Realcompactness and Uniformity

In [2] Wilansky defines realcompactness in a way which is described as a modification of the presentation in [1]. Here it is observed that Wilansky's approach can be based on uniform topology. By doing so a version of realcompactness can be defined for any compactification of a $T_{3\frac{1}{2}}$ space, not just the Stone-Čech compactification. In this subsection some results and examples based on this expanded notion are presented.

Given a $T_{3\frac{1}{2}}$ topological space (X,τ), $\mathcal{TB}(X,\tau)$ will denote the set of all separated, totally bounded uniformities generating τ. The largest such uniformity will be denoted $\mathcal{U}_M$. As noted in [3], $\mathcal{U}_M$ corresponds to the Stone-Čech compactification, $(\beta X,\iota)$. If (X,τ) is also locally compact, $\mathcal{U}_m$ will denote the element of $\mathcal{TB}(X,\tau)$ corresponding to the one-point compactification (X^+,ι^+) described in [4]. When $\mathcal{U}_m$ exists, it is the smallest element of $\mathcal{TB}(X,\tau)$. The real numbers, $\mathbb{R}$, will always have the usual metric topology; their one-point compactification, $\mathbb{R}^+$, will be assumed to have its unique uniformity.

Preliminaries

This subsection contains several well-known results, which are intermediate in the sense that they might not appear in an introductory semester of topology. For the sake of completeness, proofs are provided for results without a specific reference.

Recall that a topological space is extremely disconnected provided the closure of every open set is open.

Theorem R30.1.1 Let (X,τ) be a $T_{3\frac{1}{2}}$ topological space. Then βX is extremely disconnected if and only if X is extremely disconnected.

Proof: See [2], p. 301.

Proposition R30.1.2 Let (X,τ) be an extremely disconnected T_2 space. Let $\{x_n\}$ be a convergent sequence in X. Then $\{x_n\}$ is eventually constant.

Proof: See [2], p. 301.

Proposition R30.1.3 Let (X,τ) be a countable, compact T_2 space. Then (X,τ) is metrizable.

Proof: A finite T_2 space must be discrete and so metrizable. Thus assume X is countably infinite. Let $x\in X$. For each $y\in X-\{x\}$, there exist $O_y,G_y\in\tau$ such that $x\in O_y$, $y\in G_y$, and $O_y\cap G_y=\emptyset$. For each finite subset F of $X-\{x\}$, let $O(F)=\cap\{O_y : y\in F\}$, which is open and contains x. For $x\in O\in\tau$, by hypothesis the closed set $X-O$ is compact and so there is a finite set $F_1\subseteq X-\{x\}$ such that $X-O\subseteq\cup\{G_y : y\in F_1\}$. Then $x\in O(F_1)\subseteq O$ and so the collection of all $O(F)$ is a basis at x. Since $X-\{x\}$ is countable, the collection of its finite subsets is also countable. Thus the basis produced at x is countable. The countable union of these countable local bases is a countable basis for X, i.e., (X,τ) is second countable. Since a compact T_2 space is T_4 and so T_3, (X,τ) is also T_3. By Urysohn's metrization theorem, a second countable T_3 space must be metrizable.

RK$\mathcal{U}$ Spaces

Given a $T_{3\frac{1}{2}}$ space X, βX is characterized by the property that every bounded continuous real-valued function on X extends continuously to βX. From this follows that every continuous map from X into a compact, T_2 space extends continuously to βX. In

[2] Wilansky begins his presentation of realcompactness by observing that every continuous, real-valued map on X (possibly unbounded) can be regarded as a continuous map to $\mathbb{R}^+$and so has a continuous extension from βX to $\mathbb{R}^+$. By R7.1.1 the existence of that extension shows that the original map is uniformly continuous from $(X,\mathcal{U}_M)$ to $\mathbb{R}^+$with its unique uniformity. (That fact can also be derived more directly by using R7.2.6 and the fact that $\mathcal{U}_M$ is the largest element of $\mathcal{TB}(X,\tau)$.) This discussion motivates the first definition.

Definition R30.2.1 Let (X,τ) be a $T_{3\frac{1}{2}}$ topological space and let $\mathcal{U}$ be in $\mathcal{TB}(X,\tau)$. $C_{\mathcal{U}}(X)$ is the set of all real-valued, continuous functions on X, which are uniformly continuous functions from $(X,\mathcal{U})$ to $\mathbb{R}^+$with its unique uniformity.

From the prior discussion, $C_{\mathcal{U}_M}(X)$ is the set of all continuous real-valued functions on X. In general, for $g \in C_{\mathcal{U}}(X)$, $g[X]$ must be totally bounded in the compact space $\mathbb{R}^+$, but $g[X]$ need not be bounded in $\mathbb{R}$.

Proposition R30.2.2 Let (X,τ) be a $T_{3\frac{1}{2}}$ topological space and let $\mathcal{U},\mathcal{V}$ be in $\mathcal{TB}(X,\tau)$ with $\mathcal{U} \subseteq \mathcal{V}$. Then $C_{\mathcal{U}}(X) \subseteq C_{\mathcal{V}}(X)$.

Proof: Let $f \in C_{\mathcal{U}}(X)$. For any W in the unique uniformity for $\mathbb{R}^+$, $(f \times f)^{-1}[W]$ is in $\mathcal{U}$ and so in $\mathcal{V}$ by hypothesis. Thus $f \in C_{\mathcal{V}}(X)$.

Corollary R30.2.3 Let (X,τ) be a $T_{3\frac{1}{2}}$ topological space and let $\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ be a non-empty collection in $\mathcal{TB}(X,\tau)$. Then $\cup\{C_{\mathcal{U}_\alpha}(X) : \alpha \in \Delta\} \subseteq C_{\vee\mathcal{U}_\alpha}(X)$.

Proof: For any $\delta \in \Delta$, $\mathcal{U}_\delta \subseteq \vee\mathcal{U}_\alpha$ and the conclusion is immediate from R30.2.2.

The next four results make it possible to give a simple example showing that that the containment in R30.2.3 may be proper. $\mathbb{N}$, the set of positive integers, will always have the discrete topology. The uniformity $\mathcal{U}_E$, where E is an equivalence relation, is the collection of all supersets of E as in R5.2.1. If E is an equivalence relation on $\mathbb{N}$ with finitely many equivalence classes exactly n of which are infinite, by R5.2.4 $\mathcal{U}_m \vee \mathcal{U}_E$ is a totally bounded uniformity corresponding to an n-point compactification of $\mathbb{N}$. A function $f : \mathbb{N} \to \mathbb{R}$ is of course a sequence with $x_n = f(n)$. For an infinite subset C of $\mathbb{N}$, the subsequence determined by C will mean $\{x_{n_i}\}$ with n_i denoting the ith element of C.

