Point Spaces of the Remnant Rings

In [5] it was observed that for $k \in \boldsymbol{N}$ with $k \geq 2$, $\mathbf{R}_k$ is a complete, completely distributive lattice, in fact, a complete bounded chain. In this section the point spaces of the remnant rings are described and shown to be homeomorphic.

For future reference and in order to make this section more self-contained, the first subsection presents a detailed version of some results from Chapter II of [1]. The proof of adjunction works with details rather than following Johnstone's categorical argument.

Some Basic Facts about Frames and Locales

A frame is a complete lattice satisfying the infinite distributive law given by $a \wedge (\vee\{b_\alpha : \alpha \in \Delta\} = \vee\{a \wedge b_\alpha : \alpha \in \Delta\}$. The category **Frm** has frames as objects and functions preserving finite meets and arbitrary joins as morphisms. The requirement for morphisms is understood to include empty meets and joins, i.e., 1 must map to 1 and 0 to 0. A frame morphism also preserves the order:

Lemma R29.1.1 Let A, B be frames and let $f : A \to B$ be a frame morphism. Let $u, v \in A$ with $u \leq_A v$. Then $f(u) \leq_B f(v)$.

Proof: $f(u) = f(u \wedge v) = f(u) \wedge f(v) \leq_B f(v)$.

Note that an order preserving map need not be a frame morphism: Let $A = B = [0, 1]$ with the usual ordering and let $f(x) = x/2$. f preserves the order but $f(1) = \frac{1}{2}$ and so f is not a frame morphism.

The opposite category of **Frm** is denoted **Loc**. Objects in **Loc**, i.e. frames, are called locales. A localic morphism, say $f \in \mathrm{Hom}_{\mathbf{Loc}}(A, B)$ will be denoted $f : A \Rightarrow B$. Such f may also be considered a frame morphism $f : B \to A$.

Topoloical spaces provide an important example.

Definition R29.1.2 Let (X, τ) be a topological space. $\Omega(X)$ is the collection of open sets in X, i.e., $\Omega(X) = \tau$.

Lemma R29.1.3 Let (X, τ) be a topological space. Then $\Omega(X)$ is a frame with containment as ordering, intersection as meet, and union as join.

Proof: Containment is a partial order on any subset of the power set of X and so on $\Omega(X)$. Clearly the intersection of two subsets is the greatest lower bound, the union of two sets is the least upper bound, $\emptyset$ is the smallest element, and X is the largest element. By the definition of a topology, $\Omega(X)$ is a lattice. It is also a join-complete, since $\Omega(X)$ is closed under arbitrary unions, and so a complete lattice, i.e., a frame.

Note that by completeness $\Omega(X)$ contains infinite meets as well as joins but an infinite meet need not be the intersection.

Lemma R29.1.4 Let (X, τ) be a topological space. Let $\{O_\alpha : \alpha \in \Delta\}$ be a non-empty collection in $\Omega(X)$. Then $\bigwedge\{O_\alpha : \alpha \in \Delta\}$ =Int$(\cap\{O_\alpha : \alpha \in \Delta\})$, where Int is the interior operator in X.

Proof: Int$(\cap\{O_\alpha : \alpha \in \Delta\})$ is in $\tau = \Omega(X)$. It is clearly a lower bound of $\{O_\alpha : \alpha \in \Delta\}$. Let $G \in \Omega(X)$ be a lower bound of $\{O_\alpha : \alpha \in \Delta\}$ so that $G \subseteq \cap\{O_\alpha : \alpha \in \Delta\}$. Since the interior is the largest such open set, $G \subseteq$Int$(\cap\{O_\alpha : \alpha \in \Delta\})$,i.e., Int$(\cap\{O_\alpha : \alpha \in \Delta\})$ is the greatest lower bound.

Lemma R29.1.5 Let (X, τ) and (Y, σ) be topological spaces, and let $f : X \to Y$ be continuous. Then f^{-1} is a frame morphism from $\Omega(Y)$ to $\Omega(X)$.

Proof: That f^{-1} maps $\Omega(Y)$ to $\Omega(X)$ is immediate from continuity. General properties of the inverse image operator include: $f^{-1}[\emptyset] = \emptyset$, $f^{-1}[Y] = X$, $f^{-1}[G \cap H] = f^{-1}[G] \cap f^{-1}[H]$, and $f^{-1}[\cup\{G_\alpha : \alpha \in \Delta\}] = \cup\{f^{-1}[G_\alpha : \alpha \in \Delta\}$. Expressing these facts in lattice terminology yields the conclusion.

Definition R29.1.6 Let (X, τ) and (Y, σ) be topological spaces, and assume $f : X \to Y$ is continuous. The locale morphism $\Omega(f) : \Omega(X) \Rightarrow \Omega(Y)$ is defined by $\Omega(f) = f^{-1}$.

Motivated by the topological example, Johnstone refers to locale morphisms as continuous maps.

Notation: **Sp** denotes the category with topological spaces as objects and continuous functions as morphisms.

