Representation of Suprema

In [5] and [6], Lubben's theorem on the existence of suprema of compactifications is shown in two ways, first by using uniform space theory and then via the Stone-Čech compactification. Here a more direct representation based on products is presented. Notice that this representation could also be presented as a proof of the existence of suprema, which is exactly what Chandler does in [1]. Identification of an infinite supremum as an inverse limit, a fact which is probably known, is a refinement of Chandler's construction.

Given $\{(Y, f_\alpha) : \alpha \in \Delta\}$, a non-empty family of T_2 compactifications of some space X, the notation $(Z, g) = \bigvee\{(Y, f_\alpha) : \alpha \in \Delta\}$ will be used instead of more cumbersome but more careful expressions that the class $[(Z, g)]$ has the properties of a supremum.

1. Suprema of finite sets

The main result in this subsection presents the finite case of Chandler's proof in [1] that suprema of compactifications exist. There is no logical need to deal with the finite case separately, but it has an appealing simplicity and plays a significant role in identifying the supremum as an inverse limit. Proofs are included for the convenience of the reader.

Assumptions: Let (X, τ) be a non-compact $T_{3\frac{1}{2}}$ space and let $\{(Y_i, f_i) : i = 1 \ldots n\}$ be a finite set of T_2 compactifications of X. Let $F : X \to \prod_{i=1}^n Y_i$ be defined by $F(x) = (f_1(x), \ldots, f_n(x))$. The weight of a space is defined as in [3] and will be denoted $w(X)$.

Proposition R3.1.1 F is an embedding.

Proof: F is clearly one-to-one. For $O_1, \ldots, O_n$ where O_i is open in Y_i, $F^{-1}[\prod_{i=1}^n O_i] = \bigcap_{i=1}^n f_i^{-1}[O_i]$ so that F is continuous. For $G \in \tau$ let O_i be open in Y_i with $f_i[G] = O_i \cap f_i[X]$. Then $F[G] = (\prod_{i=1}^n O_i) \cap F[X]$ and so F is relatively open.

Comment: In the case of an infinite product, this proof misleadingly suggests that the function analogous to F might not be relatively open, because an infinite product of open sets will not in general be open in the product. A more careful analysis reveals that one needs to use only a single factor, in fact any factor for which the component function is one-to-one and relatively open. This is exactly what Chandler does in [1], and a special case of the argument appears in R3.2.4 below.

Theorem R3.1.2 Let $Y = \overline{F[X]}$. Then (Y, F) is a T_2 compactification for X and $(Y, F) = \bigvee_{i=1}^n (Y_i, f_i)$.

Proof: The first assertion is immediate from R3.1.1. Let F_i be the restriction to Y of the projection to Y_i. Clearly F_i is continuous, $F_i \circ F = f_i$, and, since $F_i[Y]$ contains the dense $f_i[X]$, F_i is onto. Thus $[(Y, F)] \geq [(Y_i, f_i)]$ for $i = 1 \ldots n$. Next let $[(Z, g)]$ be an upper bound with $g_i : Z \to Y_i$ the continuous surjection such that $g_i \circ g = f_i$. Define $G : Z \to \prod_{i=1}^n Y_i$ by $G(z) = (g_1(z), \ldots, g_n(z))$. As in the proof of R3.1.1, G is continuous and clearly $G \circ g = F$. Since $G[g[X]] = F[X]$ is dense in the image of G, $G[Z] = Y$. and so G is the map required to show $[(Z, g)] \geq [(Y, F)]$. Thus $[(Y, F)]$ is the least upper bound.

Corollary R3.1.3 Let $(Y, F) = \bigvee_{i=1}^n (Y_i, f_i)$. If Y_i is metrizable (or second countable or first countable) for $i = 1 \ldots n$, then Y is metrizable (resp. second countable, first countable).

Proof: This is immediate from R3.1.2 since these properties are preserved in finite products and inherited by subspaces.

Corollary R3.1.4 Let $(Y, F) = \bigvee_{i=1}^n (Y_i, f_i)$. Then $w(Y) = max\{w(Y_i) : i = 1 \ldots n\}$.

