Possible Applications to Number Theory

It is tempting to imagine that the interaction of algebraic, order-theoretic, and topological ideas and structures associated with $\mathbb{N}_k$ and $\mathbf{R}_k$ might yield some interesting results in number theory.

Finitary algebraic results and their extension to the p-adic numbers can be assumed to be well investigated in the century-plus since Hansen's introduction of the p-adic numbers. That $\mathbf{R}_k$ for k composite can be considered a generalization of the p-adic numbers might seem promising, but by R17.3.15 and R17.2.17 such $\mathbf{R}_k$ are morphic to a finite direct product determined by k's prime factors.

Questions involving infinite sets of numbers, perhaps of the type, "Is the set of numbers with some property P infinite?", might allow some interplay with the additional structure. As an example, the topic of prime number gaps is briefly considered.

For $n \in \mathbb{N}$, let $I(n) = \{p : p \text{ is prime and } p + n \text{ is also prime}\}$.

Definition R31.1 A positive integer g is a gap number provided $I(g)$ is infinite.

Clearly any gap number must be even. The long unsolved twin prime problem can be expressed as the question of whether 2 is a gap number.

Definition R31.2 A number $b \in \mathbb{N}$ will be called a gap bound provided the set $\{p : p \text{ is prime and there is } q \text{ prime with } p < q \leq p + b\}$ is infinite.

Since $\{p : p \text{ is prime and there is } q \text{ prime with } p < q \leq p + b\}$ equals $\cup_{n=1}^{b} I(n)$, cardinality considerations show that the existence of a gap bound implies the existence of a gap number. In a survey article [1; pp. 9-15] it is stated that Yitang Zhang has shown the existence of a gap bound and that his technique has been refined to show that 246 is a gap bound.

Lemma R31.3 Let g be a gap number. Let $\{p_i\}$ be a one-to-one sequence in $I(g)$ such that $\{f_\infty(p_i)\}$ converges to $\mathcal{F}$ in $\mathbf{R}_\infty$. If $\mathcal{F} \in f_\infty[\mathbf{Z}\ ]$, then $g = 2$ and $\mathcal{F} = f_\infty(-1)$.

Proof: Assume $\mathcal{F} = f_\infty(w)$ for some $w \in \mathbf{Z}$. By R17.Add.7 $|w| = 1$ and so w is either 1 or -1. For the one-to-one sequence $\{p_i + g\}$, since f_∞ is a continuous ring homomorphism by R16.21 and addition is continuous by R12.2.5, $\{f_\infty(p_i + g)\}$ converges to $\mathcal{F} + f_\infty(g)$, which equals $f_\infty(w + g)$. By R17.Add.7 again, $|w + g| = 1$. If $w = 1$, since g is positive, $|w + g| = w + g > 1$. Thus $w = -1$ must hold and, since $g \in \mathbb{N}$, $1 = |w + g| = g - 1$ so that $g = 2$.

The next fact is recorded for convenience of reference.

Lemma R31.4 Let (X, τ) be a first countable, T_1 topological space. Let $A \subset X$, and suppose x is a limit point of A. Then there is a one-to-one sequence in A converging to x.

Proof: Let $\{O_n\}_{n=1}^{\infty}$ be a local base at x with $O_{n+1} \subseteq O_n$ for all n. Define a sequence inductively as follows: Pick a_1 in $O_1 \cap (A - \{x\})$. Assume $a_1 \neq a_2 \neq \cdots \neq a_j$ have been chosen with each a_n in $O_n \cap (A - \{x\})$. Pick a_{j+1} in $(O_{j+1} - \{a_1, \ldots, a_j\}) \cap (A - \{x\})$. By construction $\{a_n\}$ is a one-to-one sequence in A with $a_n \in O_n$ for all n. To see that this sequence converges to x, let $x \in O \in \tau$ and pick M with $O_M \subseteq O$. For $n \geq M$, $O_n \subseteq O_M$ and so $a_n \in O$. Thus the claimed convergence holds.

Corollary R31.5 Let g be a gap number with $g \geq 3$. Let $\mathcal{F}$ in $\mathbf{R}_\infty$ be a limit point of $f_\infty[I(g)]$. Then $\mathcal{F}$ is not in $f_\infty[\mathbf{Z}\ ]$ and so not in $f_\infty[I(g)]$.

Proof: By R12.6.14 $\mathbf{R}_\infty$ is metrizable and so this is immediate from R31.4 and R31.3.

Corollary R31.6 Let g be a gap number with $g \geq 3$. Then $I(g)$ is a discrete subset of $(\mathbb{N}, \tau_\infty)$.

Proof: Let $x \in I(g)$. By R31.1.5 $f_\infty(x)$ is not a limit point of $f_\infty[I(g)]$. By definition there is O open in $\mathbf{R}_\infty$ with $f_\infty(x) \in O$ such that $O \cap (f_\infty[I(g)] - \{f_\infty(x)\}) = \emptyset$. Since τ_∞ is the topology making f_∞ an embedding, $f_\infty^{-1}[O]$ is in τ_∞. It follows that $f_\infty^{-1}[O] \cap I(g) = \{x\}$ as needed for the conclusion.

Comments: The question at this point is, what next? Infinite discrete subsets of $(\mathbb{N}, \tau_\infty)$ can be generated which do not determine a gap number, and the sets $I(n)$ do not have algebraic structure. For a gap number g with $I(g)$ discrete in $(\mathbb{N}, \tau_\infty)$, one could use an indexing of $I(g)$ and the closure of $f_\infty[I(g)]$ in $\mathbf{R}_\infty$ to generate a compactification of $\mathbb{N}$ with the discrete topology, but this object seems to have lost any significant connection to the underlying number theoretic issue.

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http://www.susanjkleinart.com/compactification/

References

1. Klarreich, Erica, Together and Alone, Closing the Prime Gap in: The Prime Number Conspiracy: The Biggest Ideas in Math from Quanta, Thomas Lin, editor, The MIT Press, 2018

2. This website, R12: Extension of Arithmetic Operations

3. This website, R16: The Remnant Rings as Compactifications

4. This website, R17: Algebraic Structure of the Remnant Rings