Normal Bases for the Remnant Rings

In [4] for $k \geq 2$ $\mathbb{N}_k$ was constructed as a Wallman compactification derived from $\mathcal{Z}_k$, a normal basis for $\mathbb{N}$ with the discrete topology. The remnant ring $\mathbf{R}_k$ was defined in [5] by removing the point-filters from $\mathbb{N}_k$.

In [7] $\mathbf{R}_k$ was shown to be a compactification of $(\mathbf{Z}\,, \tau_k)$ with an embedding f_k, where $f_k(z)$ is the non-point ultrafilter associated with $\{x_n\}_{n=1}^{\infty}$such that $x_n \equiv z \bmod k^n$ for every n. The non-discrete topology τ_k was shown to have a clopen basis $\mathcal{B}_k$ consisting of all $D_n^z(k)$ for $n \in \mathbb{N}$ and $z \in \mathbf{Z}$, where $D_n^z(k)$ is the equivalence class of z mod k^n.

Also in [7] $\mathbf{R}_\infty$, the mixed supremum of the $\mathbf{R}_k$, was shown to be a compactification of $(\mathbf{Z}\,, \tau_\infty)$, where τ_∞ is the supremum of $\{\tau_k : k \in \mathbb{N}\,\}$.

Here a normal basis for $(\mathbf{Z}\,, \tau_k)$ is described, the Wallman compactification of which is equivalent to $\mathbf{R}_k$. These bases lead to a normal basis for $(\mathbf{Z}\,, \tau_\infty)$, with the resulting Wallman compactification equivalent to $\mathbf{R}_\infty$.

A Normal Basis for $\mathbf{R}_k$

Lemma R27.1.1 Let k, l, m be in $\mathbb{N}$ with $k \geq 2$ and $l \leq m$. Let x, y be in $\mathbf{Z}$. Assume that $D_l^x(k) \cap D_m^y(k) \neq \emptyset$, then $D_m^y(k) \subseteq D_l^x(k)$.

Proof: Let $z \in D_l^x(k) \cap D_m^y(k)$ and let $a \in D_m^y(k)$. Then $a \equiv y \equiv z \bmod k^m$. Since $l \leq m$, $a \equiv z \equiv x \bmod k^l$ and so $a \in D_l^x(k)$.

Now define $\mathcal{D}_k$ to be the set of unions of finite subcollections of $\mathcal{B}_k$.

Lemma R27.1.2 Let $k \geq 2$ be in $\mathbb{N}$. Then

i) $\mathcal{B}_k \subseteq \mathcal{D}_k$ and each element of $\mathcal{D}_k$ is τ_k-clopen.

ii) $\mathcal{D}_k$ is closed under finite unions and intersections.

iii) $\mathcal{D}_k$ is closed under complementation.

Proof: Part i) follows since each $D_n^x(k)$, the union of a one-element subcollection, is clopen by R16.9 and a finite union of clopen sets is clopen. For ii) the claim about finite unions is clear from the definition. Let $D_1 = \cup_{i=1}^s D_{n_i}^{x_i}(k)$ and $D_2 = \cup_{j=1}^t D_{m_j}^{y_j}(k)$. By R27.1.1 $D_{n_i}^{x_i}(k) \cap D_{m_j}^{y_j}(k)$ is either $\emptyset$, which is in $\mathcal{D}_k$ as the union of the empty subcollection, or one of the equivalence classes. Thus $D_1 \cap D_2$ is a finite union of $\mathcal{B}_k$-sets and so in $\mathcal{D}_k$. For iii), the complement of $D_n^x(k)$ is the union of the other $k^n - 1$ equivalence classes mod k^n and so is in $\mathcal{D}_k$. The conclusion follows from this observation, DeMorgan's Law, and part ii).

Lemma R27.1.3 Let $k \geq 2$ be in $\mathbb{N}$. Then $\mathcal{D}_k$ is a normal basis for $(\mathbf{Z}\,, \tau_k)$.

Proof: Each element of $\mathcal{D}_k$ is τ_k-clopen and so τ_k-closed. By R16.8 $\mathcal{B}_k$ is a basis for τ_k and so the collection of complements of elements of $\mathcal{B}_k$, which is contained in $\mathcal{D}_k$, is a closed basis. $\mathcal{D}_k$ is closed under finite unions and intersections by the previous lemma. Let F be τ_k-closed and suppose $x \notin F$. There is $D \in \mathcal{B}_k \subseteq \mathcal{D}_k$ with $x \in D \subseteq \mathbf{Z} - F$ and so the third requirement for a normal basis holds. Finally let D_1, D_2 be in $\mathcal{D}_k$ with $D_1 \cap D_2 = \emptyset$. Let $C_1 = \mathbf{Z} - D_2$ and $C_2 = \mathbf{Z} - D_1$. By the previous lemma C_1 and C_2 are in $\mathcal{D}_k$, by DeMorgan's Law $\mathbf{Z} = C_1 \cup C_2$, and clearly $D_i \subseteq C_i$.

Lemma R27.1.4 Let $k \geq 2$ be in $\mathbb{N}$ and let $\mathcal{F}$ be in $\omega(\mathcal{D}_k)$. For every n in $\mathbb{N}$, there is a unique $x_n \in \{1, 2 \ldots k^n\}$ such that $D_n^{x_n}(k) \in \mathcal{F}$.

Proof: Fix n. The equivalence classes mod k^n are pairwise disjoint and so a $\mathcal{D}_k$-filter can contain at most one. The union of those equivalence classes is $\mathbf{Z}$ and so every $\mathcal{D}_k$-ultrafilter must contain at least one.

Lemma R27.1.5 Let $k \geq 2$ be in $\mathbb{N}$ and let $\mathcal{F}$ be in $\omega(\mathcal{D}_k)$. The sequence $\{x_n\}_{n=1}^{\infty}$ given by the previous lemma determines a unique $\mathcal{G}$ in $\mathbf{R}_k$.