Proof: Let $\aleph$ =max $\{w(Y_i) : i = 1 \ldots n\}$. Since $[(Y, F)] \geq [(Y_i, f_i)]$ for each i, by definition there are continous surjections $g_i : Y \to Y_i$. It follows that $w(Y) \geq w(Y_i)$ for each i and so $w(Y) \geq \aleph$. Also, since weight is monotone, we have by R3.1.2

$$w(Y) \leq w(\prod_{i=1}^{n} Y_i) \leq \aleph^n = \aleph.$$

Comments: It is noted in [7] that arbitrary suprema do not preserve metrizability, second countability, first countability, or weights. The extension of R3.1.3 to countably infinite suprema will be derived in the next section. In the locally compact case, the author has been unable to find a comparably simple representation for finite infima. The difficulty seems to be that even a finite infimum is defined as the supremum of a possibly infinite set of lower bounds.

2. Suprema of infinite families

A supremum of an infinite family of compactifications will be shown to be the inverse limit of a suitable inverse spectrum. Notation consistent with that in Appendix II of Dugundji [2] is used, but the presentation does not refer to Dugundji's general results.

Assumptions for this subsection: Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$ be an infinite set of T_2 compactifications of X. Let Δ^* denote the set of all non-empty, finite subsets of Δ,and let $\mathcal{C}$ be a cofinal subset of Δ^*. For $\delta \in \mathcal{C}$ of cardinality 1, say $\delta = \{\alpha\}$, let $W_\delta = Y_\alpha$ and $F_\delta = f_\alpha$. If $|\delta| \geq 2$, let $F_\delta : X \to \prod\{Y_\alpha : \alpha \in \delta\}$ be defined by $F_\delta(x)(\alpha) = f_\alpha(x)$ and let $W_\delta = \overline{F_\delta[X]}$, where the closure is in the product. Since δ is finite, section 1 applies: $(W_\delta, F_\delta) = \bigvee\{(Y_\alpha, f_\alpha) : \alpha \in \delta\}$. For $\delta, \gamma \in \mathcal{C}$ with $\delta \subseteq \gamma$, let $\pi_{\gamma\delta}$ be the restiction to W_γ of the continuous projection described by $\pi_{\gamma\delta}(y)(\alpha) = y(\alpha)$ for $\alpha \in \delta$.

Lemma R3.2.1 For $\delta \subseteq \gamma$, $\pi_{\gamma\delta}$ maps W_γ onto W_δ.

Proof: It is clear that $\pi_{\gamma\delta} \circ F_\gamma = F_\delta$ and so $\pi_{\gamma\delta}[F_\gamma[X]] = F_\delta[X]$. By continuity $\pi_{\gamma\delta}[W_\gamma] = \pi_{\gamma\delta}[\overline{F_\gamma[X]}] \subseteq \overline{F_\delta[X]} = W_\delta$. Surjectivity of $\pi_{\gamma\delta}$ follows immediately from the density of $F_\delta[X]$.

Lemma R3.2.2 Let δ,γ,ϵ be in $\mathcal{C}$ with $\delta \subseteq \gamma \subseteq \epsilon$. Then $\pi_{\gamma\delta} \circ \pi_{\epsilon\gamma} = \pi_{\epsilon\delta}$.

Proof: Clear from the definition.

Additional assumptions for this subsection: In $\prod\{W_\delta : \delta \in \mathcal{C}\}$, let $S = \{y : \delta, \gamma \in \mathcal{C}$ with $\delta \subseteq \gamma \Rightarrow y(\delta) = \pi_{\gamma\delta}(y(\gamma))\}$. For $x \in X$, y_x in $\prod\{W_\delta : \delta \in \mathcal{C}\}$ is defined by $y_x(\delta) = F_\delta(x)$.

Lemma R3.2.3 S is closed in $\prod\{W_\delta : \delta \in \mathcal{C}\}$ and, for all $x \in X$, $y_x \in S$.