Lemma R29.1.7 $\Omega : \mathbf{Sp} \to \mathbf{Loc}$ is a functor.

Proof: This follows from the definition of a functor by applying the lemmas above and general facts about the inverse image operator. In particular, given f, g with $f \circ g$ defined, the equation $(f \circ g)^{-1}[S] = g^{-1}[f^{-1}[S]]$ implies $\Omega(f \circ g) = \Omega(f)\Omega(g)$.

The next definition is motivated by the facts that, for X in **Sp**, a point in X is identified by a continuous map from a one-point space to X and that Ω applied to a one-point space is the two-point locale (i.e. frame) $\{0, 1\}$.

Definition R29.1.8 Let A be a locale. A point in A is a locale morphism $p : \{0, 1\} \Rightarrow A$. The set of all points in A will be denoted $\mathrm{pt}(A)$.

For the next definition $\mathcal{P}(\mathcal{S})$ denotes the collection of all subsets of set S. Elements of $\mathrm{pt}(A)$ are treated as frame morphisms from A to $\{0, 1\}$.

Definition R29.1.9 Let A be a locale. The map $\phi : A \to \mathcal{P}(\mathrm{pt}(A))$ is defined by $\phi(a) = \{p \in \mathrm{pt}(A) : p(a) = 1\}$.

If a context uses several frames, ϕ_A will be used instead of ϕ.

Lemma R29.1.10 Let A be a locale. Then

i) ϕ is a frame morphism.

ii) The image of ϕ is a topology on $\mathrm{pt}(A)$.

Proof: For i): For any $p \in \mathrm{pt}(A)$, $p(0) = 0$ and $p(1) = 1$. Thus $\phi(0) = \emptyset$, the smallest element in $\mathcal{P}(\mathrm{pt}(A))$, and $\phi(1) = \mathrm{pt}(A)$, the largest element in $\mathcal{P}(\mathrm{pt}(A))$. Given $a_1, a_2 \in A$, note that $p \in \phi(a_1) \cap \phi(a_2)$ if and only if both $p(a_1) = 1$ and $p(a_2) = 1$ if and only if $p(a_1 \wedge a_2) = p(a_1) \wedge p(a_2) = 1$. Since the meet in $\mathcal{P}(\mathrm{pt}(A))$ is intersection, it follows easily that $\phi(a_1 \wedge a_2) = \phi(a_1) \wedge \phi(a_2)$. Lastly let $\{a_\alpha : \alpha \in \Delta\}$ be a non-empty collection in A. Then $p \in \cup\{\phi(a_\alpha) : \alpha \in \Delta\}$ if and only if $p(a_\alpha) = 1$ for at least one α if and only if $p(\vee\{a_\alpha : \alpha \in \Delta\}) = \vee\{p(a_\alpha) : \alpha \in \Delta\} = 1$. Since the join in $\mathcal{P}(\mathrm{pt}(A))$ is union, $\phi(\vee\{a_\alpha : \alpha \in \Delta\}) = \vee\{\phi(a_\alpha) : \alpha \in \Delta\}$. Part ii) follows easily from part i).

Given a locale A, whenever $\mathrm{pt}(A)$ is treated as a topological space with no topology identified, the image of ϕ is assumed to be the topology, i.e., by definition $\Omega(\mathrm{pt}(A))$ is the image of ϕ. The first part of the previous lemma can be restated: ϕ is a morphism in **Loc**, i.e., $\phi : \Omega(\mathrm{pt}(A)) \Rightarrow A$.

Definition R29.1.11 Let A be a locale and let $p \in \mathrm{pt}(A)$. The element $a_0(p)$ in A is defined to be $\vee\{a \in A : p(a) = 0\}$.

Lemma R29.1.12 Let A be a locale and let $p \in \mathrm{pt}(A)$.
Then $p^{-1}[\{0\}] = \{a \in A : a \leq a_0(p)\}$.

Proof: By definition of $a_o(p)$, $p^{-1}[\{0\}] \subseteq \{a \in A : a \leq a_0(p)\}$. Since p preserves joins, $p(a_0(p)) = p(\vee\{a : p(a) = 0\}) = \vee\{p(a) : p(a) = 0\} = 0$, i.e, $a_0(p) \in p^{-1}[\{0\}]$. Since p is order-preserving, $p^{-1}[\{0\}] \supseteq \{a \in A : a \leq a_0(p)\}$.

In Johnstone's ideal terminology, the last lemma says that $p^{-1}[\{0\}]$ is the principal ideal generated by $a_0(p)$. Every element of A generates a principal ideal, but the following example shows that a principal ideal need not be associated with a point of A.