Proof: By the previous lemma $x_1 \in \{1, 2, ..., k\}$ and, for each n, $D_n^{x_n}(k)$ and $D_{n+1}^{x_{n+1}}(k)$ are both in $\mathcal{F}$ so that $D_n^{x_n}(k) \cap D_{n+1}^{x_{n+1}}(k) \neq \emptyset$. By R27.1.1 $D_{n+1}^{x_{n+1}}(k) \subseteq D_n^{x_n}(k)$ and so $x_{n+1} \equiv x_n \mod k^n$. Since $x_{n+1} \in \{1, 2, \ldots, k^{n+1}\}$ and $x_n \in \{1, 2, \ldots, k^n\}$, there is $t \in \{0, 1, \ldots, k-1\}$ such that $x_{n+1} = x_n + tk^n$. By R10.2.6 there is a unique $\mathcal{G}$ in $\mathbf{R}_k$ associated with $\{x_n\}_{n=1}^{\infty}$.

The last two lemmas define a function $h_k : \omega(\mathcal{D}_k) \to \mathbf{R}_k$. This function will be used in the next five results.

Lemma R27.1.6 Let $k \geq 2$ be in $\mathbb{N}$. Then h_k is one-to-one.

Proof: Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be in $\omega(\mathcal{D}_k)$ with $h_k(\mathcal{F}_1) = h_k(\mathcal{F}_2)$. This ultrafilter in $\mathbf{R}_k$ is associated with a unique sequence $\{x_n\}_{n=1}^{\infty}$. Let $D = \cup_{i=1}^{t} D_{m_i}^{s_i}(k)$ be in $\mathcal{F}_1$ and let m be the maximum of $\{m_1, \ldots, m_t\}$. By definition of the associated sequence of h_k, $D_m^{x_m}(k)$ is in both $\mathcal{F}_1$ and $\mathcal{F}_2$. Since $\mathcal{F}_1$ is a $\mathcal{D}_k$-filter, $D \cap D_m^{x_m}(k) \neq \emptyset$ and so $D_{m_i}^{s_i}(k) \cap D_m^{x_m}(k) \neq \emptyset$ for some i. By R27.1.1 and the choice of m, $D_m^{x_m}(k) \subseteq D_{m_i}^{s_i}(k) \subseteq D$. Since $\mathcal{F}_2$ is a $\mathcal{D}_k$-filter, $D \in \mathcal{F}_2$. Since $\mathcal{F}_1$ is a $\mathcal{D}_k$-ultrafilter and $\mathcal{F}_1 \subseteq \mathcal{F}_2$, $\mathcal{F}_1 = \mathcal{F}_2$. Thus h_k is one-to-one.

Lemma R27.1.7 Let $k \geq 2$ be in $\mathbb{N}$. Then h_k is onto.

Proof: Let $\mathcal{G}$ in $\mathbf{R}_k$ be associated with $\{x_n\}_{n=1}^{\infty}$. Let $\mathcal{F}$ be defined as the collection $\{D \in \mathcal{D}_k : D_n^{x_n}(k) \subseteq D \text{ for some } n \in \mathbb{N}\}$. By R10.2.5ii, if $m < n$, $k^m | (x_n - x_m)$ and so $x_n \in D_m^{x_m}(k)$. This and R27.1.1 show that $\mathcal{F}$ is closed under finite intersections. It follows easily that $\mathcal{F}$ is a $\mathcal{D}_k$-filter. To see that it is an ultrafilter, let $D_0 = \cup_{i=1}^{t} D_{m_i}^{s_i}(k)$ be in $\mathcal{D}_k$ such that $D_0 \cap D \neq \emptyset$ for every $D \in \mathcal{F}$. Let $m = \max\{m_1, \ldots, m_t\}$. Since $D_m^{x_m}(k) \in \mathcal{F}$, $D_0 \cap D_m^{x_m}(k) \neq \emptyset$. It follows easily from R27.1.1 that $D_0 \in \mathcal{F}$. Thus $\mathcal{F}$ is in $\omega(\mathcal{D}_k)$. Let $\{y_n\}_{n=1}^{\infty}$ be associated with $h_k(\mathcal{F})$. By definition, since $D_n^{x_n}(k) \in \mathcal{F}$ for every n, the uniqueness of y_n implies $x_n = y_n$ for all n. By R10.2.4 $h_k(\mathcal{F}) = \mathcal{G}$.

The next lemma uses some notation from [4]: for $k, m \in \mathbb{N}$ with $k \geq 2$, $E_m(k)$ denotes equivalence mod k^m on $\mathbb{N}$.

Lemma R27.1.8 Let $k \geq 2$ be in $\mathbb{N}$. Then h_k is continuous.

Proof: By P3.6 $\mathcal{D}_k^{\omega}$ is a normal basis for $\omega(\mathcal{D}_k)$ and $\mathcal{Z}_k^{\omega}$ a normal basis for $\mathbb{N}_k$. The second fact says that $\{Z^{\omega} \cap \mathbf{R}_k : Z \in \mathcal{Z}_k\}$ is a closed basis for $\mathbf{R}_k$ and so it is sufficient to verify that $h_k^{-1}[Z^{\omega} \cap \mathbf{R}_k]$ is closed in $\omega(\mathcal{D}_k)$ for every $Z \in \mathcal{Z}_k$. Let $Z \in \mathcal{Z}_k$. There is m in $\mathbb{N}$ such that $Z \in \mathcal{Z}(E_m(k))$, and so Z is associated with some $\Delta \subseteq \{1, \ldots, k^m\}$. Let $D_0 = \cup\{D_m^t(k) : t \notin \Delta\}$, which is in $\mathcal{D}_k$. It is sufficient to show $h_k^{-1}[Z^{\omega} \cap \mathbf{R}_k] = D_0^{\omega}$. Let $\mathcal{G}$ be in $h_k^{-1}[Z^{\omega} \cap \mathbf{R}_k]$ with $\{x_n\}_{n=1}^{\infty}$ the associated sequence of $h_k(\mathcal{G})$. By the definition of h_k above, $D_m^{x_m}(k)$ is in $\mathcal{G}$. Since $Z \in h_k(\mathcal{G})$, by the definition of associated sequence (R10.2.3), $x_m \notin \Delta$ and so D_0, a superset of $D_m^{x_m}(k)$, is in $\mathcal{G}$, i.e., $\mathcal{G} \in D_0^{\omega}$. Conversely, let $\mathcal{F} \in D_0^{\omega}$ and suppose $\mathcal{F} \notin h_k^{-1}[Z^{\omega} \cap \mathbf{R}_k]$. Let $\{y_n\}_{n=1}^{\infty}$ be the associated sequence of $h_k(\mathcal{F})$. By the definition of h_k, $D_m^{y_m}(k)$ is in $\mathcal{F}$. Since $h_k(\mathcal{F})$ is not in Z^{ω}, $Z \notin h_k(\mathcal{F})$ and so, by R10.2.3, $y_m \in \Delta$. Since the equivalence classes mod k^m are pairwise disjoint, $D_0 \cap D_m^{y_m}(k) = \emptyset$, i.e., $D_0 \notin \mathcal{F}$, which contradicts $\mathcal{F} \in D_0^{\omega}$.