Lemma R30.2.4 The unique uniformity for $\mathbb{R}^+$has a basis of sets of the form $\cup_{i=1}^n(G_i \times G_i)$, where $G_1,\ldots,G_n$ form an open cover of $\mathbb{R}^+$, $\infty \in G_1$, $\mathbb{R}-G_1$ is compact, and for $2 \leq i \leq n$ $\infty \notin G_i$ and G_i is open in $\mathbb{R}$.

Proof: Let V be in the unique uniformity for $\mathbb{R}^+$, i.e., V is a neighborhood of the diagonal. By compactness, there is a finite open cover of $\mathbb{R}^+$, $O_1,\ldots,O_n$ such that $\cup_{i=1}^n(O_i \times O_i) \subseteq V$. By re-indexing, one can assume $\infty \in O_1$ so that $\mathbb{R}-O_1$ is compact. Write $G_1 = O_1$. For $2 \leq i \leq n$, if $\infty \in O_i$, $O_i = \{\infty\} \cup G_i$, where G_i is open in $\mathbb{R}$. If $\infty \notin O_i$, let $G_i = O_i$. Then $\cup_{i=1}^n(G_i \times G_i)$ is in the uniformity of $\mathbb{R}^+$and is a subset of V as required.

Lemma R30.2.5 Let $n \in \mathbb{N}$. Let E be an equivalence relation on $\mathbb{N}$ with exactly n distinct equivalence classes, $D_1,\ldots,D_n$, each of which is infinite. Let $f \in C_{\mathcal{U}_m\vee\mathcal{U}_E}(\mathbb{N})$. Then for each $1 \leq j \leq n$, the subsequence determined by D_j has at most one cluster point in $\mathbb{R}$.

Proof: Let $1 \leq j \leq n$. Suppose $L < M$ are both cluster points of the subsequence determined by D_j. Let $\delta = M - L$. Let

$$O_1 = (-\infty, L-\delta/4) \cup (L+\delta/4, M-\delta/4) \cup (M+\delta/4,\infty).$$

Note that the complement of O_1 in $\mathbb{R}$ is compact and so $G_1 = O_1 \cup \{\infty\}$ is open in $\mathbb{R}^+$. Let $G_2 = (L - \delta/2, L + \delta/2)$ and $G_3 = (M - \delta/2, M + \delta/2)$. These form an open cover of $\mathbb{R}^+$and so $V = \cup_{k=1}^3 G_k \times G_k$ is in the unique uniformity for $\mathbb{R}^+$. By uniform continuity, $(f \times f)^{-1}[V]$ is in $\mathcal{U}_m \vee \mathcal{U}_E$. Thus there is $S \subseteq \mathbb{N}$ with a finite complement such that, for $U = S \times S \cup \{(t,t) : t \in \mathbb{N} - S\}$, $U \cap E \subseteq (f \times f)^{-1}[V]$. Since D_j is infinite and the complement of S is finite, there is K such that $i \geq K$ implies n_i (the ith element of D_j) is in S. Since L and M are both cluster points, there are $r, s \geq K$ in $\mathbb{N}$ such that $f(n_r) \in (L - \delta/8, L + \delta/8)$ and $f(n_s) \in (M - \delta/8, M + \delta/8)$. Then $(n_r, n_s) \in U$ and, since both are in D_j, in E. But $f(n_r) \in G_2 - G_3$, $f(n_s) \in G_3 - G_2$, and neither is in G_1, i.e. $(f(n_r), f(n_s)) \notin V$, a contradiction. Thus the conclusion holds.

Lemma R30.2.6 Let $n \in \mathbb{N}$. Let E be an equivalence relation on $\mathbb{N}$ with exactly n distinct equivalence classes, $D_1, \ldots, D_n$, each of which is infinite. Let $f \in C_{\mathcal{U}_m \vee \mathcal{U}_E}(\mathbb{N})$ and let $j \in \mathbb{N}$ with $1 \leq j \leq n$. If the subsequence determined by D_j has a cluster point in $\mathbb{R}$, then it converges to that cluster point.

Proof: Assume $1 \leq j \leq n$ and the subsequence determined by D_j has a cluster point L in $\mathbb{R}$. (It must be unique by R30.2.5.) Let $\epsilon > 0$ and assume the subsequence determined by D_j is not eventually in $(L-\epsilon, L+\epsilon)$. Then the set $S = \{n : n \in D_j \text{ and } f(n) \notin (L-\epsilon, L+\epsilon)\}$ is infinite. Let $O_1 = (-\infty, L-\epsilon/2) \cup (L+\epsilon/2, \infty)$. The complement of O_1 in $\mathbb{R}$ is compact and so $G_1 = O_1 \cup \{\infty\}$ is open in $\mathbb{R}^+$, as is $G_2 = (L - \epsilon, L + \epsilon)$. Since these cover $\mathbb{R}^+$, $V = \cup_{i=1}^2 (G_i \times G_i)$ is in the unique uniformity for $\mathbb{R}^+$. By uniform continuity, there is T contained in $\mathbb{N}$ such that $\mathbb{N} - T$ is finite and, for $U = (T \times T) \cup \{(t,t) : t \in \mathbb{N} - T\}$, $U \cap E \subseteq (f \times f)^{-1}[V]$. Since S is infinite and T co-finite, $S \cap T$ must be infinite. Pick $n \in S \cap T$. By definition of S, $f(n) \notin (L - \epsilon, L + \epsilon)$ and so it is not in G_2. Also, since the complement of T is finite and L is a cluster point, there is $m \in D_j \cap T$ with $f(m) \in (L - \epsilon/2, L + \epsilon/2)$ and so not in G_1. Thus $(m, n) \in T \times T \subseteq U$ and $(m, n) \in D_j \times D_j \subseteq E$, i.e., $(m, n) \in U \cap E$. But $f(m) \notin G_1$ and $f(n) \notin G_2$, a contradiction. Thus the subsequence determined by D_j is eventually in $(L-\epsilon, L+\epsilon)$, i.e., the subsequence determined by D_j converges to L.

Proposition R30.2.7 Let $n \in \mathbb{N}$. Let E be an equivalence relation on $\mathbb{N}$ with exactly n distinct equivalence classes, $D_1, \ldots, D_n$, each of which is infinite.
Then $f \in C_{\mathcal{U}_m \vee \mathcal{U}_E}(\mathbb{N})$ if and only if, for each j with $1 \leq j \leq n$, the subsequence determined by D_j either converges or has no cluster point in $\mathbb{R}$.

Proof: Assume $f \in C_{\mathcal{U}_m \vee \mathcal{U}_E}(\mathbb{N})$. By R30.2.5 the subsequence determined by D_j either has one cluster point in $\mathbb{R}$ or it has none. By R30.2.6, if it has one cluster point in $\mathbb{R}$, it converges to that cluster point. Thus the condition is necessary.