Proof: Let $\{x_t\}$ be a net in S converging to $y \in \prod\{W_\delta : \delta \in \mathcal{C}\}$, and let $\delta \subseteq \gamma$. By definition of S, for every t, $x_t(\delta) = \pi_{\gamma\delta}(x_t(\gamma))$. Since the product has the topology of pointwise convergence, $\{x_t(\delta)\}$ and $\{x_t(\gamma)\}$ converge to $y(\delta)$ and $y(\gamma)$ respectively. Since $\pi_{\gamma\delta}$ is continuous and limits are unique in a T_2 space, $y(\delta) = \pi_{\gamma\delta}(y(\gamma))$. Thus $y \in S$ and S is closed. The second assertion is clear from the definitions of y_x, $\pi_{\gamma\delta}$, and S.

In Dugundji's terminology [2], $\{W_\epsilon; \pi_{\gamma\delta}\}$ is an inverse spectrum over $\mathcal{C}$ with spaces W_ϵ and connecting maps $\pi_{\gamma\delta}$. S is the inverse limit space of the spectrum.

Additional assumptions for this subsection: Let $f : X \to S$ be defined by $f(x) = y_x$. Let ρ_ϵ denote the projection from $\prod\{W_\delta : \delta \in \mathcal{C}\}$ onto W_ϵ.

Lemma R3.2.4 f is an embedding.

Proof: First, pick any $\epsilon \in \mathcal{C}$ and let $\alpha \in \epsilon$. For $x \neq t$ in X, since f_α is one-to-one, $y_x(\epsilon)(\alpha) = f_\alpha(x) \neq f_\alpha(t) = y_t(\epsilon)(\alpha)$, so that $y_x(\epsilon) \neq y_t(\epsilon)$, i.e., $f(x) \neq f(t)$. Also, $\rho_\delta \circ f = F_\delta$ is continuous for every δ. Thus f is continuous, one-to-one, and onto $f[X]$. Now let O be open in X with $x \in O$. For the ϵ chosen above and every $\alpha \in \epsilon$, since f_α is an embedding, there exists U_α open in Y_α with $f_\alpha[O] = U_\alpha \cap f_\alpha[X]$. Then $\rho_\epsilon^{-1}[(\prod\{U_\alpha : \alpha \in \epsilon\}) \cap W_\epsilon]$ is open in $\prod\{W_\delta : \delta \in \mathcal{C}\}$ and it is easy to check that

$$f(x) \in \rho_\epsilon^{-1}[(\prod\{U_\alpha] : \alpha \in \epsilon\}) \cap W_\epsilon] \cap f[X] \subseteq f[O] \cap f[X]$$

. Thus f is also relatively open and so an embedding.

Lemma R3.2.5 Let B be a non-empty open subset of S and let $y \in B$. Then there exist $\gamma \in \mathcal{C}$ and $\{G_\alpha : \alpha \in \gamma\}$ with G_α open in Y_α such that

$$y \in \rho_\gamma^{-1}[(\prod\{G_\alpha : \alpha \in \gamma\}) \cap W_\gamma] \bigcap S \subseteq B.$$

Proof: There exist $\delta_1, \ldots, \delta_n \in \mathcal{C}$ and $O_{\delta_1}, \ldots, O_{\delta_n}$ with O_{δ_j} open in W_{δ_j} such that

$$y \in (\cap_{i=1}^n \rho_{\delta_i}^{-1}[O_{\delta_i}]) \cap S \subseteq B.$$

Since $y(\delta_i) \in O_{\delta_i}$, for each i there exists $\{\tilde{O}_\alpha^i : \alpha \in \delta_i\}$ with $\tilde{O}_\alpha^i$ open in Y_α and

$$y(\delta_i) \in [(\prod\{\tilde{O}_\alpha^i : \alpha \in \delta_i\}) \cap W_{\delta_i}] \subseteq O_{\delta_i}.$$

Note that, since $\delta_i \in \mathcal{C}$, the product is finite. Also, to include every α from δ_i in this product, $\tilde{O}_\alpha^i$ can be taken to be Y_α if necessary.

Since $\mathcal{C}$ is co-final in Δ^*, there exists $\gamma \in \mathcal{C}$ such that $\delta_i \subseteq \gamma$ for all i. For $\alpha \in \cup_{i=1}^n \delta_i$, let $G_\alpha = \cap\{\tilde{O}_\alpha^i : \alpha \in \delta_i\}$. For any other α in γ, let $G_\alpha = Y_\alpha$. Since any intersection is finite, G_α is open in Y_α for all $\alpha \in \gamma$. For convenience let V denote $(\prod\{G_\alpha : \alpha \in \gamma\}) \cap W_\gamma$, which is open in W_γ.