Corollary R27.1.9 Let $k \geq 2$ be in $\mathbb{N}$. As topological spaces, $\mathbf{R}_k$ and $\omega(\mathcal{D}_k)$ are homeomorphic and h_k is a homeomorphism.

Proof: Since both spaces are compact and T_2, by the previous three lemmas h_k is a homeomorphism.

The first part of the last corollary would also follow immediately from the following stronger result.

Proposition R27.1.10 Let $k \geq 2$ be in $\mathbb{N}$. The compactifications $\mathbf{R}_k$ and $\omega(\mathcal{D}_k)$ are equivalent.

Proof: Let ι_k denote the embedding of $(\mathbf{Z}, \tau_k)$ into $\omega(\mathcal{D}_k)$. Since h_k is a homeomorphism, it is sufficient to show $h_k \circ \iota_k = f_k$, where f_k is the embedding into $\mathbf{R}_k$ described in [7]. Let z be in $\mathbf{Z}$. By definition P3.8 $\iota_k(z)$ is the $\mathcal{D}_k$-pointfilter of z, i.e., $\{D \in \mathcal{D}_k : z \in D\}$. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence associated with $h_k(\iota_k(z))$. For each n, by definition of h_k, $D_n^{x_n}(k)$ is in $\iota_k(z)$, i.e., $z \in D_n^{x_n}(k)$ so that $x_n \equiv z \bmod k^n$. By the description of the associated sequence of $f_k(z)$ in this section and the uniqueness of the associated sequence, $h_k(\iota_k(z)) = f_k(z)$.

Directed Sets of Normal Bases

In [3] results were obtained about directed sets of normal bases under the assumption that each normal basis generated the same topology. This subsection contains modest generalizations of those results to the case in which the normal bases may generate different topologies.

Proposition R27.2.1 Let $\{\tau_\gamma : \gamma \in \Gamma\}$ be a non-empty set of topologies on a set X. For each γ let $\mathcal{Z}_\gamma$ be a normal basis for (X, τ_γ) and assume $\{\mathcal{Z}_\gamma : \gamma \in \Gamma\}$ is a directed set under containment. Let $\mathcal{Z} = \cup\{\mathcal{Z}_\gamma : \gamma \in \Gamma\}$ and let $\tau = \vee\{\tau_\gamma : \gamma \in \Gamma\}$. Then $\mathcal{Z}$ is a normal basis for (X, τ).

Proof: Each $Z \in \mathcal{Z}$ is τ_γ-closed for some γ and so is τ-closed. Let F be τ-closed and let $x \notin F$. There is a finite set $\Delta \subseteq \Gamma$ and $O_\gamma \in \tau_\gamma$ for each γ in Δ such that $x \in \cap\{O_\gamma : \gamma \in \Delta\} \subseteq X - F$. For each $\gamma \in \Delta$ there is $Z_\gamma \in \mathcal{Z}_\gamma$ such that $X - O_\gamma \subseteq Z_\gamma$ and $x \notin Z_\gamma$. By the directed set assumption there is $\delta \in \Gamma$ such that $\mathcal{Z}_\gamma \subseteq \mathcal{Z}_\delta$ for each $\gamma \in \Delta$ and so $Z = \cup\{Z_\gamma : \gamma \in \Delta\} \in \mathcal{Z}_\delta \subseteq \mathcal{Z}$. Clearly $F \subseteq \cup\{X - O_\gamma : \gamma \in \Delta\} \subseteq Z$ and $x \notin Z$. Thus $\mathcal{Z}$ is a closed basis for (X, τ). The directed set assumption similarly shows that $\mathcal{Z}$ is closed under finite unions and intersections. For the third property of a normal basis again let F be τ-closed with $x \notin F$. As before there is a finite set Δ and $\{O_\gamma \in \tau_\gamma : \gamma \in \Delta\}$ with $x \in \cap\{O_\gamma : \gamma \in \Delta\} \subseteq X - F$. For each $\gamma \in \Delta$ there is $Z_\gamma \in \mathcal{Z}_\gamma$ such that x is in Z_γ and $(X - O_\gamma) \cap Z_\gamma = \emptyset$. By the directed set property $Z = \cap\{Z_\gamma : \gamma \in \Delta\}$ is in $\mathcal{Z}$. Clearly $x \in Z$ and $Z \cap F = \emptyset$. For the fourth property, let Z_1, Z_2 be in $\mathcal{Z}$ with $Z_1 \cap Z_2 = \emptyset$. By the directed set assumption there is α in Γ such that both Z_1 and Z_2 are in $\mathcal{Z}_\alpha$. By the fourth property for the normal basis $\mathcal{Z}_\alpha$ there are C_1 and C_2 in $\mathcal{Z}_\alpha \subseteq \mathcal{Z}$ such that $X = C_1 \cup C_2$, $Z_1 \subseteq C_1$, and $Z_2 \subseteq C_2$. Thus the conclusion holds.