For sufficiency assume that, for $1 \leq j \leq n$, the subsequence determined by D_j converges or has no cluster point in $\mathbb{R}$. By R30.2.4 let $V = \cup_{i=1}^n (G_i \times G_i)$ be a basic set in the unique uniformity for $\mathbb{R}^+$, where $G_1, \ldots, G_n$ covers $\mathbb{R}^+$, $\infty \in G_1$, $\mathbb{R} - G_1$ is compact, and for $2 \leq i \leq n$ $\infty \notin G_i$ and G_i is open in $\mathbb{R}$. Assume $\mathbb{R} - G_1 \subseteq [a, b]$ and let $1 \leq j \leq n$. If the subsequence determined by D_j has no cluster point in $\mathbb{R}$, the set $\{t \in D_j : f(t) \in [a, b]\}$ is finite. Pick M_j such that $t \in D_j$ and $t \geq M_j$ imply $f(t) \notin [a, b]$. If the subsequence determined by D_j converges to L_j, pick M_j such that $t \in D_j$ and $t \geq M_j$ imply $f(t) \in \cap\{G_i : L_j \in G_i\}$. Let M be the largest of $M_1, \ldots, M_n$ and let S be the set $\{t \in \mathbb{N} : t \geq M\}$. For $U = (S \times S) \cup \{(t,t) : t \notin S\}$, which is in $\mathcal{U}_m$, it is easy to check that $U \cap E \subseteq (f \times f)^{-1}[V]$. Thus $f \in C_{\mathcal{U}_m \vee \mathcal{U}_E}(\mathbb{N})$.

Comment: Although it will not be needed here, the previous lemma can be easily generalized to the case where E has finitely many equivalence classes, exactly n of which are infinite. A finite class plays no role in determining the uniform continuity.

The following is a simple application of the pidgeonhole principle.

Corollary R30.2.8 Let $n \in \mathbb{N}$ and let $f : \mathbb{N} \to \mathbb{R}$. Let E be an equivalence relation on $\mathbb{N}$ with exactly n distinct equivalence classes, $D_1, \ldots, D_n$, each of which is infinite. Assume there are $n+1$ distinct real numbers $L_1, \ldots, L_{n+1}$ and infinite subsets $S_1, \ldots, S_{n+1}$ of $\mathbb{N}$ such that the subsequence determined by S_i converges to L_i for $1 \leq i \leq n+1$. Then $f \notin C_{\mathcal{U}_m \vee \mathcal{U}_E}(\mathbb{N})$.

Proof: For each i with $1 \leq i \leq n+1$, there is $j(i)$ such that $T_i = S_i \cap D_{j(i)}$ is infinite. The subsequence determined by T_i converges to L_i and so L_i is a cluster point of the subsequence determined by $D_{j(i)}$. Since $n < n+1$, there must be $i \neq k$ such that $j(i) = j(k)$. Then the subsequence determined by $D_{j(k)}$ has two distinct cluster points, L_i and L_k. By R30.2.7 $f \notin C_{\mathcal{U}_m \vee \mathcal{U}_E}(\mathbb{N})$.

Example R30.2.9 Let E be equivalence mod 2 and F be equivalence mod 3, both on $\mathbb{N}$. By basic number theory, $E \cap F$ is equivalence mod 6. Write $G = E \cap F$. It follows that $\mathcal{U}_E \vee \mathcal{U}_F = \mathcal{U}_G$ and so $(\mathcal{U}_m \vee \mathcal{U}_E) \vee (\mathcal{U}_m \vee \mathcal{U}_F) = \mathcal{U}_m \vee \mathcal{U}_G$. For $1 \leq i \leq 6$ let f map the mod 6 equivalence class of i to i. By R30.7 $f \in C_{\mathcal{U}_m \vee \mathcal{U}_G}(\mathbb{N})$. Because the image of $\mathbb{N}$ under f has too many subsequence limits, by R30.2.8 $f \notin C_{\mathcal{U}_m \vee \mathcal{U}_E}(\mathbb{N})$ and $f \notin C_{\mathcal{U}_m \vee \mathcal{U}_F}(\mathbb{N})$.

Because suprema of totally bounded uniformities correspond to suprema of compactifications, the previous example suggests that results in this section about suprema of compactifications, if any, are likely to be negative. Of course, for a finite chain of uniformities, equality would hold in R30.2.3. Next an example is presented that shows equality can fail in the case of a countable chain. This only serves as a counterexample and does not play any role in the subsequent development starting with R30.2.13.

The promised example will use specific equivalence relations on $\mathbb{N}$: for each n E_n will denote equivalence mod 2^n. The E_n-equivalence class of $x \in \mathbb{N}$ will be written $Cl_n(x)$.

Lemma R30.2.10 The collection $\{Cl_n(2^{n-1}) : n \in \mathbb{N}\}$ is a partition of $\mathbb{N}$.

Proof: First, the sets are disjoint: suppose $j \in Cl_n(2^{n-1}) \cap Cl_m(2^{m-1})$ with $n < m$. Then $j = 2^{n-1} + \alpha 2^n = 2^{m-1} + \beta 2^m$ for some integers α, β. Then $2^{n-1} - 2^{m-1} = \beta 2^m - \alpha 2^n$ so that $1 - 2^{m-n} = \beta 2^{m-n+1} - 2\alpha$. The left term of the last equation is odd and the right even, a contradiction. To see that the sets cover $\mathbb{N}$, let $x \in \mathbb{N}$. From number theory x has a unique binary representation $x = \Sigma_{i=1}^{M} c_i 2^{i-1}$, where each $c_i \in \{0, 1\}$ and $c_M = 1$. Let j be the smallest element of $\{i : c_i = 1\}$. Then $x = 2^{j-1} + \Sigma_{i=j+1}^{M} c_i 2^{i-1}$ and $x \in Cl_j(2^{j-1})$.

Lemma R30.2.11 Let $a, b, M \in \mathbb{N}$ with $a \equiv b \bmod 2^M$. Assume $a \in Cl_j(2^{j-1})$ and $b \in Cl_l(2^{l-1})$ with $j \neq l$. Then j, l are both greater than M.

Proof: Without loss of generality, assume $j < l$. Suppose the conclusion is false so that $j \leq M$. By hypothesis there exist integers c, d, e such that $a = b + c2^M$, $a = 2^{j-1} + d2^j$, and $b = 2^{l-1} + e2^l$. Combining these equations yields

$$2^{j-1} + d2^j = 2^{l-1} + e2^l + c2^M.$$

Now divide by 2^{j-1} to obtain

$$1 + 2d = 2^{l-j} + e2^{l-j+1} + c2^{M-j+1}.$$

That equation yields a contradiction, because the left expression is odd and the right even.