First, for any $\alpha \in \delta_i$, since $y \in S$, $\pi_{\gamma\delta_i}(y(\gamma)) = y(\delta_i)$ and so $y(\gamma)(\alpha) = y(\delta_i)(\alpha) \in \tilde{O}_\alpha^i$. It follows easily that $y(\gamma) \in V$.

Now let $w \in \rho_\gamma^{-1}[V] \cap S$. To show $w \in B$, since $(\cap_{i=1}^n \rho_{\delta_i}^{-1}[O_{\delta_i}]) \cap S \subseteq B$, it is sufficient to verify $w \in \rho_{\delta_i}^{-1}[O_{\delta_i}]$ for each i between 1 and n. For such i let $\alpha \in \delta_i$. Since $w \in S$, $\pi_{\gamma\delta_i}(w(\gamma)) = w(\delta_i)$ so that $w(\delta_i)(\alpha) = w(\gamma)(\alpha) \in G_\alpha$. Therefore $w(\delta_i)(\alpha) \in \tilde{O}_\alpha^i$ and $w(\delta_i) \in [\prod\{\tilde{O}_\alpha^i : \alpha \in \delta_i\}] \cap W_{\delta_i} \subseteq O_{\delta_i}$. To summarize, $y \in \rho_\gamma^{-1}[V] \cap S \subseteq B$, as required for this lemma.

Proposition R3.2.6 (S, f) is a T_2 compactification of X.

Proof: S is T_2 and compact by virtue of its position as a closed subspace of a compact, T_2 space. Since f is an embedding from above, only the density of $f[X]$ remains to be shown. Let B be a non-empty open subset of S. For any $y \in B$, by R3.2.5 there exist $\gamma \in \mathcal{C}$ and $\{G_\alpha : \alpha \in \gamma\}$ with G_α open in Y_α such that

$$y \in \rho_\gamma^{-1}[(\prod\{G_\alpha : \alpha \in \gamma\}) \cap W_\gamma] \bigcap S \subseteq B.$$

Since $F_\gamma[X]$ is dense in W_γ, there exists $x \in X$ with $F_\gamma(x) = \rho_\gamma(y_x)$ in the open set $(\prod\{G_\alpha : \alpha \in \gamma\}) \cap W_\gamma$. Thus $y_x = f(x)$ is in B and so $f[X]$ is dense in S.

Theorem R3.2.7 $(S, f) = \bigvee\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$.

Proof: For each $\alpha^* \in \Delta$, since $\mathcal{C}$ is co-final, there exists $\gamma \in \mathcal{C}$ with $\alpha^* \in \gamma$. Let $g_\gamma = \rho_\gamma|_S$. Clearly $g_\gamma : S \to W_\gamma$ is continuous and $g_\gamma \circ f = F_\gamma$. Therefore the closed image of g_γ must contain the dense $F_\gamma[X]$ so that g_γ is onto. Then g_γ is the map required by the definition to show that $(S, f) \geq (W_\gamma, F_\gamma)$. By R3.1.2, since γ is finite, $(W_\gamma, F_\gamma) = \bigvee\{(Y_\alpha, f_\alpha) : \alpha \in \gamma\}$ and so by transitivity $(S, f) \geq (Y_{\alpha^*}, f_{\alpha^*})$,i.e.,$(S, f)$ is an upper bound. Now suppose the T_2 compactification (Z, h) is also an upper bound. Then for every $\gamma \in \mathcal{C}$ R3.1.2 shows $(Z, h) \geq (W_\gamma, F_\gamma)$ and so there exists $\phi_\gamma : Z \to W_\gamma$, the continuous surjection such that $\phi_\gamma \circ h = F_\gamma$. Define $\phi : Z \to \prod\{W_\gamma : \gamma \in \mathcal{C}\}$ by $\phi(z)(\gamma) = \phi_\gamma(z)$. The continuity of ϕ is immediate since $\rho_\gamma \circ \phi = \phi_\gamma$ for all γ. In addition, $\rho_\gamma \circ \phi \circ h = \phi_\gamma \circ h = F_\gamma$, which implies $\phi \circ h = f$. The last equation also shows that $\overline{f[X]} = S$ must be a subset of the closed set $\phi[Z]$. To show that this image is contained in S, let $\delta, \epsilon \in \mathcal{C}$ with $\delta \subseteq \epsilon$. First note that $\pi_{\epsilon\delta} \circ \phi_\epsilon \circ h = \pi_{\epsilon\delta} \circ F_\epsilon = F_\delta = \phi_\delta \circ h$. Since $\pi_{\epsilon\delta} \circ \phi_\epsilon$ and ϕ_δ agree on a dense set and map into a T_2 space, they must be equal. Now let $z \in Z$. To show $\phi(z) \in S$ requires $\pi_{\epsilon\delta}(\phi(z)(\epsilon)) = \phi(z)(\delta)$ or, equivalently, $\pi_{\epsilon\delta}(\phi_\epsilon(z)) = \phi_\delta(z)$, which has just been demonstrated. The map ϕ shows that $(Z, h) \geq (S, f)$ and so (S, f) represents the least upper bound, as claimed.

