Order-Reversing Involutions for the Remnant Rings

The remnant rings, defined in [4], will be denoted $\mathbf{R}_k$, where $k \in \mathbb{N}$ with $k \geq 2$. In [6] a linear order $<_k$ was described and in R19.1.7 the order topology was shown to be the topology of the compact T_2 space $\mathbf{R}_k$. In [5] the map f_k from $\mathbf{Z}$ to $\mathbf{R}_k$ was defined and in R19.1.8 it was shown that $f_k(1)$ is the $<_k$-smallest in $\mathbf{R}_k$ and $f_k(0)$ the $<_k$-largest. By R21.23 $\mathbf{R}_k$ is order-complete and so is a complete, completely distributive lattice.

In this section $\mathbf{R}_k$ will be shown to have an order-reversing involution. This means that $\mathbf{R}_k$ is available as an underlying lattice for certain results in fuzzy topology, e.g., [2] or the generalization of fuzzy unit intervals in [1].

This section makes extensive use of definitions, notations, and results from [3] and [4]. In particular, the sequence associated with a non-point ultrafilter in $\mathbb{N}_k$ (defined in R10.2.3) is repeatedly employed, as well as properties of these sequences: the recursive relationship of terms (R10.2.5) and the recursive method of defining a non-point ultrafilter in $\mathbb{N}_k$ (R10.2.6). For any $z \in \mathbf{Z}$, by R12.5.9ii and R16.1 $f_k(z)$ corresponds to the sequence $\{x_n\}$, where $x_n \in \{1, 2, \ldots, k^n\}$ and $x_n \equiv z \bmod k^n$ for all n.

For $L = [0, 1]$ the map $x \mapsto 1 - x$ is an order-reversing involution. For $\mathbf{R}_k$, it will be shown that an analogous map, $\mathcal{F} \mapsto f_k(1) - \mathcal{F}$, is an order-reversing involution, but the argument for that is non-trivial because $<_k$ is not compatible with the algebraic operations of $\mathbf{R}_k$, as the following example shows.

Example R28.1 Let $k \in \mathbb{N}$ with $k \geq 2$. Recall that $f_k(0)$, which is the largest element of $\mathbf{R}_k$, is also the additive identity. Let $\mathcal{F}, \mathcal{G}$ be in $\mathbf{R}_k$ with $\mathcal{F} <_k \mathcal{G} <_k f_k(0)$. Then $\mathcal{F} + f_k(0) = \mathcal{F} <_k \mathcal{G} = \mathcal{G} + f_k(0)$, i.e., adding $f_k(0)$ preserves order. On the other hand, $\mathcal{F} + (-\mathcal{F}) = f_k(0) >_k \mathcal{G} + (-\mathcal{F})$, i.e., adding $-\mathcal{F}$ reverses order. Thus addition does not have a predictable effect on order. Moreover, since $-f_k(0) = f_k(0)$, $\mathcal{F} <_k f_k(0)$ and $-\mathcal{F} <_k f_k(0)$, i.e., negation is not order-reversing in $\mathbf{R}_k$. Since f_k is a homomorphism, $-f_k(1) = f_k(-1)$ and $-f_k(-1) = f_k(1)$. Because $f_k(1)$ is the smallest, $f_k(1) <_k f_k(-1)$ and $-f_k(1) >_k -f_k(-1)$. Thus negation is not in general order-preserving.

Throughout the rest of this section k will denote a fixed natural number with $k \geq 2$. For an integer x and $a \in \mathbb{N}$, $x(a) = y$, where y is the unique element of $\{1, 2, \ldots, k^a\}$ such that $x \equiv y \bmod k^a$. For $x \in \{1, 2, \ldots, k^{a+1}\}$, $c(x)$ will denote the unique coefficient in $\{0, 1, \ldots, k-1\}$ such that $x = x(a) + c(x)k^a$.