Example R30.2.12 Use the partition above to define $f : \mathbb{N} \to \mathbb{R}$ by $f(x) = n$ for every $x \in Cl_n(2^{n-1})$. For any $t \in \mathbb{N}$, the subsequence determined by $Cl_t(2^{t-1})$ is constant with value t, i.e. it converges to t. Thus there are infinitely many infinite subsets determining subsequences with distinct limits. By R30.2.8 $f \notin C_{\mathcal{U}_m \vee \mathcal{U}_{E_n}}(\mathbb{N})$ for any positive integer n. Next it will be shown that $f \in C_{\vee\{\mathcal{U}_m \vee \mathcal{U}_{E_n}\}}(\mathbb{N})$. Let V be a basic entourage in the unique uniformity for $\mathbb{R}^+$of the form described in R30.2.4: $V = \cup_{i=1}^{n}(G_i \times G_i)$, where $G_1, \ldots, G_n$ form an open cover of $\mathbb{R}^+$, $\infty \in G_1$, $\mathbb{R}-G_1$ is compact, and for $2 \leq i \leq n$ $\infty \notin G_i$ and G_i is open in $\mathbb{R}$. There is $M \in \mathbb{N}$ such that $n > M$ implies $n \in G_1$. Since $E_M \in \vee\{\mathcal{U}_m \vee \mathcal{U}_{E_n}\}$, it is sufficient to check that $E_M \subseteq (f \times f)^{-1}[V]$, which yields the claimed uniform continuity of f. Let $(a, b) \in U_M$. Since V contains the diagonal, assume $f(a) \neq f(b)$. Since $(a, b) \in E_M$, $a \equiv b \bmod 2^M$. Apply R30.2.11 with $j = f(a)$ and $l = f(b)$ to see that $f(a), f(b)$ are both greater M. By choice of M, $(f(a), f(b)) \in G_1 \times G_1 \subseteq V$.

Comment: Since $E_{n+1} \subseteq E_n$ for all $n, \mathcal{U}_{E_n} \subseteq \mathcal{U}_{E_{n+1}}$ and so the sequence $\{\mathcal{U}_m \vee \mathcal{U}_{E_n}\}$ is increasing. Thus the example e shows that the containment of R30.3 may be proper even if the collection of uniformities is linearly ordered. Note that by R5.2.4 and R5.3.8 $\mathcal{U}_m \vee \mathcal{U}_{E_n}$ corresponds to the class of the Wallman compactification $\omega(\mathcal{Z}(E_n))$ and so, by R1.5 and R10.1.4ii, $\vee_{n=1}^{\infty}(\mathcal{U}_m \vee \mathcal{U}_{E_n})$ corresponds to $\mathbb{N}_2$.

The next definition follows the pattern in [2;p.160]. Notation: Let (X, τ) be a $T_{3\frac{1}{2}}$ topological space and let $\mathcal{U}$ be in $\mathcal{TB}(X, \tau)$ correspond to the compactification class $[(Y, f)]$. For $g \in C_{\mathcal{U}}(X)$, $\overline{g}$ denotes the extension of g from Y to $\mathbb{R}^+$, which is guaranteed by R7.1.3.

Definition R30.2.13 Let (X, τ) be a $T_{3\frac{1}{2}}$ topological space and let $\mathcal{U}$ be in $\mathcal{TB}(X, \tau)$. Assume $\mathcal{U}$ corresponds to the compactification class $[(Y, f)]$. Define $\nu_{\mathcal{U}}^{Y}(X)$ as the set of $t \in Y$ such that, for every $g \in C_{\mathcal{U}}(X)$, $\overline{g}(t)$ is in $\mathbb{R}$.

Lemma R30.2.14 Let (X, τ) be a $T_{3\frac{1}{2}}$ topological space and let $\mathcal{U}$ be in $\mathcal{TB}(X, \tau)$. Assume $\mathcal{U}$ corresponds to the compactification class $[(Y, f)]$. Then $f[X]$ is a dense subspace of $\nu_{\mathcal{U}}^{Y}(X)$.

Proof: Let $f(x) \in f[X]$. For any $g \in C_{\mathcal{U}}(X)$, $\overline{g}(f(x)) = \iota^+ \circ g(x) = g(x)$, which is in $\mathbb{R}$. Thus $f[X] \subseteq \nu_{\mathcal{U}}^{Y}(X)$. That containment and the fact that $f[X]$ is dense in Y imply $f[X]$ is also dense in $\nu_{\mathcal{U}}^{Y}(X)$ with the subspace topology.

Corollary R30.2.15 Let (X, τ) be a $T_{3\frac{1}{2}}$ topological space and let $\mathcal{U}$ be in $\mathcal{TB}(X, \tau)$. Assume $\mathcal{U}$ corresponds to the compactification class $[(Y, f)]$. Then the following are equivalent:

i) $\nu_{\mathcal{U}}^{Y}(X)$ is closed in Y.
ii) $\nu_{\mathcal{U}}^{Y}(X)$ is compact.
iii) $\nu_{\mathcal{U}}^{Y}(X) = Y$.

Proof: The first two are equivalent because Y is compact and T_2. Clearly iii) implies i) and by the density of $f[X]$ i) implies iii).

Lemma R30.2.16 Let (X, τ) be a $T_{3\frac{1}{2}}$ topological space and let $\mathcal{U}, \mathcal{V}$ be in $\mathcal{TB}(X, \tau)$ with $\mathcal{U} \subseteq \mathcal{V}$. Assume $\mathcal{U}, \mathcal{V}$ correspond to the compactification classes $[(Y, f)], [(Z, h)]$ respectively. Then there exists $\phi : Z \to Y$, a continuous surjection, such that $\phi \circ h = f$ and $\phi[\nu_{\mathcal{V}}^{Z}(X)] \subseteq \nu_{\mathcal{U}}^{Y}(X)$.

Proof: By R1.5, since $\mathcal{U} \subseteq \mathcal{V}$, $(Y, f) \leq (Z, h)$ and so a unique continuous surjection

ϕ exists with $\phi \circ h = f$. Let $g \in C_{\mathcal{U}}(X)$ have extension $\overline{g} : Y \to \mathbb{R}^+$, i.e., $\overline{g} \circ f = g$. By R30.2.2 g has an extension $\overline{\overline{g}} : Z \to \mathbb{R}^+$with $\overline{\overline{g}} \circ h = g$. Since the extensions are unique and $g = \overline{g} \circ f = \overline{g} \circ \phi \circ h$, $\overline{\overline{g}} = \overline{g} \circ \phi$. Now let $t \in \nu^Z_{\mathcal{V}}(X)$ and let $s = \phi(t)$. Then $\overline{g}(s) = \overline{\overline{g}}(t)$, which is in $\mathbb{R}$. Since this holds for any g, $s \in \nu^Y_{\mathcal{U}}(X)$.