Corollary R3.2.8 Let $(Y, F) = \bigvee_{i=1}^{\infty}(Y_i, f_i)$. If Y_i is metrizable (or second countable or first countable) for every i, then Y is also metrizable (resp. second countable, first countable).

Proof: Since these properties are preserved in countable products and inherited by subspaces, this is immediate from R3.2.7.

Corollary R3.2.9 Let $(Y, F) = \bigvee_{i=1}^{\infty}(Y_i, f_i)$. If $w(Y_i) \leq \aleph$ for all i, then $w(Y) \leq \aleph$.

Proof: Let $\mathcal{C}$ be the collection of all non-empty initial intervals of natural numbers and, for $\delta = \{1, \ldots, j\}$, write W_j instead of W_δ. Clearly $w(W_j) \leq \aleph^j = \aleph$. By R3.2.7, Y is homeomorphic to a subspace of the product of the W_i and so, since weight is monotone, $w(Y) \leq w(\prod_{i=1}^{\infty} W_i) \leq \aleph^{\aleph_0} = \aleph$.

Lifting Extensions to Suprema

The results in this subsection make use of the representation of suprema as inverse limits, as presented above. They could be obtained as an application of a theorem on continuous maps of inverse spectra, which appears in Dugundji [2; Thm. 2.5, p. 430]. The proofs given here are specializations of the arguments in Dugundji.

The word 'extension' will be used in the following not uncommon way.

Definition R3.3.1 Let (X, τ) and (W, σ) be a $T_{3\frac{1}{2}}$ spaces, let $h : X \to W$ be continuous, and let (Y, f) and (Z, g) be T_2 compactifications of X and W respectively. A continuous map $H : Y \to Z$ will be called an extension of h provided $H \circ f = g \circ h$. If such an H exists, it will be said that h extends to a map from Y to Z.

The first lemma records some simple properties and the second shows that extendability does not depend on the choice of representatives from the compactification classes.

Lemma R3.3.2 (X, τ) and (W, σ) be a $T_{3\frac{1}{2}}$ spaces, let $h : X \to W$ be continuous, and let (Y, f) and (Z, g) be T_2 compactifications of X and W respectively. Let $H : Y \to Z$ be an extension of h. Then

i) H is the unique extension of h from Y to Z.

ii) If h is onto, then H is also onto.

Proof: For i): Another extension would equal H on the dense set $f[X]$. Since Z is T_2, it would equal H on all of Y. For ii): If h is onto, the closed set $H[Y]$ must contain $g[h[X]] = g[X]$, which is dense in Z.

Lemma R3.3.3 Let (X, τ) and (W, σ) be a $T_{3\frac{1}{2}}$ spaces, let $h : X \to W$ be continuous. Let (Y, f) and (Y^*, f^*) be equivalent T_2 compactifications of X and Let (Z, g) and (Z^*, g^*) be equivalent T_2 compactifications of W. If h extends to a map from Y to Z, then h extends to a map from Y^* to Z^*.