For the next three lemmas assume the following: $\mathcal{F}, \mathcal{G} \in \mathbf{R}_k$ with $\mathcal{F} <_k \mathcal{G}$. Let $\mathcal{F}$ be associated with $\{x_n\}$, $-\mathcal{F}$ with $\{y_n\}$, and $f_k(1) - \mathcal{F}$ with $\{z_n\}$. Let $\mathcal{G}$ be associated with $\{a_n\}$, $-\mathcal{G}$ with $\{b_n\}$, and $f_k(1) - \mathcal{G}$ with $\{c_n\}$. Since $\mathcal{F} \neq \mathcal{G}$, by R10.2.4 $x_n \neq a_n$ for some n. Let M be the smallest of the non-empty set $\{n : x_n \neq a_n\}$. Since $f_k(0)$ is associated with the sequence $\{k^n\}$ and $f_k(1)$ is associated with the constant sequence $\{1\}$, by R12.4.3, $x_n + y_n \equiv 0 \bmod k^n$ and $1 + y_n \equiv z_n \bmod k^n$ for all n. For the same reasons, $a_n + b_n \equiv 0 \bmod k^n$ and $1 + b_n \equiv c_n \bmod k^n$ for all n.

Lemma R28.2 For $1 \leq n < M$, $y_n = b_n$ and $z_n = c_n$. In addition $y_M \neq b_M$ and $z_M \neq c_M$.

Proof: Let $1 \leq n < M$. Since $x_n = a_n$ and $x_n + y_n \equiv 0 \equiv a_n + b_n \bmod k^n$, $y_n \equiv b_n \bmod k^n$. Since $y_n, b_n \in \{1, 2, \ldots, k^n\}$, which contains a unique representative of each equivalence class, $y_n = b_n$. Similarly, $y_n = b_n$ implies $z_n \equiv c_n \bmod k^n$, which implies $z_n = c_n$. For the second assertion, suppose $Y_M = b_M$. As above, $x_M + y_M \equiv a_M + b_M$

mod k^M and so $x_M \equiv a_M \bmod k^M$. Since $x_M, a_M \in \{1, 2, \ldots, k^M\}$, $x_M = a_M$, which contradicts the choice of M. Thus $y_M \neq b_M$. Lastly, suppose $z_M = c_M$. Similarly, this would imply $y_M = b_M$, a contradiction.

The fact that $x_n, y_n, z_n, a_n, b_n, c_n$ are all in $\{1, 2, \ldots, k^n\}$ for any $n \in \mathbb{N}$ will be used repeatedly in the next two proofs to identify unique solutions of congruences.

Lemma R28.3 If $M = 1$, then $f_k(1) - \mathcal{F} >_k f_k(1) - \mathcal{G}$.

Proof: Since $\mathcal{F} <_k \mathcal{G}$ and $x_1 \neq a_1$, by definition of $<_k$, $x_1 < a_1$. By R28.2 $z_1 \neq c_1$ and so the $<_k$-order of $f_k(1) - \mathcal{F}$ and$f_k(1) - \mathcal{G}$ is determined by the usual order of z_1 and c_1. Since $x_1 < a_1$, $x_1 < k$. Moreover, since $x_1 + y_1 \equiv 0 \bmod k$, $y_1 = k - x_1 < k$. Since $z_1 \equiv y_1 + 1 \bmod k$, $z_1 = y_1 + 1 = k - x_1 + 1$. As a first case, suppose $a_1 < k$. Similarly, $c_1 = k - a_1 + 1$. By routine algebra, $x_1 < a_1$ implies $z_1 > c_1$ and so by definition $f_k(1) - \mathcal{F} >_k f_k(1) - \mathcal{G}$. Secondly, suppose $a_1 = k$. Since $a_1 + b_1 \equiv 0 \bmod k$, $b_1 = k$. Since $c_1 \equiv b_1 + 1 \bmod k$, $c_1 = 1$. Since $z_1 \neq c_1$ and $1 \leq z_1 \leq k$, $z_1 > c_1$ and so $f_k(1) - \mathcal{F} >_k f_k(1) - \mathcal{G}$.