Example R29.1.13 Let X be a set with the discrete topology and assume $|X| \geq 3$. Let x, y, z be three distinct points of X. Suppose $f \in \mathrm{pt}(\Omega(X))$ with $a_0(f) = \{x\}$, i.e., $f(G) = 0$ if and only if $G \subseteq \{x\}$and $f(G) = 1$ otherwise. Then $f(\{x, y\} \cap \{x, z\}) = 0$ but $f(\{x, y\}) = f(\{x, z\}) = 1$. Thus $f(\{x, y\} \wedge \{x, z\}) \neq f(\{x, y\}) \wedge f(\{x, z\})$, which contradicts the assumption that f is a frame morphism.

Lemma R29.1.14 Let A be a locale and let p, q be in $\mathrm{pt}(A)$. Then $a_0(p) = a_0(q)$ if and only if $p = q$.

Proof: Assume $a_0(p) = a_0(q)$ and let $a \in A$. Clearly $p(a) = 0$ if and only if $q(a) = 0$. Since the only possible values of p, q are 0 and 1, it follows that $p(a) = 1$ if and only if $q(a) = 1$. Thus $p = q$. The converse is clear from the definition.

The next few results focus on the example $\Omega(X)$.

Definition R29.1.15 Let (X, τ) be a topological space and let $x \in X$. The map $p_x : \Omega(X) \to \{0, 1\}$ is defined by $p_x(G) = 1$ if $x \in G$ and $p_x(G) = 0$ if $x \notin G$.

Lemma R29.1.16 Let (X, τ) be a topological space and let $x \in X$. Then

i) $p_x \in \mathrm{pt}(\Omega(X))$.

ii) $a_0(p_x) = X - c(\{x\})$, where c is the closure operator in X.

Proof: For i): It is necessary to verify that the map p_x is a frame morphism. The largest and smallest elements of $\Omega(X)$ are X and $\emptyset$. By definition $p_x(X) = 1$ and $p_x(\emptyset) = 0$. Likewise, $p_x(G_1 \cap G_2) = 1$ if and only if $x \in G_1 \cap G_2$ if and only if $p_x(G_1) = 1$ and $p_x(G_2) = 1$. It follows that $p_x(G_1 \cap G_2) = p_x(G_1) \wedge p_x(G_2)$. Finally, $p_x(\cup\{G_\alpha : \alpha \in \Delta\}) = 1$ if and only if $x \in \cup\{G_\alpha : \alpha \in \Delta\}$ if and only if x is in at least one G_α. Thus $p_x(\cup\{G_\alpha : \alpha \in \Delta\}) = \vee\{p_x(G_\alpha) : \alpha \in \Delta\}$. For ii): If $G \in \Omega(X)$ and $p_x(G) = 0$, then $c(\{x\})$ is contained in the closed set $X - G$ so that $G \subseteq X - c(\{x\})$. Clearly $X - c(\{x\}) \in \Omega(X)$ and $p_x(X - c(\{x\})) = 0$. The result follows from the definition.

The next four lemmas are true in general but some of the proofs assume a space with at least two elements. The omitted cases can be dealt with as follows: If X is a singleton, there is only one one topology, $\Omega(X)$ is the two-element lattice, and $\mathrm{pt}(\Omega(X))$ is also a singleton. If $X = \emptyset$, there is only one topology, $\Omega(X)$ is the one-element lattice (i.e., $0 = 1$), and $\mathrm{pt}(\Omega(X))$ is also the empty set. In both cases the conclusions of the lemmas are easily verified.

Lemma R29.1.17 Let (X, τ) be a topological space. The map $x \mapsto p_x$ is one-to-one if and only is (X, τ) is T_0.

Proof: Assume X is T_0 and let $x \neq y$ be in X. There is an open G with $x \in G$ and $y \notin G$ or $x \notin G$ and $y \in G$. Then $x \notin c(\{y\})$ or $y \notin c(\{x\}$ and so $X - c(\{x\}) \neq X - c(\{y\})$, i.e., $a_0(p_x) \neq a_0(p_y)$. Thus $p_x \neq p_y$. Now assume the map is one-to-one and let $x \neq y$ be in X. Then $p_x \neq p_y$ and so $a_0(p_x) \neq a_0(p_y)$, i.e. $X - c(\{x\}) \neq X - c(\{y\})$. This implies $x \notin c(\{y\})$ or $y \notin c(\{x\})$, which easily yields T_0.

Lemma R29.1.18 Let (X, τ) be a topological space. The map $x \mapsto p_x$ is onto if and only if for every $q \in \text{pt}(\Omega(X))$ there is $x \in X$ with $a_0(q) = X - c(\{x\})$.

Proof: Assume the map is onto. Given $q \in \text{pt}(\Omega(X))$, for any x with $p_x = q$, $a_0(q) = a_0(p_x) = X - c(\{x\})$. Conversely, let $q \in \text{pt}(\Omega(X))$. By hypothesis there is $x \in X$ with $a_0(q) = X - c(\{x\}) = a_0(p_x)$. By J.14 $p_x = q$.

Lemma R29.1.19 Let (X, τ) be a T_2 topological space. Then the map $x \mapsto p_x$ is one-to-one and onto.