The next few results examine the possibility of comparing compactifications generated by nested normal bases. In the next lemma $\mathcal{F}_x^i$ denotes the point-filter of x for the normal basis $\mathcal{Z}_i$.

Lemma R27.2.2 Let τ_1 and τ_2 be topologies for a set X and let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be normal bases for τ_1, respectively τ_2, with $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$. Then

i) If $\mathcal{F}$ is a $\mathcal{Z}_2$-filter, then $\mathcal{F} \cap \mathcal{Z}_1$ is a $\mathcal{Z}_1$-filter.

ii) For every $x \in X$, $\mathcal{F}_x^2 \cap \mathcal{Z}_1 = \mathcal{F}_x^1$.

iii) Assume for every $\mathcal{F}$ in $\omega(\mathcal{Z}_2)$ that $\mathcal{F} \cap \mathcal{Z}_1$ is in $\omega(\mathcal{Z}_1)$.

Then $[(\omega(\mathcal{Z}_2), \iota_2)] \geq [(\omega(\mathcal{Z}_1), \iota_1)]$.

Proof: The proof is identical to the proof of R9.1.1, despite the apparent complication of different topologies. The generalized definition of $\leq$ for compactifications [6] is also required.

The next lemma could be expressed with what seems like more generality (cf. R9.1.5), but it covers the examples which will be of interest here.

Lemma R27.2.3 Let τ_1 and τ_2 be topologies for a set X and let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be normal bases for τ_1, respectively τ_2, with $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$. Assume $\mathcal{Z}_1$ is closed under complementation. If $\mathcal{F}$ in $\omega(\mathcal{Z}_2)$, then $\mathcal{F} \cap \mathcal{Z}_1$ is in $\omega(\mathcal{Z}_1)$.

Proof: Let $\mathcal{F}$ in $\omega(\mathcal{Z}_2)$. By the previous lemma $\mathcal{F} \cap \mathcal{Z}_1$ is a $\mathcal{Z}_1$-filter and so it is only necessary to show that it is an ultrafilter. Let $A \in \mathcal{Z}_1$ with $A \cap F \neq \emptyset$ for every $F \in \mathcal{F} \cap \mathcal{Z}_1$. Suppose A is not in the $\mathcal{Z}_2$-ultrafilter $\mathcal{F}$. By P3.3iii, there is $C \in \mathcal{F}$ such that $C \subseteq X - A$. By hypothesis $X - A$ is in $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$ and so, by the superset property of the $\mathcal{Z}_2$-filter $\mathcal{F}$, $X - A$ is in $\mathcal{F}$, thus in $\mathcal{F} \cap \mathcal{Z}_1$. But $A \cap (X - A) = \emptyset$, a contradiction. Thus $A \in \mathcal{F} \cap \mathcal{Z}_1$ and so by P3.3 $\mathcal{F} \cap \mathcal{Z}_1$ is a $\mathcal{Z}_1$-ultrafilter.

Lemma R27.2.4 Let $\{\tau_\gamma : \gamma \in \Gamma\}$ be a non-empty set of topologies on a set X. For each γ let $\mathcal{Z}_\gamma$ be a normal basis for (X, τ_γ) and assume $\{\mathcal{Z}_\gamma : \gamma \in \Gamma\}$ is a directed set under containment. Let $\mathcal{Z} = \cup\{\mathcal{Z}_\gamma : \gamma \in \Gamma\}$. For each $\gamma \in \Gamma$ let $\mathcal{F}_\gamma$ be in $\omega(\mathcal{Z}_\gamma)$. Assume further that $\alpha, \beta \in \Gamma$ and $\mathcal{Z}_\alpha \subseteq \mathcal{Z}_\beta$ imply $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$. Then $\cup\{\mathcal{F}_\gamma : \gamma \in \Gamma\}$ is in $\omega(\mathcal{Z})$.

Proof: Since each $\mathcal{F}_\gamma$ is a $\mathcal{Z}_\gamma$-filter, $\cup\{\mathcal{F}_\gamma : \gamma \in \Gamma\}$ is a non-empty collection of non-empty $\mathcal{Z}$-sets. Let A and B be in $\cup\{\mathcal{F}_\gamma : \gamma \in \Gamma\}$ with $A \in \mathcal{F}_\alpha$ and $B \in \mathcal{F}_\beta$. By the directed set assumption, there is $\delta \in \Gamma$ with $\mathcal{Z}_\alpha \cup \mathcal{Z}_\beta \subseteq \mathcal{Z}_\delta$. By hypothesis both A and B are in $\mathcal{F}_\delta$. Thus $A \cap B$ is in $\mathcal{F}_\delta$ and so in the union. The superset property for $\cup\{\mathcal{F}_\gamma : \gamma \in \Gamma\}$ follows from the assumptions in a similar way. Finally let $W \in \mathcal{Z}$ with $W \cap F \neq \emptyset$ for every $F \in \cup\{\mathcal{F}_\gamma : \gamma \in \Gamma\}$. For some γ, $W \in \mathcal{Z}_\gamma$ and $W \cap F \neq \emptyset$ for every $F \in \mathcal{F}_\gamma$. Since $\mathcal{F}_\gamma$ is a $\mathcal{Z}_\gamma$-ultrafilter, W is in $\mathcal{F}_\gamma$ and so in the union. By P3.3 $\cup\{\mathcal{F}_\gamma : \gamma \in \Gamma\}$ is a $\mathcal{Z}$-ultrafilter.

In the next lemma $\mathcal{F}_x$ and $\mathcal{F}_x^\gamma$ refer to point-filters of x in $\mathcal{Z}$ and $\mathcal{Z}_\gamma$ respectively.