Lemma R28.4 If $M > 1$, then $f_k(1) - \mathcal{F} >_k f_k(1) - \mathcal{G}$.

Proof: By R10.2.5i $x_M = x_{M-1} + c(x_M)k^{M-1}$ and $a_M = a_{M-1} + c(a_M)k^{M-1}$, where $c(x_M), c(a_M) \in \{0, 1, \ldots, k-1\}$. By definition of $<_k$, $c(x_M) < c(a_M)$. Since $x_{M-1} = a_{M-1}$, $x_M < a_M \leq k^M$. Since $x_M + y_M \equiv 0 \bmod k^M$, $y_M = k^M - x_M$, which is smaller than k^M because $x_M \geq 1$. Since $1 + y_M \equiv z_M \bmod k^M$, $z_M = 1 + y_M$. As a first case, suppose $a_M < k^M$ so that similarly $b_M = k^M - a_M$ and $c_M = 1 + b_M$. Thus $z_M - c_M = y_M - b_M = a_M - x_M = (c(a_M) - c(x_M))k^{M-1} > 0$. Again by R10.2.5i, $z_M = z_{M-1} + c(z_M)k^{M-1}$ and $c_M = c_{M-1} + c(c_M)k^{M-1}$, where $c(z_M), c(c_M) \in \{0, 1, \ldots, k-1\}$. Since $z_{M-1} = c_{M-1}$, $z_M - c_M = (c(z_M) - c(c_M))k^{M-1} > 0$ so that $c(z_M) - c(c_M) > 0$. By definition of $<_k$, $c(z_M) > c(c_M)$ implies $f_k(1) - \mathcal{F} >_k f_k(1) - \mathcal{G}$. Finally suppose $a_M = k^M$. Now $a_M + b_M \equiv 0 \bmod k^M$ implies $b_M = k^M$ and $1 + b_M \equiv c_M \bmod k^M$ implies $c_M = 1$. Since $c_M = c_{M-1} + c(c_M)k^{M-1}$, $c_{M-1} \geq 1$, and $c(c_M) \geq 0$, it must be that $c(c_M) = 0$. Since $b_M = b_{M-1} + c(b_M)k^{M-1}$, $b_{M-1} \leq k^{M-1}$, $c(b_M) \leq k - 1$, and $b_M = k^M$, it must be that $c(b_M) = k - 1$ and $b_{M-1} = k^{M-1}$. Since $y_{M-1} = b_{M-1}$ and $1 + y_{M-1} \equiv z_{M-1} \bmod k^{M-1}$, $z_{M-1} = 1$. Thus $z_M = 1 + c(z_M)k^{M-1}$. From above, $z_M = 1 + y_M \geq 1 + y_{M-1} = 1 + k^{M-1}$ and so $c(z_M)k^{M-1} \geq k^{M-1}$. Thus $c(z_M) > 0$, i.e, $c(z_M) > c(c_M)$. By definition of $<_k$, $f_k(1) - \mathcal{F} >_k f_k(1) - \mathcal{G}$.

Corollary R28.5 Let $k \in \mathbb{N}$ with $k \geq 2$. Let $\mathcal{F}, \mathcal{G}$ be in $\mathbf{R}_k$ with $\mathcal{F} <_k \mathcal{G}$. Then $f_k(1) - \mathcal{F} >_k f_k(1) - \mathcal{G}$.

Proof: This summarizes the previous two lemmas.

Corollary R28.6 Let $k \in \mathbb{N}$ with $k \geq 2$. The map $\mathcal{F} \mapsto f_k(1) - \mathcal{F}$ is an order-reversing involution on $\mathbf{R}_k$.

Proof: Since $\mathbf{R}_k$ is a ring, the map is clearly an involution. R28.5 shows that it is order-reversing.

Finally, the self-involute elements will be described. There are two cases.

Lemma R28.7 Let $k \in \mathbb{N}$ with $k \geq 2$. If k is even, there does not exist $\mathcal{F} \in \mathbf{R}_k$ such that $f_k(1) - \mathcal{F} = \mathcal{F}$.