Proof: The map is one-to-one by R29.1.17. Let $q \in \text{pt}(\Omega(X))$ and suppose $q \neq p_x$ for every $x \in X$, i.e., $a_0(q) \neq X - \{x\}$ for every x. Assuming $|X| \geq 2$, either $|X - a_0(q)| \geq 2$ or $X - a_0(q) = \emptyset$. The latter is not possible since $q(X) = 1$ and so pick $x \neq y$ in $X - a_0(q)$. There exist $G_1, G_2 \in \tau$ with $x \in G_1$, $y \in G_2$, and $G_1 \cap G_2 = \emptyset$. Then $q(G_1 \cap G_2) = q(\emptyset) = 0$. Since $G_i \not\subseteq a_0(q)$ for both i, $q(G_i) = 1$. But $q(G_1 \cap G_2) \neq q(G_1) \wedge q(G_2)$, i.e., q does not preserve finite meets. That contradicts the choice of q in $\text{pt}(\Omega(X))$.

In [1] a sober topological space is defined as one for which the map $x \mapsto p_x$ is one-to-one and onto. The previous lemma shows that every T_2 space is sober. R29.1.17 shows that every sober space must be T_0. Johnstone also shows that, given any topological space X, $\text{pt}(\Omega(X))$ must be sober. That fact and R29.2.5i below give examples of a sober spaces which are not T_1.

Lemma R21.1.20 Let (X, τ) be a regular topological space. Then the map $x \mapsto p_x$ is onto.

Proof: Let $q \in \text{pt}(\Omega(X))$ and suppose $q \neq p_x$ for every $x \in X$, i.e., $a_0(q) \neq X - c(\{x\})$ for every x. Since $q(X) = 1$, $X - a_0(q) \neq \emptyset$. Pick $x \in X - a_0(q)$. Then $c(\{x\})$ is contained in the closed $X - a_0(q)$. Since $X - a_0(q) \neq c(\{x\})$, there is $y \in X - a_0(q)$ with $y \notin c(\{x\})$. By regularity, there exist $G_1, G_2 \in \tau$ with $c(\{x\}) \subseteq G_1$, $y \in G_2$, and $G_1 \cap G_2 = \emptyset$. Then $q(G_1 \cap G_2) = q(\emptyset) = 0$. Since $G_i \not\subseteq a_0(q)$ for both i, $q(G_i) = 1$. But $q(G_1 \cap G_2) \neq q(G_1) \wedge q(G_2)$, which contradicts the fact that q preserves finite meets.

Proposition R29.1.21 Let (X, τ) be a T_2 topological space. Then the map $x \mapsto p_x$ is a homeomorphism.

Proof: By R21.1.19 the map is one-to-one and onto. Let F denote the map, i.e., $F(x) = p_x$. For $G \in \tau$, $F[G] = \{p_x : x \in G\} = \phi(G)$ and so F is open. Each open set in $\text{pt}(\Omega(X))$ is of the form $\phi(O)$ for some $O \in \tau$. Since $F^{-1}[\phi(O)] = \{x : p_x \in \phi(O)\} = O$, F is continuous.

In the category of Hausdorff topological spaces the maps of the proposition induce a bijection of Hom-sets in a routine way. That will not be described in detail, but the following corollary gives a partial indication of it.

Corollary R21.1.22 Let $(X, \tau), (Y, \sigma)$ be T_2 topological spaces. Let $f : X \to Y$. Then f is continuous if and only if the map $p_x \mapsto p_{f(x)}$ from $\text{pt}(\Omega(X))$ to $\text{pt}(\Omega(Y))$ is continuous.

Proof: Let F, G and $\overline{f}$ be defined by $F(x) = p_x$, $G(y) = p_y$, and $\overline{f}(p_x) = p_{f(x)}$. Then $f = G^{-1} \circ \overline{f} \circ F$. Since F and G are homeomorphisms, the conclusion follows.

The point-space of a locale allows the introduction of a second functor.

Lemma R29.1.23 Let $g : A \Rightarrow B$ be a morphism in **Loc** and let $p \in \mathrm{pt}(A)$. Then $p \circ g$ is in $\mathrm{pt}(B)$ and the map $p \mapsto p \circ g$ is continuous.

Proof: The locale morphism g is also a frame morphism from B to A. Since the composition of two frame morphisms is a frame morphism, $p \circ g : B \to \{0, 1\}$ is a frame morphism, i.e., the first conclusion holds. For continuity, let F denote the map $p \mapsto p \circ g$. A typical open set in $\mathrm{pt}(B)$ has the form $\phi_B(b)$. Now $p \circ g \in \phi_B(b)$ if and only if $p \circ g(b) = 1$ if and only if $p \in \phi_A(g(b))$. Thus $F^{-1}[\phi_B(b)] = \phi_A(g(b)$, which shows the continuity of F.

Definition R29.1.24 $H : \mathbf{Loc} \to \mathbf{Sp}$ takes a locale A to the topological space $\mathrm{pt}(A)$ and a locale morphism $g : A \Rightarrow B$ to $H(g) : \mathrm{pt}(A) \to \mathrm{pt}(B)$, where $H(g)(p) = p \circ g$.

