal word, and the ordering of u, v is determined by the ordering of y_1, w so that $y_1 <_D w$. Again by the induction hypothesis, $\phi_{n+1}(y_1) = \phi_n(y_1) <_k \phi_n(w) = \phi_{n+1}(w)$. By property x) for ϕ_{n+1}, $\phi_{n+1}(u) \subseteq \phi_{n+1}(y_1)$ and $\phi_{n+1}(v) \subseteq \phi_{n+1}(w)$, and so $\phi_{n+1}(u) <_k \phi_{n+1}(v)$.

Corollary R26.Add.12 Let k be in $\mathbb{N}$ with $k \geq 2$. Let ϕ be the auxiliary function for $\mathbf{R}_k$ constructed as in the proof of R26.Add.10. If u, v are equal-length elements of M_2 with $u <_D v$, then $\phi(u) <_k \phi(v)$.

Proof: Continue with the notation in the proof of R26.Add.10. Let u, v be equal-length elements of M_2 with $u <_D v$ and pick n greater than their common length so that $u, v \in D_n$. By R26.Add.11 $\phi(u) = \phi_n(u) <_k \phi_n(v) = \phi(v)$.

Proposition R26.Add.13 Let k be in $\mathbb{N}$ with $k \geq 2$. There is an order-preserving bijection from $(\mathcal{M}, <_D)$ to $(\mathbf{R}_k, <_k)$.

Proof: Let ϕ be the auxiliary function constructed as in the proof of R26.Add.10. For $w \in \mathcal{M}$, let $g(w) = \cap_{n=1}^{\infty} \phi(w_n)$. As in the proof of R26.13 g is a function and every element of $\mathbf{R}_k$ is the image of some infinite word, i.e., g is onto. Let u, w be in $\mathcal{M}$ with $u <_D w$. By definition the dictionary order is determined by the first position where they differ and so there is n with $u_n <_D w_n$. By R26.Add.12 $\phi(u_n) <_k \phi(w_n)$. Because $g(u) \in \phi(u_n)$ and $g(w) \in \phi(w_n)$, $g(u) <_k g(w)$. Thus g is order-preserving. Lastly, if $u \neq w$, either $u <_D w$ or $w <_D u$ so that $g(u) <_k g(w)$ or $g(w) <_k g(u)$. In either case, because the inequality is strict, $g(u) \neq g(w)$ and so g is one-to-one.

The next lemma records the simple and undoubtedly known fact that an order-preserving bijection between linearly ordered spaces must be an order isomorphism.

Lemma R26.Add.14 Let $(X, <)$ and $(Y, \prec)$ be linearly ordered spaces. Let the map $h : X \to Y$ be one-to-one, onto, and order- preserving. Then h^{-1} is also order-preserving.

Proof: Let $y_1, y_2 \in Y$ with $y_1 \prec y_2$ and let $x_i = h^{-1}(y_i)$. If $x_1 \leq x_2$, then, since h is order-preserving, $h(x_1) \preceq h(x_2)$, i.e., $y_1 \preceq y_2$, a contradiction. Thus $x_1 < x_2$ as required.

Corollary R26.Add.15 Let k, l be in $\mathbb{N}$ with $k, l \geq 2$. There is an order-preserving bijection from $(\mathbf{R}_k, <_k)$ to $(\mathbf{R}_l, <_l)$.

Proof: Let g_k and g_l be an order-preserving bijections from $(\mathcal{M}, <_D)$ to $(\mathbf{R}_k, <_k)$, respectively $(\mathbf{R}_l, <_l)$. Then $g_l \circ g_k^{-1}$ is the required map.

Next two more certainly known facts are noted.

Lemma R26.Add.16 Let $(X, <)$ and $(Y, \prec)$ be linearly ordered spaces. Let the map $h : X \to Y$ be one-to-one, onto, and order- preserving. Let a, b be a consecutive pair in X with b larger. Then $h(a), h(b)$ are consecutive in Y with $h(b)$ larger.

Proof: Since $a < b$, $h(a) \prec h(b)$. Suppose the images are not consecutive, i.e., there is $y \in Y$ with $h(a) \prec y \prec h(b)$. Then $a < h^{-1}(y) < b$, which contradicts the assumption that a, b are consecutive.

Lemma R26.Add.17 Let $(X, <)$ and $(Y, \prec)$ be linearly ordered spaces. Let the map $h : X \to Y$ be an order-preserving bijection. Then $h : (X, \tau(<)) \to (Y, \tau(\prec))$ is a homeomorphism.

Proof: Let $y \in Y$. It is easy to check that $h^{-1}[(-\infty, y)] = (-\infty, h^{-1}(y))$ and

$h^{-1}[(y, \infty)] = (h^{-1}(y), \infty)$. Thus the inverse image of every subbasic open set in Y is open in X and so h is continuous. Similarly, h^{-1} is continuous and the result follows.

Note that R26.25 could be derived from R26.Add.15, R26.Add.17, and R19.1.7.

Lemma R26.Add.18 Let k, l be in $\mathbb{N}$ with $k, l \geq 2$ and let h be an order-preserving bijection from $(\mathbf{R}_k, <_k)$ to $(\mathbf{R}_l, <_l)$. Then $h[f_k[\mathbb{N}]] = f_l[\mathbb{N}]$.

Proof: Since an order isomorphism maps the smallest element to the smallest, $h(f_k(1))$ is $f_l(1)$. Now let $j \in \mathbb{N}$ with $j \geq 2$. By R19.1.15 $f_k(j)$ is the larger of a consecutive pair. By R26.Add.15 $h(f_k(j))$ is the larger of a consecutive pair in $\mathbf{R}_l$. By R19.1.19 there is m in $\mathbb{N}$ with $m \geq 2$ such that $h(f_k(j)) = f_l(m)$. Thus $h[f_k[\mathbb{N}]] \subseteq f_l[\mathbb{N}]$. The same argument applied to h^{-1} yields the reverse containment.

Corollary R26.Add.19 Let k, l be in $\mathbb{N}$ with $k, l \geq 2$. Then the topological spaces $\mathbb{N}_k$ and $\mathbb{N}_l$ are homeomorphic.

Proof: This is immediate from R26.Add.15, R26.Add.17, R26.Add.18, and R26.28.

Additional Reference

9. This website, R19: Ordering the Remnant Rings