Proposition R30.2.17 Let (X, τ) be a $T_{3\frac{1}{2}}$ topological space and let $\mathcal{U} \in \mathcal{TB}(X, \tau)$. Assume $\mathcal{U}$ corresponds to the compactification class $[(Y, f)]$. Assume (Z, h) is equivalent to (Y, f) and let $\psi : Y \to Z$ be the unique homeomorphism such that $\psi \circ f = h$. Then $\mathcal{U}$ also corresponds to $[(Z, h)]$. Moreover, ψ restricted to $\nu^Y_{\mathcal{U}}(X)$ is the unique homeomorphism onto $\nu^Z_{\mathcal{U}}(X)$ such that the homeomorphism maps $f[X]$ onto $h[X]$ pointwise, i.e., $f(x) \mapsto h(x)$.

Proof: The equivalence means that (Y, f) and (Z, h) generate the same compactification class, i.e., $\mathcal{U}$ also corresponds to $[(Z, h)]$. Applying R30.2.16 twice, once for ψ and once for ψ^{-1}, yields $\psi[\nu^Y_{\mathcal{U}}(X)] = \nu^Z_{\mathcal{U}}(X)$. The restriction of ψ is one-to-one, onto its image, and continuous. Similarly the restriction of ψ^{-1} to $\nu^Z_{\mathcal{U}}(X)$ is also continuous. This homeomorphism maps $f[X]$ onto $h[X]$ pointwise because $\psi \circ f = h$ and is the unique such because $f[X]$,$h[X]$ are dense in the T_2 spaces $\nu^Y_{\mathcal{U}}(X)$, respectively $\nu^Z_{\mathcal{U}}(X)$.

In view of R30.2.17, $\nu^Y_{\mathcal{U}}(X)$ will be written as $\nu_{\mathcal{U}}(X)$ from this point on. R30.2.17 also shows that the next definition is independent (up to homeomorphism) of the choice of compactification class representative.

Definition R30.2.18 Let (X, τ) be a $T_{3\frac{1}{2}}$ topological space and let $\mathcal{U}$ be in $\mathcal{TB}(X, \tau)$. Assume $\mathcal{U}$ corresponds to the compactification class $[(Y, f)]$. The space (X, τ) is RK$\mathcal{U}$ if and only if $\nu_{\mathcal{U}}(X) = f[X]$.

The previous definition generalizes the definition of realcompactness presented in [2]. As a result we have the following:

Proposition R30.2.19 Let (X, τ) be a $T_{3\frac{1}{2}}$ topological space. Then (X, τ) is RK$\mathcal{U}_M$ if and only if (X, τ) is realcompact.

Proof: By R1.8 $\mathcal{U}_M$ corresponds to $[(\beta X, \iota)]$, i.e., the class of the Stone-Čech compactification, and, as following R30.2.1, $C_{\mathcal{U}_M}(X)$ is the set of all continuous real-valued functions on X. Thus $\nu_{\mathcal{U}_M}(X)$ as defined here is identical to Wilansky's definition of $\nu(X)$, the realcompactification of X. Wilansky's definition of realcompact is $\nu(X) = \iota[X]$, i.e., $\nu_{\mathcal{U}_M}(X) = \iota[X]$. The conclusion follows.

The next result is a generalization of the fact that $\nu(X)$ is realcompact.

Proposition R30.2.20 Let (X, τ) be a $T_{3\frac{1}{2}}$ topological space, let $\mathcal{U}$ be in $\mathcal{TB}(X, \tau)$, and let $\mathcal{U}$ correspond to the compactification class $[(Y, f)]$. Assume $\mathcal{W}$ is the subspace uniformity on $\nu_{\mathcal{U}}(X)$ from the unique uniformity for Y. Then $\nu_{\mathcal{U}}(X)$ is RK$\mathcal{W}$.

Proof: Since $f[X] \subseteq \nu_{\mathcal{U}}(X) \subseteq Y$ and $f[X]$ is dense in Y, $\nu_{\mathcal{U}}(X)$ is also dense in Y. As a result, the compactification class corresponding to $\mathcal{W}$ is $[(Y, i)]$, where i is the inclusion map. Let $t \in Y - \nu_{\mathcal{U}}(X)$. Then there is $g \in C_{\mathcal{U}}(X)$with extension $\overline{g}$ from Y to $\mathbb{R}^+$such that $\overline{g}(t) = \infty$. Let h be the restriction of $\overline{g}$ to $\nu_{\mathcal{U}}(X)$, which maps into $\mathbb{R}$ by definition of $\nu_{\mathcal{U}}(X)$. Since $\mathcal{W}$ is the subspace uniformity, h is uniformly continuous into $\mathbb{R}^+$. The extension of h is $\overline{g}$ and so $t \notin \nu_{\mathcal{W}}(\nu_{\mathcal{U}}(X))$. The conclusion follows.

Proposition R30.2.21 Let (X, τ) be a $T_{3\frac{1}{2}}$ topological space and let $\mathcal{U}, \mathcal{V}$ be in $\mathcal{TB}(X, \tau)$ with $\mathcal{U} \subseteq \mathcal{V}$. If $(X, \tau))$ is RK$\mathcal{U}$, then (X, τ) is RK$\mathcal{V}$.

Proof: Assume (X, τ) is RK$\mathcal{U}$. Assume $\mathcal{U}, \mathcal{V}$ correspond to the compactification classes

$[(Y,f)],[(Z,h)]$ respectively. By R30.2.16 there exists $\phi : Z \to Y$, a continuous surjection, such that $\phi \circ h = f$ and $\phi[\nu_{\mathcal{V}}(X)] \subseteq \nu_{\mathcal{U}}(X) = f[X]$. Let $t \in \nu_{\mathcal{V}}(X)$ so that $\phi(t) = f(x)$ for some $x \in X$. Let $\{x_\alpha\}$ be a net in X such that $\{h(x_\alpha)\}$ converges to t. By continuity $\{\phi(h(\alpha)) = f(x_\alpha)\}$ converges to $\phi(t) = f(x)$. Since $f : X \to f[X]$ is a homeomorphism, $\{x_\alpha\}$ converges to x and so $\{h(x_\alpha)\}$ converges to $h(x)$. Since Y is T_2, $h(x) = t$ and so $\nu_{\mathcal{V}}(X) \subseteq h[X]$. The conclusion follows from R30.2.14 and the definition of RK$\mathcal{V}$.

Corollary R30.2.22 Let (X,τ) be a locally compact, T_2, non-compact topological space. Let $\mathcal{U}_m$ be the separated, totally bounded uniformity corresponding to the compactification class of the one-point compactification of (X,τ). Assume (X,τ) is RK$\mathcal{U}_m$. Then, for every $\mathcal{V} \in \mathcal{TB}(X,\tau)$, (X,τ) is RK$\mathcal{V}$.