Lemma R29.1.25 $H : \mathbf{Loc} \to \mathbf{Sp}$ is a functor.

Proof: Clearly $H(id_A) = id_{H(A)}$. Let $f : A \Rightarrow B$ and $g : B \Rightarrow C$ be locale morphisms. The equation $H(fg) = H(f) \circ H(g)$ follows via point-chasing from the associativity of function composition.

The rest of this subsection verifies that H is a right adjoint of Ω, in categorical notation $\Omega \dashv H$. (A succinct definition of adjunction can be found Johnstone, with more detail in MacLane [2].) The presentation here is non-categorical: details of the definition are verified by working with the objects and morphisms. Johnstone's categorical argument will be discussed afterward.

Lemma R29.1.26 Let $A \in \mathbf{Sp}$ and let $B \in \mathbf{Loc}$. Let $f : \Omega(A) \Rightarrow B$ be a locale morphism. Let $a \in A$. Then $p_a \circ f \in H(B)$.

Proof: As a frame morphism, f maps B to $\Omega(A)$ and so $p_a \circ f$, the composition of two frame morphisms, is also a frame morphism from B to $\{0, 1\}$, i.e., $p_a \circ f$ is in $\mathrm{pt}(B) = H(B)$.

Definition R29.1.27 Let $A \in \mathbf{Sp}$ and let $B \in \mathbf{Loc}$. For a locale morphism $f : \Omega(A) \Rightarrow B$, $\psi_{A,B}(f)$ is the function from A to $H(B)$ given by $\psi_{A,B}(f)(a) = p_a \circ f$.

Lemma R29.1.28 Let $A \in \mathbf{Sp}$ and let $B \in \mathbf{Loc}$. Then $\psi_{A,B}$ is a function from $\mathrm{Hom}_{\mathbf{Loc}}(\Omega(A), B)$ to $\mathrm{Hom}_{\mathbf{Sp}}(A, H(B))$.

Proof: Let $f : \Omega(A) \Rightarrow B$ be a locale morphism. It is necessary to show that $\psi_{A,B}(f)$ is continuous. For simplicity let $G = \psi_{A,B}(f)$. Let O be open in $H(B)$. Then there is $b \in B$ such $O = \phi_B(b)$. $G^{-1}[O] = \{a : p_a \circ f \in \phi_B(b)\}$, which is $\{a : p_a \circ f(b) = 1\} = \{a : a \in f(b)\} = f(b)$. Since $f(b) \in \Omega(A)$, $f(b)$ is open in A.

Lemma R29.1.29 Let $A \in \mathbf{Sp}$ and let $B \in \mathbf{Loc}$. Then $\psi_{A,B}(f)$ is one-to-one and onto.

Proof: For one-to-one: Let $f_1, f_2 \in \mathrm{Hom}_{\mathbf{Loc}}(\Omega(A), B)$ with $f_1 \neq f_2$. Then there is $b \in B$ with $f_1(b) \neq f_2(b)$. Let a be in the symmetric difference $f_1(b) \Delta f_2(b)$. By definition of p_a, $p_a \circ f_1(b) \neq p_a \circ f_2(b)$, i.e., $\psi_{A,B}(f_1) \neq \psi_{A,B}(f_2)$. For onto: Let $g \in \mathrm{Hom}_{\mathbf{Sp}}(A, H(B))$. Then g is continuous and by R29.1.5 and R29.1.10i, $f = g^{-1} \circ \phi$ is a frame morphism from B to $\Omega(A)$, i.e, f is in $\mathrm{Hom}_{\mathbf{Loc}}(\Omega(A), B)$. For any $a \in A$ and any $b \in B$, $p_a(f(b)) = 1$ if and only if $a \in f(b)$ if and only if $g(a) \in \phi(b)$. By definition of ϕ, the last is equivalent to $g(a)(b) = 1$. Thus $p_a \circ f = g(a)$. It follows that $\psi_{A,B}(f) = g$.

The next definition and R29.1.32 use notation as in [2].

Definition R29.1.30 Let **C** be a category, let X, Y, Y' be objects in **C**, and let $f : Y \to Y'$ be a morphism in **C**. The map f_* from $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ to $\mathrm{Hom}_{\mathbf{C}}(X, Y')$ is defined by $f_*(t) = ft$ for each t.

Lemma R29.1.31 Let A be in **Sp**, let B, B' be locales, and let $k : B \Rightarrow B'$ be a locale morphism. Then the following diagram commutes.