Lemma R27.2.5 Let $\{\tau_\gamma : \gamma \in \Gamma\}$ be a non-empty set of topologies on a set X. For each γ let $\mathcal{Z}_\gamma$ be a normal basis for (X, τ_γ) and assume $\{\mathcal{Z}_\gamma : \gamma \in \Gamma\}$ is a directed set under containment. Let $\mathcal{Z} = \cup\{\mathcal{Z}_\gamma : \gamma \in \Gamma\}$. Then, for every $x \in X$, $\mathcal{F}_x = \cup\{\mathcal{F}_x^\gamma : \gamma \in \Gamma\}$.

Proof: $A \in \mathcal{F}_x$ if and only if $x \in A$ and $A \in \mathcal{Z}$ if and only if $x \in A$ and $A \in \mathcal{Z}_\gamma$ for some $\gamma \in \Gamma$ if and only if $A \in \mathcal{F}_x^\gamma$ for some $\gamma \in \Gamma$.

The next proposition is the main result of this subsection. The generalized definition of $\leq$ from [6] is again used.

Proposition R27.2.6 Let $\{\tau_\gamma : \gamma \in \Gamma\}$ be a non-empty set of topologies on a set X. For each γ let $\mathcal{Z}_\gamma$ be a normal basis for (X, τ_γ) and assume $\{\mathcal{Z}_\gamma : \gamma \in \Gamma\}$ is a directed set under containment. Let $\mathcal{Z} = \cup\{\mathcal{Z}_\gamma : \gamma \in \Gamma\}$. Assume that, for $\gamma \in \Gamma$, $\mathcal{Z}_\gamma$ is closed under complementation. Then $[(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})]$ acts as a supremum of the collection $\{[(\omega(\mathcal{Z}_\gamma), \iota_\gamma)] : \gamma \in \Gamma\}$.

Proof: By R27.2.3 and R27.2.2iii $[(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})] \geq [(\omega(\mathcal{Z}_\gamma), \iota_\gamma)]$ for every $\gamma \in \Gamma$, i.e it is an upper bound. Assume $[(Y, f)]$ is also an upper bound. For each $\gamma \in \Gamma$, let $h_\gamma : Y \to \omega(\mathcal{Z}_\gamma)$

be the continuous surjection with $h_\gamma \circ f = \iota_\gamma$. Suppose $\alpha, \beta \in \Gamma$ with $\mathcal{Z}_\alpha \subseteq \mathcal{Z}_\beta$. By the proof of R27.2.2iii the unique continuous surjection $g : \omega(\mathcal{Z}_\beta) \to \omega(\mathcal{Z}_\alpha)$ such that $g \circ \iota_\beta = \iota_\alpha$ is given by $g(\mathcal{F}) = \mathcal{F} \cap \mathcal{Z}_\alpha$. Then $g \circ h_\beta \circ f = g \circ \iota_\beta = \iota_\alpha$. The uniqueness of h_α implies that $g \circ h_\beta = h_\alpha$. Thus, for any $y \in Y$, $h_\alpha(y) = h_\beta(y) \cap \mathcal{Z}_\alpha \subseteq h_\beta(y)$ and by R27.2.4 $\cup\{h_\gamma(y) : \gamma \in \Gamma\}$ is in $\omega(\mathcal{Z})$. Now define $h : Y \to \omega(\mathcal{Z})$ by $h(y) = \cup\{h_\gamma(y) : \gamma \in \Gamma\}$. For $x \in X$, $h \circ f(x) = \cup\{h_\gamma(f(x)) : \gamma \in \Gamma\} = \cup\{\iota_\gamma(x) : \gamma \in \Gamma\} = \cup\{\mathcal{F}_x^\gamma : \gamma \in \Gamma\}$, which is $\mathcal{F}_x = \iota_\mathcal{Z}(x)$ by R27.2.5. Thus $h \circ f = \iota_\mathcal{Z}$. To see that h is continuous, let $Z \in \mathcal{Z}$. There is $\delta \in \Gamma$ such that $Z \in \mathcal{Z}_\delta$. Since $h_\delta^{-1}[Z^\omega]$ is closed, it is sufficient to show $h^{-1}[Z^\omega] = h_\delta^{-1}[Z^\omega]$. For $y \in h_\delta^{-1}[Z^\omega]$, $Z \in h_\delta(y) \subseteq h(y)$ and so $h(y) \in Z^\omega$, i.e., $y \in h^{-1}[Z^\omega]$. Conversely, let $y \in h^{-1}[Z^\omega]$, i.e., $Z \in h(y)$. For every $W \in h_\delta(y) \subseteq h(y)$, $W \cap Z \neq \emptyset$. Since $Z \in \mathcal{Z}_\delta$ and $h_\delta(y)$ is a $\mathcal{Z}_\delta$ ultrafilter, by P3.3 $Z \in h_\delta(y)$, i.e., $y \in h_\delta^{-1}[Z^\omega]$. Thus $h^{-1}[Z^\omega] = h_\delta^{-1}[Z^\omega]$ as claimed. Since $h \circ f = \iota_\mathcal{Z}$, the image of h contains the dense $\iota_\mathcal{Z}[X]$ and so, since Y is compact and $\omega(\mathcal{Z})$ is T_2, by continuity h is onto. In summary, $[(Y, f)] \geq [(\omega(\mathcal{Z}), \iota_\mathcal{Z})]$, i.e., $[(\omega(\mathcal{Z}), \iota_\mathcal{Z})]$ acts as a least upper bound as claimed.

It might be appealing to write the conclusion of the last proposition in a more concise form as $(\omega(\mathcal{Z}), \iota_\mathcal{Z}) = \vee\{(\omega(\mathcal{Z}_\gamma, \iota) : \gamma \in \Gamma\}$, but that "equation" is set-theoretically muddled.

A Normal Basis for $\mathbf{R}_\infty$

Lemma R27.3.1 Let $j, l, q \in \mathbb{N}$ with $j, l \geq 2$ and $l = jq$. Let $z \in \mathbf{Z}$. For any $n \in \mathbb{N}$ $D_n^z(j) = \cup_{i=0}^{q^n-1} D_n^{z+ij^n}(l)$.