Proof: Since $\mathcal{U}_m$ is the smallest element of $\mathcal{TB}(X,\tau)$, the result is immediate for any such $\mathcal{V}$ from the previous proposition.

Corollary R30.2.23 Let (X,τ) be a locally compact, T_2, non-compact topological space. Let $\mathcal{U}_m$ be the separated, totally bounded uniformity corresponding to the compactification class of the one-point compactification of (X,τ). If (X,τ) is RK$\mathcal{U}_m$, then (X,τ) is realcompact.

Proof: $\mathcal{U}_M$ is in $\mathcal{T}\lfloor(X,\tau)$ and so by R30.2.22 (X,τ) is RK$\mathcal{U}_M$. By R30.2.19 (X,τ) is realcompact.

Proposition R30.2.24 Let (X,τ) be a $T_{3\frac{1}{2}}$ topological space and let $\mathcal{U},\mathcal{V} \in \mathcal{TB}(X,\tau)$ with $\mathcal{U} \subseteq \mathcal{V}$. Assume $\mathcal{U},\mathcal{V}$ correspond to the compactification classes $[(Y,f)],[(Z,h)]$ respectively. If $\nu_{\mathcal{V}}(X) = Z$, then $\nu_{\mathcal{U}}(X) = Y$.

Proof: By R30.2.16 there is $\phi : Z \to Y$, a continuous surjection, such that $\phi \circ h = f$ and $\phi[\nu^Z_{\mathcal{V}}(X)] \subseteq \nu^Y_{\mathcal{U}}(X)$. If $\nu_{\mathcal{V}}(X) = Z$, since ϕ is onto, $Y \subseteq \nu_{\mathcal{U}}(X) \subseteq Y$.

Before proceeding let's review some facts and terminology. First, as noted above, every subset of $\mathbb{R}^+$is totally bounded, which includes subsets of $\mathbb{R}$ unbounded relative to the usual metric. Secondly, a topological space (X,τ) is defined to be pseudo-compact provided every continuous real-valued map on X has a bounded image relative to the usual metric. Thirdly, given directed sets E, D, a map $T : E \to D$ is finalizing (or has the subnet property) provided for any $d \in D$ there is $e_0 \in E$ such that $e \geq e_0$ implies $T(e) \geq d$. Lastly, given a net $S : D \to X$, a subnet of S is a function $S \circ T$, where E is a directed set and $T : E \to D$ is finalizing.

Proposition R30.2.25 Let (X,τ) be a $T_{3\frac{1}{2}}$ topological space, let $\mathcal{U}$ be in $\mathcal{TB}(X,\tau)$, and let $\mathcal{U}$ correspond to the compactification class $[(Y,f)]$. If (X,τ) is pseudo-compact, then $\nu_{\mathcal{U}}(X) = Y$.

Proof: Assume (X,τ) is pseudo-compact. Let $t \in Y$ and suppose $g \in C_{\mathcal{U}}(X)$ with extension $\overline{g} : Y \to \mathbb{R}^+$. Pick a net $\{x_\alpha\}$ in X such that $\{f(x_\alpha)\}$ converges to t. By continuity $\{\overline{g}(f(x_\alpha))\}$ converges to $\overline{g}(t)$. By assumption there are $a, b \in \mathbb{R}$ such that $g[X] \subseteq [a,b]$. Because $[a,b]$ is compact in $\mathbb{R}$, it is closed in $\mathbb{R}^+$. The net $\{\overline{g}(f(x_\alpha)) = g(x_\alpha)\}$ is in the closed set $[a,b]$ and so its limit $\overline{g}(t)$ is also in $[a,b]$, i.e., in $\mathbb{R}$. Thus $t \in \nu_{\mathcal{U}}(X)$ and so $Y \subseteq \nu_{\mathcal{U}}(X) \subseteq Y$.

The previous result could also be derived from R30.2.24 and a result in [2]: A space X is pseudocompact if and only if its realcompactification is βX.

Corollary R30.2.26 Let (X,τ) be a $T_{3\frac{1}{2}}$, non-compact topological space. Let $\mathcal{U}$ be in $\mathcal{TB}(X,\tau)$. If (X,τ) is RK$\mathcal{U}$, then (X,τ) is not pseudo-compact.

Proof: Let $\mathcal{U}$ correspond to the compactification class $[(Y, f)]$ and assume (X, τ) is RK$\mathcal{U}$. By non-compactness and R30.2.15 $\nu_{\mathcal{U}}(X) \neq Y$, and so by the previous result (X, τ) is not pseudo-compact.

The next result is somewhat in the direction of a converse of R30.2.25.

Proposition R30.2.27 Let (X, τ) be a $T_{3\frac{1}{2}}$ topological space, let $\mathcal{U}$ be in $\mathcal{TB}(X, \tau)$, and let $\mathcal{U}$ correspond to the compactification class $[(Y, f)]$. If $\nu_{\mathcal{U}}(X) = Y$, then every $g \in C_{\mathcal{U}}(X)$ is bounded in $\mathbb{R}$.

Proof: Assume $\nu_{\mathcal{U}}(X) = Y$ and suppose $g \in C_{\mathcal{U}}(X)$ with extension $\overline{g} : Y \to \mathbb{R}^+$. Suppose $g[X]$ is unbounded. For each $n \in \mathbb{N}$, pick $x_n \in X$ such that $g(x_n) \notin [-n, n]$. By compactness of Y the sequence $\{f(x_n)\}$ has a subnet converging to some $t \in Y$, i.e., there is a directed set E and a finalizing map $T : E \to \mathbb{N}$ such that the subnet $\{f(x_{T(e)})\}$ converges to t. By the assumption $\overline{g}(t) \in \mathbb{R}$. Pick $m \in \mathbb{N}$ such that $\overline{g}(t) \in (-m, m)$, which is open in $\mathbb{R}^+$. By convergence there is $e_1 \in E$ such that $e \geq e_1$ implies $\overline{g}(f(x_{T(e)}) \in (-m, m)$. Since T is finalizing, there is $e_0 \in E$ such that $e \geq e_0$ implies $T(e) \geq m$. In the directed set E pick e with $e \geq e_0$ and $e \geq e_1$. Since $\overline{g} \circ f = g$, $e \geq e_1$ implies $g(x_{T(e)}) \in (-m, m)$. But $e \geq e_0$ implies $T(e) \geq m$ so that by construction $g(x_{T(e)}) \notin [-m, m]$, a contradiction. Thus g has a bounded image as claimed.

The next two results, both known, will be used to provide some additional examples. Recall that a topological space is countably compact if and only if every countable open cover has a finite subcover.

Lemma R30.2.28 Let (X, τ) be a countably compact topological space. Then (X, τ) is pseudo-compact.