Proof: Suppose $H : Y \to Z$ is an extension of h. Let $\phi : Y \to Y^*$ and $\psi : Z \to Z^*$ be homeomorphisms such that $\phi \circ f = f^*$ and $\psi \circ g = g^*$. Then $\psi \circ H \circ \phi^{-1}$ is an extension of h from Y^* to Z^*.

The next lemma is the finite case of the general result.

Lemma R3.3.4 Let (X, τ) and (W, σ) be a $T_{3\frac{1}{2}}$ spaces. Let $n \in \mathbb{N}$. For $i = 1, \ldots, n$, assume (Y_i, f_i) is a T_2 compactification of X and (Z_i, g_i) is a T_2 compactification of W. Suppose that $h : X \to W$ is continuous and that $h_i : Y_i \to Z_i$ is an extension of h for each i. Then there exists $H : \vee_{i=1}^n Y_i \to \vee_{i=1}^n Z_i$ which is an extension of h. If each h_i is one-to-one, then H is one-to-one.

Proof: By R3.1.2 $S = \overline{\{(f_1(x), \ldots, f_n(x)) : x \in X\}}$ with embedding $x \mapsto (f_1(x), \ldots, f_n(x))$ represents the supremum of $\{(Y_i, f_i) : i = 1, \ldots, n\}$. Similarly $T = \overline{\{(g_1(w), \ldots, g_n(w)) : w \in W\}}$ with embedding $w \mapsto (g_1(w), \ldots, g_n(w))$ represents the supremum of $\{(Z_i, g_i) : i = 1, \ldots, n\}$. Let H be the restriction of the continuous map $(y_1, \ldots, y_n) \mapsto (h_1(y_1), \ldots, h_n(y_n))$ to S. For any $x \in X$, $H(f_1(x), \ldots, f_n(x)) = (h_1 \circ f_1(x), \ldots, h_n \circ f_n(x)) = (g_1 \circ h(x), \ldots, g_n \circ h(x))$. This calculation shows that the functional equation needed for an extension holds. It also shows, for the dense set $D = \{(f_1(x), \ldots, f_n(x)) : x \in X\}$, $H[D] \subseteq T$ so that, by continuity, H maps S into T. Thus H is the required extension. If each h_i is one-to-one, so is the map $(x_1, \ldots, x_n) \mapsto (h_1(x_1), \ldots, h_n(x_n))$ and consequently its restriction H.

Proposition R3.3.5 Let (X, τ) and (W, σ) be a $T_{3\frac{1}{2}}$ spaces. Let Δ be a non-empty set. Let $\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$ and $\{(Z_\alpha, g_\alpha) : \alpha \in \Delta\}$ be collections of T_2 compactifications of X and W respectively. Let $h : X \to W$ be continuous and assume, for every $\alpha \in \Delta$, h has an extension $h_\alpha : Y_\alpha \to Z_\alpha$. Then there exists $H : \vee Y_\alpha \to \vee Z_\alpha$, which is an extension of h. If every h_α is one-to-one, then H is one-to-one.