Proof: Since $z + ij^n \equiv z \bmod j^n$ and $j^n | l^n$, by transitivity each equivalence class in the union is contained in $D_n^z(j)$. Conversely, let $w \in D_n^z(j)$. If $w \geq z$, $w - z = tj^n$ for $t \geq 0$. By the divisor theorem $t = aq^n + i$ where a, i are a non-negative integers and $0 \leq i \leq q^n - 1$. Then $w - z = (aq^n + i)j^n = al^n + ij^n$. It follows that $w \equiv z + ij^n \bmod l^n$, i.e., $w \in D_n^{z+ij^n}(l)$. If $w < z$, pick m such that $w + ml^n \geq z$. By the first case, $w + ml^n$ is in $D_n^{z+ij^n}(l)$ for some $0 \leq i \leq q^n - 1$, as is w since $w \equiv w + ml^n \bmod l^n$.

Lemma R27.3.2 The collection of normal bases $\{\mathcal{D}_k : k \geq 2\}$ is a directed set under containment.

Proof: Let i, j be in $\mathbb{N}$ with $i, j \geq 2$. It is immediate from the previous lemma that $D_n^z(i)$ and $D_n^z(j)$ are in $\mathcal{D}_{ij}$ for all $n \in \mathbb{N}$ and $z \in \mathbf{Z}$ and so $\mathcal{D}_i \cup \mathcal{D}_j \subseteq \mathcal{D}_{ij}$.

Definition R27.3.3 $\mathcal{D}_\infty = \cup\{\mathcal{D}_k : k \geq 2\}$.

As shown in R16.31 $\tau_\infty = \vee\{\tau_k : k \geq 2\}$.

Corollary R27.3.4 $\mathcal{D}_\infty$ is a normal basis for $(\mathbf{Z}, \tau_\infty)$.

Proof: This is immediate from R27.2.1.

In the next result, ι_∞ denotes the embedding of $(\mathbf{Z}, \tau_\infty)$ into $\omega(\mathcal{D}_\infty)$. As in R16.16, f_∞ is the embedding into $\mathbf{R}_\infty$.

Corollary R27.3.5 The compactifications $(\omega(\mathcal{D}_\infty), \iota_\infty)$ and $(\mathbf{R}_\infty, f_\infty)$ are equivalent.

Proof: By R16.29 $[(\mathbf{R}_\infty, f_\infty)]$ acts as a supremum for $\{[(\mathbf{R}_k, f_k)] : k \geq 2\}$. By R27.1.2iii and R27.2.6, $[(\omega(\mathcal{D}_\infty), \iota_\infty)]$ acts as a supremum for $\{[(\omega(\mathcal{D}_k), f_k)] : k \geq 2\}$. By R27.1.10 $[(\mathbf{R}_k, f_k)] = [(\omega(\mathcal{D}_k), f_k)]$ for all $k \geq 2$. Thus $[(\mathbf{R}_\infty, f_\infty)] = [(\omega(\mathcal{D}_\infty), \iota_\infty)]$, i.e., the conclusion holds.

The Remnant Rings as Compactifications of $\mathbb{N}$

The following notation will be used in the this subsection: Let X be a set with $A \subseteq X$. If τ is a topology for X, τ_A will denote the relative topology on A. If $\mathcal{U}$ is a uniformity on X, $\mathcal{U}_A$ will denote the relative uniformity on A.

Lemma R27.4.1 Let (Y, f) be a T_2 compactification of the topological space (X, τ). Let $A \subseteq X$ and assume $f[A]$ is dense in Y. Then $(Y, f|_A)$ is a T_2 compactification of (A, τ_A).

Proof: Since $f|_A[A] = f[A]$ is dense, it is sufficient to verify that $f|_A$ is an embedding, i.e., a homeomorphism onto its image. As a restriction of a one-to-one, continuous map, $f|_A$ is one-to-one and continuous. For $O \in \tau$, $f[O] = G \cap f[X]$ for some G open in Y. Since f is one-to-one, $f|_A[O \cap A] = G \cap f[A]$.

Comment: The hypothesis of the last lemma is set up to use a previously derived result. It's easy to check that $f[A]$ dense in Y is equivalent to A dense in X.

Lemma R27.4.2 Let (Y, f) and (Z, g) be equivalent T_2 compactifications of the topological space (X, τ). Let A be a dense subset of X. Then $(Y, f|_A)$ is equivalent to $(Z, g|_A)$.

Proof: As noted, A dense in X implies $f[A]$ is dense in Y and $g[A]$ is dense in Z. Let $h : Y \to Z$ be the homeomorphism such that $h \circ f = g$. Clearly $h \circ f|_A = g|_A$ and so the conclusion holds.

Lemma R27.4.3 Let $(X,\mathcal{U})$ be a separated, totally bounded uniform space and let $\mathcal{U}$ correspond to the compactification class of (Y, f). Let $A \subseteq X$ with $f[A]$ dense in Y. Then $\mathcal{U}_A$ corresponds to the compactification class of $(Y, f|_A)$.

Proof: By R1.6a it is sufficient to check that $f|_A$ is a uniform embedding, i.e., it is a unimorphism onto its image. As a restriction, $f|_A$ is one-to-one and uniformly continuous. For $U \in \mathcal{U}$, $(f \times f)[U] = V \cap (f[X] \times f[X])$ for some V in the unique uniformity for Y. Since f is one-to-one, $(f|_A \times f|_A)[U \cap (A \times A)] = V \cap (f[A] \times f[A])$.

For $k \in \mathbb{N}$ with $k \geq 2$, let σ_k be the relative topology on $\mathbb{N}$ from τ_k and let g_k be the restriction of f_k to $\mathbb{N}$.

Corollary R27.4.4 Let $k \in \mathbb{N}$ with $k \geq 2$. Then $(\mathbf{R}_k, g_k)$ is a T_2 compactification of $(\mathbb{N}, \sigma_k)$.

Proof: By R12.6.9 $f_k[\mathbb{N}]$ is dense in $\mathbf{R}_k$ and so this follows from R16.15 and R27.4.1.