Proof: Let $f : X \to \mathbb{R}$ be continuous. $X = \cup_{n=1}^{\infty} f^{-1}[(-n, n)]$ and so, by hypothesis, there is a poitive integer M such that $X = \cup_{n=1}^{M} f^{-1}[(-n, n)]$. Then $|f(x)| \leq M$ for all $x \in X$.

Lemma R30.2.29 Let (X, τ) be an infinite discrete topological space and let S be a non-empty finite subset of $\beta X - \iota[X]$. Then $\beta X - S$ with the subspace topology is countably compact but not compact.

Proof: The open set $\beta X - S$ is a proper subset of βX, which contains the dense $\iota[X]$. Thus the closure (in βX) of $\beta X - S$ is βX, i.e., $\beta X - S$ is not closed in the compact, T_2 space βX so that $\beta X - S$ is not compact. Now let $\beta X - S = \cup_{n=1}^{\infty} G_n$ where G_n is open in the relative topology on $\beta X - S$ for every n. Suppose this open cover has no finite subcover. Form a countably infinite set A by picking $a_k \in \beta X - S$ with $a_k \notin \cup_{n=1}^{k} G_n$ and letting $A = \{a_k : k \in \mathbb{N}\}$. Let t be an accumulation point of A in βX. Claim: $t \in S$. If not, $t \in G_j$ for some j. By the choice of a_k, $a_k \notin G_j$ for all $k \geq j$ so that $A \cap G_j$ is finite, which contradicts the assumption that t is an accumulation point of A. Thus the claim holds and so $\overline{A}$ (closure in βX) is a subset of the countable set $A \cup S$. Note that $A \neq \overline{A}$ because A is not compact: $\{G_n : n \in \mathbb{N}\}$ induces an open cover of A with no finite subcover. By R30.1.3 the compact, countable $\overline{A}$ is metrizable. Let $s \in \overline{A} - A$. Since $\overline{A}$ is metrizable, there is a sequence $\{x_n\}$ in A converging to s in $\overline{A}$ (and so in βX). By R30.1.1 and R30.1.2 $\{x_n\}$ is eventually constant so that $s \in A$, a contradiction. Thus $\beta X - S$ is countably compact.

Corollary R30.2.30 Let (X, τ) be an infinite discrete topological space and let S be a non-empty finite subset of $\beta X - \iota[X]$. Let $\mathcal{U}$ be a separated, totally bounded uniformity for $\beta X - S$ with the subspace toplogy, and let (Y, f) be in the compactification class corresponding to $\mathcal{U}$. Then $\nu_{\mathcal{U}}(\beta X - S) = Y$ and $\beta X - S$ is not RK$\mathcal{U}$.

Proof: By the previous lemma $\beta X - S$ is pseudo-compact but not compact. By R30.2.25 $\nu_{\mathcal{U}}(\beta X - S) = Y$ and so $\beta X - S$ is not RK$\mathcal{U}$.

The next result collects some facts about the compactifications of $\beta X - S$.

Proposition R30.2.31 Let (X, τ) be an infinite discrete topological space and let S be a non-empty finite subset of $\beta X - \iota[X]$. Let $\beta X - S$ have the subspace topology. Then

i) The Stone-Čech compactification of $\beta X - S$ is βX with the inclusion map.

ii) The Stone-Čech compactification of $\beta X - S$ is an n-point compactification, where $n = |S|$.

iii) For every positive integer $m \leq |S|$, $\beta X - S$ has an m-point compactification.

iv) Every compactification of $\beta X - S$ is a finite-point compactification.

v) $\beta X - S$ has finitely many distinct compactification classes.

vi) $\beta X - S$ has finitely many distinct separated, totally bounded uniformities.

Proof: R5.Add.4 implies i), which in turn implies ii). Part iii) follows from R5.1.3. For iv), let (Y, f) be a compactification of $\beta X - S$. Because $(Y, f) \leq (\beta X, in)$, where in is inclusion, there is a continuous surjection $\phi : \beta X \to Y$ such that $\phi \circ in = f$. This says $\phi[\beta X - S] = f[\beta X - S]$ so that the finitely many elements of S map onto $Y - f[\beta X - S]$. Thus (Y, f) is a finite-point compactification. R5.Add.10 implies v), and vi) follows from the one-to-one correspondence between compactification classes and separated, totally bounded uniformities for $\beta X - S$.

Lemma R30.2.32 Let (X, τ) be a $T_{3\frac{1}{2}}$ topological space and let $\mathcal{U}, \mathcal{V}$ be in $\mathcal{TB}(X, \tau)$. Assume $\mathcal{U}$ corresponds to the compactification class $[(Y, f)]$ and $\mathcal{V}$ to the class $[(Z, h)]$. Let $u : \nu_{\mathcal{U}}(X) \to \nu_{\mathcal{V}}(X)$ be a unimorphism, where the spaces have the subspace uniformities from the unique uniformities for X, Y. Assume $u(f(x)) = u(h(x))$ for all $x \in X$. Then (Y, f) and (Z, h) are equivalent and $\mathcal{U} = \mathcal{V}$.

Proof: By assumption, u induces a unimorphism from $(X, \mathcal{U})$ to $(X, \mathcal{V})$ and the induced map is the identity. Thus $\mathcal{U} = \mathcal{V}$ and by R1.5 the corresponding compactifications are equivalent.

R30.2.30 provides examples of spaces which are not RK$\mathcal{U}$. By R30.2.31 $\beta X - S$ has an m-point compactification for every positive integer $m \leq |S|$. For $m \neq n$ an m-point compactification cannot be equivalent to an n-point compactification by R5.1.4, and so by R30.2.32 the underlying ν-spaces are not unimorphic by a map that leaves the embedded subspaces pointwise fixed.

In general, however, non-equivalent compactifications may have underlying spaces homeomorphic in the general or unrestricted sense. For example, by R10.3.6 $\mathbb{N}_p$ and $\mathbb{N}_q$, where p, q are distinct primes, are not equivalent but the underlying spaces are homeomorphic by R26.Add.19. Thus the question of whether some non-equivalent ν-spaces of $\beta X - S$ are still homeomorphic is still unanswered.

Before presenting a final example, a few results on the coproduct need to be recalled. To simplify the notation, disjointness of the underlying sets will be assumed so that the disjoint union of two sets is simply their union. The disjoint union topology and uniformity are defined as in [7] and [8], where it is shown that the topology of the disjoint union of uniformities is the disjoint union of the uniform topologies. The preservation of completeness and, for two uniformities, total boundedness are also verified there. These facts could be extended to finite collections of spaces.

Naturally, as a functor, the disjoint union acts on functions as well. For $u : X \to Z$ and $v : Y \to Z$, the disjoint union of u and v is $u \coprod v : X \coprod Y \to Z$ defined by $u \coprod v(t) = u(t)$ if $t \in X$ and $u \coprod v(t) = v(t)$ if $t \in Y$. The disjoint union of two continuous (uniformly continuous) maps is also continuous (uniformly continuous). A map on a disjoint union can regarded as the disjoint union of its restrictions to the appropriate subspaces.