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{Loc}}(\Omega A, B) & \xrightarrow{\psi_{A,B}} & \mathrm{Hom}_{\mathbf{Sp}}(A, HB) \\
\downarrow k_* & & \downarrow (Hk)_* \\
\mathrm{Hom}_{\mathbf{Loc}}(\Omega A, B') & \xrightarrow[\psi_{A,B'}]{} & \mathrm{Hom}_{\mathbf{Sp}}(A, HB')
\end{array}$$

Proof: Let $f : \Omega A \Rightarrow B$. To verify that the functions $(Hk)_*\psi_{A,B}(f)$ and $\psi_{A,B'}(k_*f)$ are equal, their values will be checked at an arbitrary $a \in A$. As a frame morphism, k maps B' to B and so $(Hk)_*\psi_{A,B}(f)(a) = \psi_{A,B}(f)(a) \circ k = (p_a \circ f) \circ k$. In the other direction, $k_*f = f \circ k$ and so $\psi_{A,B'}(k_*f)(a)$ equals $\psi_{A,B'}(f \circ k)(a)$, which is $p_a \circ (f \circ k)$. By associativity equality holds.

Definition R29.1.32 Let **C** be a category, let X, X', Y be objects in **C**, and let $f : X' \to X$ be a morphism in **C**. The map f^* from $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ to $\mathrm{Hom}_{\mathbf{C}}(X', Y)$ is defined by $f^*(t) = tf$ for each t.

Lemma R29.1.33 Let A, A' be in **Sp**, let $l : A' \to A$ be a morphism, and let B be a locale. Then the following diagram commutes.

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{Loc}}(\Omega A, B) & \xrightarrow{\psi_{A,B}} & \mathrm{Hom}_{\mathbf{Sp}}(A, HB) \\
\downarrow (\Omega l)^* & & \downarrow l^* \\
\mathrm{Hom}_{\mathbf{Loc}}(\Omega A', B) & \xrightarrow[\psi_{A',B}]{} & \mathrm{Hom}_{\mathbf{Sp}}(A', HB)
\end{array}$$

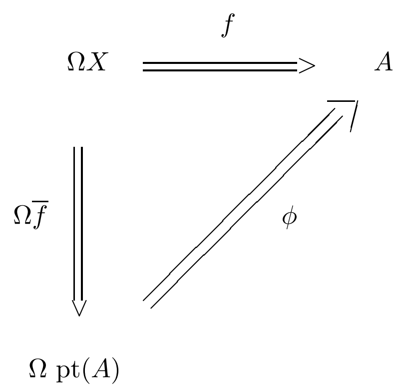
Proof: Let $f : \Omega A \Rightarrow B$. To verify that the functions $l^*(\psi_{A,B}(f))$ and $\psi_{A',B}((\Omega l)^* f)$ are equal, their values will be checked at an arbitrary $x \in A'$. By definition, $l^*(\psi_{A,B}(f)) = \psi_{A,B}(f) \circ l$ and so $l^*(\psi_{A,B}(f))(x)$ is the element $p_{l(x)} \circ f$ in $\mathrm{pt}(B)$. Thus, for $b \in B$, $p_{l(x)} \circ f(b) = 1$ if and only if $l(x) \in f(b)$. Along the other edge, $(\Omega l)^* f$ is the frame morphism $(\Omega l) \circ f = l^{-1} \circ f$. The B-point $\psi_{A',B}((\Omega l) \circ f)(x)$ can be evaluated for $b \in B$ as follows: $\psi_{A',B}((\Omega l) \circ f)(x)(b) = 1$ if and only if $x \in l^{-1}[f(b)]$ if and only if $l(x) \in f(b)$. Since elements of $\mathrm{pt}(B)$ are maps into $\{0, 1\}$, two points which have the same inverse image of $\{1\}$ must be equal, i.e., $l^*(\psi_{A,B}(f))(x) = \psi_{A',B}((\Omega l) \circ f)(x)$. Since this holds for all $x \in A'$, it follows that $l^*(\psi_{A,B}(f)) = \psi_{A',B}((\Omega l)^* f)$.

Theorem R29.1.34 The functor H is right adjoint to Ω, i.e. $\Omega \dashv H$.

Proof: Lemmas R29.1.28, R29.1.29, R29.1.31, and R29.1.33 verify the definition.

The previous proof uses some categorical language but not category theory as preferred by Johnstone, who begins with the next proposition. ϕ is the map defined in R29.1.9.

Proposition R29.1.35 Let X be in **Sp** and A be in **Loc**. Let $f : \Omega X \Rightarrow A$ be a morphism in **Loc**. Then f factors uniquely through ϕ, i.e., there is a unique continuous map $\overline{f} : X \to \mathrm{pt}(A)$ such that the following diagram commutes.