Proof: Let k be even and suppose $\mathcal{F}$ is in $\mathbf{R}_k$ with $\mathcal{F} = f_k(1) - \mathcal{F}$. Let $\mathcal{F}$ be associated with the sequence $\{x_n\}$. Recall that $f_k(1)$ is associated with the constant sequence $\{1\}$. Since $\mathcal{F} + \mathcal{F} = f_k(1)$, by R12.4.4, for each n, $2x_n \equiv 1 \bmod k^n$. When $n = 1$, since k is even, the congruence $2x_1 \equiv 1 \bmod k$ has no solution in $\mathbf{Z}$, which contradicts the existence of $\mathcal{F}$.

Lemma R28.8 Let $k \in \mathbb{N}$ with $k \geq 2$. If k is odd, there is a unique $\mathcal{F} \in \mathbf{R}_k$ such that $f_k(1) - \mathcal{F} = \mathcal{F}$. The unique such $\mathcal{F}$ is associated with the sequence $\{x_n\}$ where $x_n = \frac{k^n+1}{2}$.

Proof: Let k be odd so that $x_n = \frac{k^n+1}{2}$ is a positive integer for each n. Clearly $x_1 \in \{1, \ldots, k\}$. For any $n \geq 1$, $x_{n+1} - x_n = k^n(\frac{k-1}{2})$ and $\frac{k-1}{2}$ is in $\{0, 1, \ldots, k-1\}$. By R10.2.6, there is a unique $\mathcal{F}$ in $\mathbf{R}_k$ associated with the sequence $\{x_n\}$. For each n, the choice of x_n yields $2x_n \equiv 1 \bmod k^n$ so that by R12.4.4 and R10.2.4 $\mathcal{F} + \mathcal{F} = f_k(1)$, i.e., $\mathcal{F}$ is self-involute. Suppose $\mathcal{G}$ associated with $\{y_n\}$ is self-involute in $\mathbf{R}_k$. As above, for each n, $2y_n \equiv 1 \bmod k^n$ and so $2y_n \equiv 2x_n \bmod k^n$. Since k is odd, k^n and 2 are relatively prime so that 2 is invertible mod k^n. Thus $y_n \equiv x_n \bmod k^n$ for each n. Since x_n, y_n are both in $\{1, 2, \ldots, k^n\}$, a set which contains a unique representative of each equivalence class, $x_n = y_n$ for each n. By R10.2.4 $\mathcal{G} = \mathcal{F}$.

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1. Klein, A. J., Generalizing the L-fuzzy Unit Interval, Fuzzy Sets and Systems, 12(1984), 271-279.

2. Kubiak, T, Separation Axioms: Extensions of Mappings and Embedding of Spaces, in: Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory, Ulrich Höhle and Stephen E. Rodabaugh, editors, Kulwer Academic Publishers, 1999.

3. This Website, R10: Some Metric Compactifications of $\mathbb{N}$

4. This Website, R12: Extension of Arithmetic Operations

5. This Website, R16: The Remnant Rings as Compactifications

6. This Website, R19: Ordering the Remnant Rings

7. This Website, R21: Order Compactifications

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Let $k, j \in \mathbb{N}$ with $k \geq 2$ and $j \geq 2$. In R26.Add.15 and R26.Add.14 it was shown that $(\mathbf{R}_k, <_k)$ and $(\mathbf{R}_j, <_j)$ are order-isomorphic. The next result shows that, if k is odd and j is even, no order isomorphism preserves the involutions described above. Subsequently, it is shown that $\mathbf{R}_j$ does not have a unique order-reversing involution.

Lemma R28.Add.1 Let $k, j \in \mathbb{N}$ with $k \geq 2$ and $j \geq 2$. Assume k is odd and j is even. Let ϕ be an order isomorphism from $(\mathbf{R}_k, <_k)$ to $(\mathbf{R}_j, <_j)$. Then there is $\mathcal{F}$ in $\mathbf{R}_k$ such that $\phi(f_k(1) - \mathcal{F}) \neq f_j(1) - \phi(\mathcal{F})$.