For $k \in \mathbb{N}$ with $k \geq 2$, in R16.24 $\mathcal{V}_k$ was used to denote the separated, totally bounded uniformity on $\mathbf{Z}$ corresponding to the compactification class of $(\mathbf{R}_k, f_k)$. Let $\mathcal{W}_k$ denote the relative uniformity on $\mathbb{N}$ from $\mathcal{V}_k$.

Corollary R27.4.5 Let $k \in \mathbb{N}$ with $k \geq 2$. Then $\mathcal{W}_k$ corresponds to the compactification class of $(\mathbf{R}_k, g_k)$.

Proof: This follows from R12.6.9 and R27.4.3.

In R16.30 τ_∞ was used to denote the topology on $\mathbf{Z}$ such that $(\mathbf{R}_\infty, f_\infty)$ is a T_2 compactification of $(\mathbf{Z}, \tau_\infty)$ and in R16.23 $\mathcal{V}_\infty$ denoted the separated, totally bounded uniformity on $\mathbf{Z}$ corresponding to the compactification class of $(\mathbf{R}_\infty, f_\infty)$. Let g_∞ denote the restriction of f_∞ to $\mathbb{N}$, let σ_∞ be the the relative topology on $\mathbb{N}$ from τ_∞, and let $\mathcal{W}_\infty$ be the relative uniformity on $\mathbb{N}$ from $\mathcal{V}_\infty$.

Corollary R27.4.6 $(\mathbf{R}_\infty, g_\infty)$ is a T_2 compactification of $(\mathbb{N}, \sigma_\infty)$.

Proof: This follows from R12.6.18 and R27.4.1.

Corollary R27.4.7 $\mathcal{W}_\infty$ corresponds to the compactification class of $(\mathbf{R}_\infty\,, g_\infty)$.

Proof: This follows from R12.6.18 and R27.4.3.

Finally, it will be shown that these compactifications are equivalent to Wallman compactifications. For $A \subseteq X$ and $\mathcal{Z}$ a normal basis for a topology on X, $\mathcal{Z}_A$ denotes $\{Z \cap A : Z \in \mathcal{Z}\}$. Given a $\mathcal{Z}$-filter $\mathcal{F}$, $\mathcal{F}_A = \{F \cap A : F \in \mathcal{F}\}$.

Lemma R27.4.8 Let $\mathcal{Z}$ be a normal basis for the topological space (X, τ) and let $A \subseteq X$. Assume that, for every non-empty $Z \in \mathcal{Z}$, $Z \cap A \neq \emptyset$. Then $\mathcal{Z}_A$ is a normal basis for (A, τ_A).

Proof: The first 3 requirements for a normal basis in definition P3.1 clearly hold for any subset, without the additional hypothesis. For the fourth, let $(Z \cap A) \cap (W \cap A) = \emptyset$, where $Z, W \in \mathcal{Z}$. By hypothesis, $A \cap (Z \cap W) = \emptyset$ implies $Z \cap W = \emptyset$ and so there exist C, D in $\mathcal{Z}$ such that $Z \subseteq X - C$, $W \subseteq X - D$, and $C \cup D = X$. Clearly, $Z \cap A \subseteq A - (C \cap A)$, $(W \cap A) \subseteq A - (D \cap A)$, and $(C \cap A) \cup (D \cap A) = A$.

Lemma R27.4.9 Let $\mathcal{Z}$ be a normal basis for the topological space (X, τ) and let $A \subseteq X$. Assume that, for every non-empty $Z \in \mathcal{Z}$, $Z \cap A \neq \emptyset$. Then A is dense in X.

Proof: Let $O \in \tau$ be non-empty. Let $x \in O$. By P3.1iii there is $Z \in \mathcal{Z}$ such that $x \in Z$ and $Z \cap (X - O) = \emptyset$. Then $Z \subseteq O$, $Z \cap A \neq \emptyset$, and so $A \cap O \neq \emptyset$.

Comment: The converse of the previous lemma is false. Let $X = [0, 1]$ with the usual topology, let $\mathcal{Z}$ be the zero-sets of X (cf. P3.11), and let $A = (0, 1)$.

Lemma R27.4.10 Let $\mathcal{Z}$ be a normal basis for the topological space (X, τ) and let $A \subseteq X$. Assume that, for every non-empty $Z \in \mathcal{Z}$, $Z \cap A \neq \emptyset$. Let $\mathcal{F}$ be a $\mathcal{Z}$-filter. Then

i) $\mathcal{F}_A$ is a $\mathcal{Z}_A$-filter.

ii) If $\mathcal{F} \in \omega(\mathcal{Z})$, then $\mathcal{F}_A \in \omega(\mathcal{Z}_A)$.

Proof: The hypotheses say that $F \cap A \neq \emptyset$ for every $F \subset \mathcal{F}$. The other requirements for a filter can be easily checked and so part i) holds. For part ii), suppose $\mathcal{F}$ is a $\mathcal{Z}$-ultrafilter. Assume $Z \in \mathcal{Z}$ with $(Z \cap A) \cap (F \cap A) \neq \emptyset$ for every $F \cap A \in \mathcal{F}_A$. Then $Z \cap F \neq \emptyset$ for every $F \in \mathcal{F}$ and so by P3.3 $Z \in \mathcal{F}$. Thus $Z \cap A \in \mathcal{F}_A$ and by P3.3 again $\mathcal{F}_A$ is a $\mathcal{Z}_A$-ultrafilter.

In the following ι_X and ι_A denote the standard embeddings induced by the normal bases $\mathcal{Z}$ and $\mathcal{Z}_A$.

Lemma R27.4.11 Let $\mathcal{Z}$ be a normal basis for the topological space (X, τ) and let $A \subseteq X$. Assume that, for every non-empty $Z \in \mathcal{Z}$, $Z \cap A \neq \emptyset$. Then $(\omega(\mathcal{Z}_A), \iota_A)$ is equivalent to $(\omega(\mathcal{Z}), \iota_X|_A)$.