Lemma R30.2.33 Let (X, τ) and (Y, σ) be $T_{3\frac{1}{2}}$ topological spaces with $X \cap Y = \emptyset$. Let $\mathcal{U}$ be in $\mathcal{TB}(X, \tau)$ and $\mathcal{V}$ be in $\mathcal{TB}(Y, \sigma)$. Assume $\mathcal{U}, \mathcal{V}$ correspond to the compactification classes $[(Z, g)]$ and $[(W, h)]$ respectively, where Z and W are chosen to be disjoint. Then $\nu_{\mathcal{U} \coprod \mathcal{V}}(X \coprod Y) = \nu_{\mathcal{U}}(X) \coprod \nu_{\mathcal{V}}(Y)$.

Proof: By R24.15 $\mathcal{U} \coprod \mathcal{V}$ corresponds to the class of $(Z \coprod W, f)$ where $f = g \coprod h$. Let $t \in \nu_{\mathcal{U}}(X)$ and $u \in C_{\mathcal{U} \coprod \mathcal{V}}(X \coprod Y)$. Let u_1 be the extension of $u|_X$ to Z and u_2 the extension of $u|_Y$ to W. By uniqueness of extensions, $u_1 \coprod u_2$ is the extension of u to $Z \coprod W$. Since $t \in \nu_{\mathcal{U}}(X) \subseteq Z$, $(u_1 \coprod u_2)(t) = u_1(t)$, which is in $\mathbb{R}$. Thus $t \in \nu_{\mathcal{U} \coprod \mathcal{V}}(X \coprod Y)$. Similarly, $\nu_{\mathcal{V}}(Y) \subseteq \nu_{\mathcal{U} \coprod \mathcal{V}}(X \coprod Y)$. Now let $s \in \nu_{\mathcal{U} \coprod \mathcal{V}}(X \coprod Y)$. If $s \in Z$, let $v \in C_{\mathcal{U}}(X)$ with v_1 its extension to Z. Then $v \coprod 1$ is in $C_{\mathcal{U} \coprod \mathcal{V}}(X \coprod Y)$, where 1 here denotes the constant function with value 1. By uniqueness again, the extension of $v \coprod 1$ to $Z \coprod W$ is $v_1 \coprod 1$. Since $s \in Z$, $v_1(s) = (v_1 \coprod 1)(s)$, which is in $\mathbb{R}$. thus $s \in \nu_{\mathcal{U}}(X)$. Similarly, if $s \in W$, $s \in \nu_{\mathcal{V}}(Y)$.

Lemma R30.2.34 Let $\mathbb{N}$ have the discrete topology and let $\mathcal{U}_m$ be the separated, totally bounded uniformity corresponding to the one-point compactification of $\mathbb{N}$. Then $\nu_{\mathcal{U}_m}(\mathbb{N}) = \mathbb{N}$, i.e., $\mathbb{N}$ is RK$\mathcal{U}_m$.

Proof: Because $\mathbb{N}^+ = \mathbb{N} \cup \{\infty\}$, either $\nu_{\mathcal{U}_m}(\mathbb{N}) = \mathbb{N}$ or $\nu_{\mathcal{U}_m}(\mathbb{N}) = \mathbb{N}^+$. Let g be defined on $\mathbb{N}$ by $g(n) = n$ and let $E = \mathbb{N} \times \mathbb{N}$. Note that $\mathcal{U}_m \vee \mathcal{U}_E = \mathcal{U}_m$ and so by R30.2.7 $g \in C_{\mathcal{U}_m}(\mathbb{N})$. By R30.2.27 $\nu_{\mathcal{U}_m}(\mathbb{N}) \neq \mathbb{N}^+$.

Corollary R30.2.35 Let $k \in \mathbb{N}$ with $k \geq 2$ and let $\mathcal{U}_k$ be the separated, totally bounded uniformity corresponding to $\mathbb{N}_k$. Then $\mathbb{N}$ with the discrete topology is RK$\mathcal{U}_k$.

Proof: Let $\mathcal{U}_m$ be the separated, totally bounded uniformity corresponding to the one-point compactification of $\mathbb{N}$ with the discrete topology. As noted in [6] $\mathbb{N}_k$ is also a compactification of $\mathbb{N}$ with the discrete topology and so $\mathcal{U}_m \subseteq \mathcal{U}_k$. The conclusion follows from R30.2.34 and R30.2.22.

Comment: $\mathbf{R}_k$ is topologically a remainder space of $\mathbb{N}_k$ but it is a compactification of $\mathbb{N}$ with a non-discrete, i.e., strictly smaller, topology. As a result, neither the conclusion of the last result nor the technique used to derive it provide any answers about the similar question for $\mathbf{R}_k$.

Example R30.2.36 Let (X, τ) be a pseudo-compact, non-compact topological space and let (Y, σ) be $\mathbb{N}$ with the discrete topology. Assume $X \cap Y = \emptyset$. Let $\mathcal{U}_m$ correspond to the class of (Y^+, ι^+) and let $\mathcal{U} \in \mathcal{TB}(X, \tau)$ correspond to the compactification class $[(Z, g)]$, where the representative Z is chosen with $Z \cap Y^+ = \emptyset$. The compactification $(Z \coprod Y^+, g \coprod \iota^+)$ represents the class corresponding to $\mathcal{U} \coprod \mathcal{U}_m$. By R30.2.25 $\nu_{\mathcal{U}}(X) = Z$ and by R30.2.34 $\nu_{\mathcal{U}_m}(Y) = Y$. By R30.2.33, $\nu_{\mathcal{U} \coprod \mathcal{U}_m}(X \coprod Y)$ is $Z \cup Y$, which is a proper subset of $Z \coprod Y^+$. Since X is non-compact, it is also a proper superset of $g[X] \coprod Y$.

Comment: The previous example is the only one here where the ν-space is properly between the embedded space and the compactification.

Albert J. Klein 2020
http://www.susanjkleinart.com/compactification/

References

1. Gillman, L., and Jerison, M.,Rings of Continuous Functions, Springer Verlag reprint, 1960
2. Wilansky, A., Topology for Analysis, Ginn and Co., 1970
3. This Website, R1: Existence of Suprema via Uniform Space Theory
4. This Website, R5: Finite-Point Compactifications
5. This Website, R7: Uniform Continuity and Extension of Maps
6. This Website, R10: Some Metric Compactifications of $\mathbb{N}$
7. This Website, R15: S-maps
8. This Website, R24: Disjoint Unions of Uniform Spaces
9. This Website, R26: The Remnant Rings Are Homeomorphic