Proof: By R28.8 there is $\mathcal{F}$ in $\mathbf{R}_k$ such that $f_k(1) - \mathcal{F} = \mathcal{F}$. Let $\mathcal{G} = \phi(\mathcal{F})$. Since j is even, by R28.7 $\mathcal{G} \neq f_j(1) - \mathcal{G}$, i.e., $\phi(f_k(1) - \mathcal{F}) \neq f_j(1) - \phi(\mathcal{F})$.

Lemma R28.Add.2 Let $k, j \in \mathbb{N}$ with $k \geq 2$ and $j \geq 2$. Let ϕ be an order isomorphism from $(\mathbf{R}_k, <_k)$ to $(\mathbf{R}_j, <_j)$. For $\mathcal{G} \in \mathbf{R}_j$, let $\mathcal{G}^* = \phi(f_k(1) - \phi^{-1}(\mathcal{G}))$. Then the map $\mathcal{G} \mapsto \mathcal{G}^*$ is an order-reversing involution on $\mathbf{R}_j$.

Proof: For $\mathcal{G} \in \mathbf{R}_j$, $\mathcal{G}^{**} = \phi(f_k(1) - \phi^{-1}(\mathcal{G}^*))$. Also $\phi^{-1}(\mathcal{G}^*) = f_k(1) - \phi^{-1}(\mathcal{G}))$ so that $f_k(1) - \phi^{-1}(\mathcal{G}^*) = \phi^{-1}(\mathcal{G})$. Thus $\mathcal{G}^{**} = \phi(\phi^{-1}(\mathcal{G})) = \mathcal{G}$, i.e., the map is an involution. Now let $\mathcal{G}, \mathcal{H}$ be in $\mathbf{R}_j$ with $\mathcal{G} <_j \mathcal{H}$. Then $\phi^{-1}(\mathcal{G}) <_k \phi^{-1}(\mathcal{H})$, which implies $f_k(1) - \phi^{-1}(\mathcal{G}) >_k f_k(1) - \phi^{-1}(\mathcal{H})$ by R28.5. Thus $\phi(f_k(1) - \phi^{-1}(\mathcal{G})) >_j \phi(f_k(1) - \phi^{-1}(\mathcal{H}))$, i.e., $\mathcal{G}^* >_j \mathcal{H}^*$.

Proposition R28.Add.3 Let $j \in \mathbb{N}$ with $j \geq 2$. Then $(\mathbf{R}_j \ , <_j)$ has an order-reversing involution, but it is not unique,

Proof: By R28.6 there is one. Let $k = j + 1$ and let ϕ be an order isomorphism from $(\mathbf{R}_k \ , <_k)$ to $(\mathbf{R}_j \ , <_j)$. By R28.Add.2 the map $\mathcal{G} \mapsto \mathcal{G}^*$, where $\mathcal{G}^* = \phi(f_k(1) - \phi^{-1}(\mathcal{G}))$, is an order-reversing involution on $\mathbf{R}_j$. If j is even, by R28.Add.1 there is $\mathcal{F}$ in $\mathbf{R}_k$ such that $\phi(f_k(1) - \mathcal{F}) \neq f_j(1) - \phi(\mathcal{F})$. Let $\mathcal{G} = \phi(\mathcal{F})$ so that $\mathcal{F} = \phi^{-1}(\mathcal{G})$. The non-equality says that $\mathcal{G}^* \neq f_j(1) - \mathcal{G}$. If j is odd, apply R28.Add.1 to ϕ^{-1}: there is $\mathcal{G}$ in $\mathbf{R}_j$ such that $\phi^{-1}(f_j(1) - \mathcal{G}) \neq f_k(1) - \phi^{-1}(\mathcal{G})$ so that $f_j(1) - \mathcal{G} \neq \mathcal{G}^*$. In either case, the map $\mathcal{G} \mapsto \mathcal{G}^*$ is not the involution described in R28.6.

Added Reference

8. This website, R26: The Remnant Rings are Homeomorphic