Proof: Let Δ^* denote the set of all non-empty, finite subsets of Δ and let $\delta \in \Delta^*$. If δ is a singleton, say $\delta = \{\alpha\}$, let $(Y_\delta^*, f_\delta^*) = (Y_\alpha, f_\alpha)$, $(Z_\delta^*, g_\delta^*) = (Z_\alpha, g_\alpha)$, and $h_\delta^* = h_\alpha$. If $|\delta| \geq 2$, let $f_\delta^* : X \to \prod\{Y_\alpha : \alpha \in \delta\}$ be defined by $f_\delta^*(x)(\alpha) = f_\alpha(x)$ and let $Y_\delta^* = \overline{f_\delta^*[X]}$, where the closure is in the product. Similarly, let $g_\delta^* : X \to \prod\{Z_\alpha : \alpha \in \delta\}$ be defined by $g_\delta^*(x)(\alpha) = g_\alpha(x)$ and let $Z_\delta^* = \overline{g_\delta^*[X]}$, where the closure is in the product. Since δ is finite, R3.1.2 applies: $(Y_\delta^*, f_\delta^*) = \vee\{(Y_\alpha, f_\alpha) : \alpha \in \delta\}$ and $(Z_\delta^*, g_\delta^*) = \bigvee\{(Z_\alpha, g_\alpha) : \alpha \in \delta\}$. Also, by R3.3.4, there is $h_\delta^* : Y_\delta^* \to Z_\delta^*$ which is an extension of h. Next, for $\delta, \gamma \in \Delta^*$ with $\delta \subseteq \gamma$, let $\pi_{\gamma\delta}$ be the restiction to Y_γ^* of the continuous projection described by $\pi_{\gamma\delta}(y)(\alpha) = y(\alpha)$ for $\alpha \in \delta$. The map $\rho_{\gamma\delta} : Z_\gamma^* \to Z_\delta^*$ is defined analogously. Finally, in $\prod\{Y_\delta^* : \delta \in \Delta^*\}$ let $S_1 = \{p : \delta, \gamma \in \Delta^* \text{ with } \delta \subseteq \gamma \Rightarrow p(\delta) = \pi_{\gamma\delta}(p(\gamma))\}$ and in $\prod\{Z_\delta^* : \delta \in \Delta^*\}$ let $S_2 = \{q : \delta, \gamma \in \Delta^* \text{ with } \delta \subseteq \gamma \Rightarrow q(\delta) = \rho_{\gamma\delta}(q(\gamma))\}$.

Note that, for $\delta, \gamma \in \Delta^*$ with $\delta \subseteq \gamma$, since $\pi_{\gamma\delta}$ and $\rho_{\gamma\delta}$ are projections, the extensions constructed as in the proof of R3.3.4 satisfy the equation $\rho_{\gamma\delta} \circ h^*_\gamma = h^*_\delta \circ \pi_{\gamma\delta}$.

By R3.2.7 $(S_1, f) = \vee\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$ and $(S_2, g) = \vee\{(Z_\alpha, g_\alpha) : \alpha \in \Delta\}$, where $f(x)(\delta) = f^*_\delta(x)$ and $g(w)(\delta) = g^*_\delta(w)$.

Define H on S_1 by $H(p)(\delta) = h^*_\delta(p(\delta))$. Because the product topology is the topology of pointwise convergence and each h^*_δ is continuous, H is continuous. For $p \in S_1$ and $\delta, \gamma \in \Delta^*$ with $\delta \subseteq \gamma$, $\rho_{\gamma\delta}(H(p)(\gamma)) = \rho_{\gamma\delta}(h^*_\gamma(p(\gamma))) = h^*_\delta(\pi_{\gamma\delta}(p(\gamma))) = h^*_\delta(p(\delta)) = H(p)(\delta)$. Thus H maps S_1 into S_2. Finally, for $x \in X$ and $\delta \in \Delta^*$, $H(f(x))(\delta) = h^*_\delta(f(x)(\delta)) = h^*_\delta(f^*_\delta(x) = g^*_\delta(h(x)) = g(h(x)(\delta)$. Thus H is an extension of h.

Finally, if each h_α is one-to-one, R3.3.4 shows that each h^*_δ is also one-to-one. If $H(p_1) = H(p_2)$, then, for every $\delta \in \Delta^*$, $H(p_1)(\delta) = H(p_2)(\delta)$ implies $h^*_\delta(p_1(\delta)) = h^*_\delta(p_2(\delta))$ and so $p_1(\delta) = p_2(\delta)$. Thus H is one-to-one.

Example R3.3.6 In [7] it is shown that, for $\mathbb{N}$ with the discrete topology, $\beta\mathbb{N}$ is the supremum of the two-point compactifications of $\mathbb{N}$. Let $f : \mathbb{N} \to \mathbb{N}$ by $f(n) = i$, where $i \in \{1, 2, 3\}$ and $n \equiv i$ mod 3. It is easy to check that f does not have an extension to any two-point compactification, but, because of the universal extension property of the Stone-Čech compactification, f does extend to $\beta\mathbb{N}$. Thus the converse of R3.3.5 is false.

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