Proof: For $x \in X$, let $\overline{f}(x) = p_x \circ f$, which, as a composition of two frame morphisms, is a frame morphism. It also maps A to $\{0, 1\}$ and so is in $\mathrm{pt}(A)$. Note first that, for every $a \in A$, $x \in (\overline{f})^{-1}[\phi(a)]$ if and only if $p_x \circ f \in \phi(a)$ if and only if $p_x(f(a)) = 1$ if and only if $x \in f(a)$. Thus $(\overline{f})^{-1}[\phi(a)] = f(a)$. This set equation shows that $\overline{f}$ is continuous and, since $\Omega\overline{f}$ is $(\overline{f})^{-1}$, that $\Omega\overline{f} \circ \phi = f$, which, interpreted in **Loc**, says that the diagram commutes. For uniqueness, suppose $g : X \to \mathrm{pt}(A)$ is continuous with $\Omega g \circ \phi = f$, i.e., for every $a \in A$, $g^{-1}[\phi(a)] = f(a)$. Let $x \in X$ and $a \in A$. Then $\overline{f}(x)(a) = 1$ if and only if $p_x \circ f(a) = 1$ if and only if $x \in f(a)$ if and only if $g(x) \in \phi(a)$ if and only if $g(x)(a) = 1$. Since $\overline{f}$ and g map into $\{0, 1\}$, this shows $\overline{f} = g$.

The proposition says that $<\mathrm{pt}(A), \phi_A>$ is a universal arrow from Ω to A as defined on p. 58 of [2]. By the dual of theorem 2ii on p.83 of [2], H is a functor and $\Omega \dashv H$.

Application to the Remnant Rings

A complete bounded chain is a linearly ordered set, which is order-complete and has a smallest and a largest element. As a lattice, a complete bounded chain must also be completely distributive, i.e., both arbitrary distributive laws hold so that it is automatically an object in **Loc** or **Frm** .

Definition R29.2.1 Let C be a complete bounded chain and let $c \in C - \{1\}$. The map $q_c : C \to \{0, 1\}$ is given by $q_c(x) = 0$ if $x \leq c$ and $q_c(x) = 1$ if $x > c$.

Lemma R29.2.2 Let C be a complete bounded chain and let $c, d \in C - \{1\}$ with $c \neq d$. Then $q_c \neq q_d$.

Proof: If $c < d$, then $q_c(d) = 1$ and $q_d(d) = 0$ so that $q_c \neq q_d$. Similarly, $d < c$ implies $q_c \neq q_d$. By linearity those are the only two cases.

Lemma R29.2.3 Let C be a complete bounded chain and let $c \in C - \{1\}$. Then $q_c \in \text{pt}(C)$.

Proof: Clearly $q_c(0) = 0$ and $q_c(1) = 1$. Let $x, y \in C$. Since $x \wedge y = \min\{x, y\}$, $q_c(x \wedge y) = 1$ implies $q_c(x) = 1$ and $q_c(y) = 1$. Likewise, $q_c(x \wedge y) = 0$ implies $q_c(x) = 0$ or $q_c(y) = 0$. In either case, $q_c(x \wedge y) = q_c(x) \wedge q_c(y)$. For joins, let $\{x_\alpha : \alpha \in \Delta\} \subseteq C$, where $\Delta \neq \emptyset$. If $q_c(\vee\{x_\alpha : \alpha \in \Delta\}) = 0$, then by definition $\vee\{x_\alpha : \alpha \in \Delta\} \leq c$ and so c is an upper bound of $\{x_\alpha : \alpha \in \Delta\}$, i.e., $x_\alpha \leq c$ for every α so that $q_c(x_\alpha) = 0$ for every α. Thus $\vee\{q_c(x_\alpha) : \alpha \in \Delta\} = 0$. If $q_c(\vee\{x_\alpha : \alpha \in \Delta\}) = 1$ then $c < \vee\{x_\alpha : \alpha \in \Delta\}$ so that c is not a lower bound of $\{x_\alpha : \alpha \in \Delta\}$. By linearity, there exists $\gamma \in \Delta$ such that $c < x_\gamma$. Thus $q_c(x_\gamma) = 1$ and so $\vee\{q_c(x_\alpha) : \alpha \in \Delta\} = 1$.

Lemma R29.2.4 Let C be a complete bounded chain and let $q \in \text{pt}(C)$. Then there exists $c \in C - \{1\}$ such that $q = q_c$.

Proof: Let $c = a_0(q) = \vee\{x : q(x) = 0\}$. Because q preserves arbitrary joins and $q(1) = 1$, $q(c) = 0$ and $c < 1$. By definition of c, $x > c$ implies $q(x) = 1 = q_c(x)$. Since q is non-decresing, $x \leq c$ implies $q(x) = 0 = q_c(x)$. Thus $q = q_c$.

Proposition R29.2.5 Let C be a complete bounded chain. Then

i) q_0 is in every non-empty open subset of $\text{pt}(C)$.

ii) If G is open in $\text{pt}(C)$, then there is a unique $x \in C$ such that $G = \{q_c : c \in C - \{1\} \text{ and } c < x\}$.

iii) The open sets of $\text{pt}(C)$ form a chain.