Proof: By R27.4.9 A is dense in X and so $\iota_X|_A[A]$ is dense in $\omega(\mathcal{Z})$. By R27.4.10ii a map $h : \omega(\mathcal{Z}) \to \omega(\mathcal{Z}_A)$ is defined by $h(\mathcal{F}) = \mathcal{F}_A$. To see that h is one-to-one, suppose $\mathcal{F}_A = \mathcal{G}_A$ and let $Z \in \mathcal{F}$. Since $F \cap A$ is in $\mathcal{G}_A$, for every $W \in \mathcal{G}$, $(Z \cap A) \cap (W \cap A) \neq \emptyset$, i.e., $Z \cap W \neq \emptyset$. Since $\mathcal{G}$ is an ultrafilter, $Z \in \mathcal{G}$. Since $\mathcal{F}$ is an ultrafilter and $\mathcal{F} \subseteq \mathcal{G}$, $\mathcal{F} = \mathcal{G}$. To see that h is continuous, let $Z \in \mathcal{Z}$. It is sufficient to show $h^{-1}[(Z \cap A)^\omega] = Z^\omega$. Let $\mathcal{F} \in Z^\omega$, i.e., $Z \in \mathcal{F}$. Then $Z \cap A$ is in $h(\mathcal{F})$ and so $\mathcal{F} \in h^{-1}[(Z \cap A)^\omega]$. Conversely, let $\mathcal{F} \in h^{-1}[(Z \cap A)^\omega]$, i.e., $Z \cap A \in \mathcal{F}_A$. For every $W \in \mathcal{F}$, $(W \cap A) \cap (Z \cap A) \neq \emptyset$ and so $Z \cap W \neq \emptyset$. Since $\mathcal{F}$ is an ultrafilter, $Z \in \mathcal{F}$, i.e., $\mathcal{F} \in Z^\omega$. Next let $a \in A$ with point-filter $\mathcal{F}_a$. For every $Z \in \mathcal{F}_a$, $a \in Z \cap A$ and so $h(\mathcal{F}_a)$ is contined in the $\omega(\mathcal{Z}_A)$ point-filter of a. Since $h(\mathcal{F}_a)$ is an ultrafilter, equality holds. Thus, $h \circ \iota_X|_A = \iota_A$. Finally, since $\omega(\mathcal{Z})$ is compact, $\omega(\mathcal{Z}_A)$ is T_2, and the image of the contiuous h contains the dense $\iota_A[A]$, h is onto and a homeomorphism. Thus the conclusion holds.

Lemma R27.4.12 For every $k \in \mathbb{N}$ with $k \geq 2$, if D is a non-empty element of $\mathcal{D}_k$, then $D \cap \mathbb{N} \neq \emptyset$. If D is a non-empty element of $\mathcal{D}_\infty$, then $D \cap \mathbb{N} \neq \emptyset$.

Proof: Let $n, k \in \mathbb{N}$ with $k \geq 2$. For any $z \in \mathbf{Z}$, there is $l \in \mathbb{N}$ such that $z + lk^n \geq 1$. Then $z + lk^n \in D_n^z(k) \cap \mathbb{N}$. Any non-empty element of $\mathcal{D}_k$ is the union of one or more equivalence classes and so has a non-empty intersection with $\mathbb{N}$. Because of the definition of $\mathcal{D}_\infty$, the second assertion is immediate from the first.

For $k, n, j \in \mathbb{N}$ with $k \geq 2$, let $C_n^j(k)$ denote the equivalence class of j mod k^n and let $\mathcal{C}_k$ be the set of unions of finite subcollections of $\{C_n^j(k) : n, j \in \mathbb{N}\}$. Let $\mathcal{C}_\infty = \cup\{\mathcal{C}_k : k \geq 2\}$.

Lemma R27.4.13 For every $k \in \mathbb{N}$ with $k \geq 2$, $\mathcal{C}_k = \{D \cap \mathbb{N} : D \in \mathcal{D}_k\}$. Moreover, $\mathcal{C}_\infty = \{D \cap \mathbb{N} : D \in \mathcal{D}_\infty\}$.

Proof: Let $k, n \in \mathbb{N}$ with $k \geq 2$ and let $z \in \mathbf{Z}$. For $j \in D_n^z(k) \cap \mathbb{N}$, $D_n^z(k) = D_n^j(k)$ and $D_n^j(k) \cap \mathbb{N} = C_n^j(k)$. The first claim now follows from the distributive law for intersection over unions. The second assertion follows from the first and the definitions of $\mathcal{C}_\infty$ and $\mathcal{D}_\infty$.

Proposition R27.4.14 Let $k \in \mathbb{N}$ with $k \geq 2$. Then $\mathcal{C}_k$ is a normal basis for the space $(\mathbb{N}, \sigma_k)$. Moreover, $(\omega(\mathcal{C}_k), \epsilon_k)$, where ϵ_k is the standard embedding induced by the normal basis, is equivalent to $(\mathbf{R}_k, g_k)$.

Proof: The first assertion follows from R27.4.13, R27.4.12, and R27.4.8. By R27.1.10 $(\omega(\mathcal{D}_k), \iota_k)$ is equivalent to $(\mathbf{R}_k, f_k)$. Let δ_k denote the restriction of ι_k to $\mathbb{N}$. By R27.4.11 $(\omega(\mathcal{C}_k), \epsilon_k)$ is equivalent to $(\omega(\mathcal{D}_k), \delta_k)$. By R27.4.2 $(\omega(\mathcal{D}_k), \delta_k)$ is equivalent to $(\mathbf{R}_k, g_k)$. The second conclusion now follows by transitivity.

Proposition R27.4.15 $\mathcal{C}_\infty$ is a normal basis for $(\mathbb{N}, \sigma_\infty)$. Moreover, $(\omega(\mathcal{C}_\infty), \epsilon_\infty)$, where ϵ_∞ is the embedding induced by the normal basis, is equivalent to $(\mathbf{R}_\infty, g_\infty)$.

Proof: Similar to the proof of R27.4.14 with R27.3.5 used in place of R27.1.10.

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