Proof: An open set in $\text{pt}(C)$ must be of the form $\phi(x)$ for some $x \in C$, where $\phi(x) = \{q \in \text{pt}(C) : q(x) = 1\}$ as in R29.1.9. By the previous lemmas $\phi(x) = \{q_c : c \in C - \{1\} \text{ and } q_c(x) = 1\}$. By the definition of q_c, $q_c(x) = 1$ if and only if $c < x$. Thus $\phi(x) = \{q_c : c \in C - \{1\} \text{ and } c < x\}$. Uniqueness follows because, for $x < y$, $q_x \in \phi(y)$ but $q_x \notin \phi(x)$. Thus ii) holds. Given $x, y \in C$, by ii) $x \leq y$ implies $\phi(x) \subseteq \phi(y)$. By the linearity of C, iii) holds. Clearly, if $0 < x$, $q_0(x) = 1$ and so $q_0 \in \phi(x)$. Since $\phi(0) = \emptyset$, i) holds.

Note that, if $|C| \geq 3$, $\{q_0\}$ cannot be closed in $\text{pt}(C)$ and so $\text{pt}(C)$ is not T_1. Since [1] shows that the point space of any locale is sober, this provides examples of sober spaces which are not T_1.

Proposition R29.2.6 Let C be a complete bounded chain. Then C and $\Omega(\text{pt}(C))$ are order isomorphic and $\phi : C \to \Omega(\text{pt}(C)$ is an order isomorphism.

Proof: The map $\phi : C \to \Omega(\mathrm{pt}(C))$ is onto by definition and, as noted in the previous proof, $\phi(x) = \{q_c : c \in C - \{1\} \text{ and } c < x\}$ so that ϕ is order preserving. Clearly $a < b$ implies $q_a \in \phi(a)$ but $q_a \notin \phi(a)$ so that ϕ is one-to-one. The inverse of an order preserving bijection between chains is automatically order preserving.

Corollary R29.2.7 Let C be a complete bounded chain. Then C and $\Omega(\mathrm{pt}(C))$ are isomorphic objects in **Loc** and $\phi : \Omega(\mathrm{pt}(C)) \Rightarrow C$ is an invertible arrow.

Proof: Order isomorphisms of complete bounded chains are automatically frame morphisms. As in the previous proof, ϕ and ϕ^{-1} are both frame morphisms and consequently $\phi : C \to \Omega(\mathrm{pt}(C)$ is an invertible arrow in **Frm.** The conclusion follows by definition.

In [1] Johnstone defines a spatial locale A as one for which $\phi : A \to \Omega(\mathrm{pt}(A))$ is a frame isomorphism. The last corollary shows that a complete bounded chain must be spatial. A lattice theoretical approach might begin with one of Johnstone's characterizations of spatial locales: A is spatial if and only if $a, b \in A$ with $a \not\leq b$ implies there is $p \in \mathrm{pt}(A)$ such that $p(a) = 1$ and $p(b) = 0$. For a complete bounded chain C, it would then follow easily that C is spatial: given $c \not\leq d$ use q_d. R29.2.7 is immediate from the fact that C is spatial. R29.2.6 follows from R29.2.7 because a frame morphism must preserve order.

The characterization also shows that, given any topological space X, the locale $\Omega(X)$ is spatial: given $O \not\subseteq G$, use p_x for any $x \in O - G$.

Proposition R29.2.8 Let C_1, C_2 be complete bounded chains. Then C_1 and C_2 are order-isomorphic if and only if $\mathrm{pt}(C_1)$ and $\mathrm{pt}(C_2)$ are homeomorphic.

Proof: If C_1 and C_2 are order-isomorphic, they are isomorphic objects in **Loc.** Since any functor maps invertible arrows to invertible arrows, their images under H are isomorphic objects in **Sp,** , i.e., $\mathrm{pt}(C_1)$ and $\mathrm{pt}(C_2)$ are homeomorphic. If $\mathrm{pt}(C_1)$ and $\mathrm{pt}(C_2)$ are homeomorphic, the general fact about functors implies that $\Omega(\mathrm{pt}(C_1))$ and $\Omega(\mathrm{pt}(C_2))$ are isomorphic objects in **Loc.** As noted above, complete bounded chains are spatial so that C_i and $\Omega(\mathrm{pt}(C_i))$ are isomorphic. By transitivity, C_1 and C_2 are isomorphic objects in **Loc** and so order-isomorphic.

Now let k, j be in $\boldsymbol{N}$ with $k, j \geq 2$. As noted above, $\mathbf{R}_k$ with the order $<_k$ is a complete, bounded chain and so the results above apply, in particular the description of its point space.

At first glance, it seems plausible that $\mathrm{pt}(\mathbf{R}_k)$ and $\mathrm{pt}(\mathbf{R}_j)$ might be homeomorphic, but that seems difficult to verify directly. The result does follow as a corollary of R29.2.8.

Corollary R29.2.9 Let $k \neq j$ be natural numbers with $k \geq 2$ and $j \geq 2$. Then $\mathrm{pt}(\mathbf{R}_k)$ and $\mathrm{pt}(\mathbf{R}_j)$ are homeomorphic.

Proof: By R26.Add.15 and R29.Add.14 $(\mathbf{R}_k, <_k)$ and $(\mathbf{R}_j, <_j)$ are order-isomorphic. The result now follows from R29.2.8.

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