Normal Bases for Finite-Point Compactifications

In [4] it is shown that every finite-point compactification of an infinite discrete space is equivalent to a compactification generated from a normal basis. Here that result and the technique used to prove it are generalized to finite-point compactifications of an arbitrary non-compact $T_{3\frac{1}{2}}$ space. It is also shown that a supremum of finite-point compactifications must be a Wallman compactification.

Note that all compactifications considered are T_2 compactifications and that a non-compact space has a finite-point compactification if and only if it is locally compact and T_2. For $A \subseteq X$, $\overline{A}$ denotes the closure of A in X.

Let (X, τ) be a non-compact T_2 topological space. A pairwise disjoint family $\{G_i : i = 1, \ldots, n\}$ of open sets whose union has a compact complement K such that $K \cup G_i$ is not compact for each i will be called an n-star of (X, τ). In what follows, when an n-star is given as a pairwise disjoint family $\{G_i : i = 1, \ldots, n\}$, the compact set $X - \cup_{i=1}^n G_i$ will be implicit unless needed. In R5.1.1 it is shown that a T_2 space with an n-star is locally compact and n-stars determine n-point compactifications.

When (X, τ) is discrete, infinitely many examples of n-stars are provided by n-compatible equivalence relations. See R5.1.9 and R5.1.10. For an arbitrary non-compact locally compact T_2 space (X, τ), $\{X\}$ is a 1-star, which determines the one-point compactification. In $\mathbb{R}$ both $\{(-\infty, 0), (0, \infty)\}$ and $\{-\infty, -1)), (1, \infty)\}$ are a 2-stars, which determine equivalent two-point compactifications.

The Normal Basis Generated by an n-star

Definition R33.1.1 Let $\{G_i : i = 1, \ldots, n\}$ be an n-star for the non-compact T_2 topological space (X, τ) and let $S \subseteq X$. Let $\Delta \subseteq \{1, 2 \ldots, n\}$. S is associated with Δ if and only if $i \in \Delta$ implies $S \cap G_i$ has compact closure in X and $i \notin \Delta$ implies $(X - S) \cap G_i$ has compact closure in X.

When (X, τ) is discrete, this definition reduces to R5.3.1.

Lemma R33.1.2 Let $\{G_i : i = 1, \ldots, n\}$ be an n-star for the non-compact T_2 topological space (X, τ) and let $S \subseteq X$. Assume S is associated with Δ_1 and with Δ_2, both subsets of $\{1, 2, \ldots, n\}$. Then $\Delta_1 = \Delta_2$

Proof: Let $K = X - \cup_{i=1}^n G_i$. First note that $K \cup G_i$ is closed for each $1 \le i \le n$ because its complement is the open set $\cup\{G_j : j \ne i\}$. Now let $i \in \Delta_1$ so that $S \cap G_i$ has compact closure in X. Suppose $i \notin \Delta_2$ so that $(X - S) \cap G_i$ also has compact closure in X. Since $G_i = (S \cap G_i) \cup ((X - S) \cap G_i)$, $\overline{G_i}$ is the union of two compact sets and so compact. Since $K \cup G_i$ is closed, $\overline{G_i} \subseteq K \cup G_i$. Then the closed set $K \cup G_i$ is contained in $K \cup \overline{G_i}$, a union of two compact sets, so that $K \cup G_i$ is compact. But by definition of an n-star $K \cup G_i$ is not compact, a contradiction. Thus $i \in \Delta_2$. Similarly $\Delta_2 \subseteq \Delta_1$.

Lemma R33.1.3 Let (X, τ) be a topological space, let $S \subseteq X$, and let $G \in \tau$. Then $\overline{S \cap G} = \overline{\overline{S} \cap G}$.

Proof: Since $\overline{\overline{S} \cap G}$ is a closed set containing $S \cap G$, $\overline{S \cap G} \subseteq \overline{\overline{S} \cap G}$. Now let $x \in \overline{\overline{S} \cap G}$ and let $x \in O \in \tau$. Pick $t \in (\overline{S} \cap G) \cap O$. Then t is in $\overline{S}$ and in the open set $G \cap O$ so that there is s in $S \cap (G \cap O) = (S \cap G) \cap O$. Thus $x \in \overline{S \cap G}$.

Lemma R33.1.4 Let $\{G_i : i = 1, \ldots, n\}$ be an n-star for the non-compact T_2 topological space (X, τ) and let $S \subseteq X$. Then

i) S is associated with $\{1, 2, \ldots, n\}$ if and only if $\overline{S}$ is compact.

ii) If S is associated with Δ, then $X - S$ is associated with $\{1, 2, \ldots, n\} - \Delta$.

iii) If S is associated with Δ, $\overline{S}$ is also associated with Δ.

Proof: Let $K = X - \cup_{i=1}^{n} G_i$ and write X as $K \cup (\cup_{i=1}^{n} G_i)$ so that one can write $S = (S \cap K) \cup (\cup_{i=1}^{n} (S \cap G_i))$ and $\overline{S} = \overline{S \cap K} \cup (\cup_{i=1}^{n} \overline{S \cap G_i})$. Since K is compact, so is $\overline{S \cap K} \subseteq K$. Thus, if S is associated with $\{1, 2, \ldots, n\}$, the definition and second equation show that $\overline{S}$ is compact. Conversely, if $\overline{S}$ is compact, for $1 \le i \le n$, $S \cap G_i \subseteq \overline{S}$ and so has compact closure. Thus i) holds. Part ii) follows from the definition because $X - (X - S) = S$. For part iii) assume S is associated with Δ. If $i \in \Delta$, because $\overline{\overline{S} \cap G_i} = \overline{S \cap G_i}$ is compact, $\overline{S} \cap G_i$ has compact closure in X. If $i \notin \Delta$, because $X - \overline{S} \subseteq X - S$, $(X - \overline{S}) \cap G_i$ has compact closure in X. The conclusion follows from the definition.

Lemma R33.1.5 Let $\{G_i : i = 1, \ldots, n\}$ be an n-star for the non-compact T_2 topological space (X, τ) and let S, T be subsets of X. Assume S is associated with Δ and T is associated with Γ. Then $S \cup T$ is associated with $\Delta \cap \Gamma$ and $S \cap T$ is associated with $\Delta \cup \Gamma$.

Proof: For $i \in \Delta \cap \Gamma$, because $S \cap G_i$ and $T \cap G_i$ both have compact closure in X, so does $(S \cup T) \cap G_i$. For $i \notin \Delta \cap \Gamma$, at least one of $(X - S) \cap G_i, (X - T) \cap G_i$ has compact closure in X, which implies that $(X - (S \cup T)) \cap G_i = ((X - S) \cap (X - T)) \cap G_i$ does as well. Thus the first assertion holds. For $i \in \Delta \cup \Gamma$, at least one of $S \cap G_i, T \cap G_i$ has compact closure in X so that $(S \cap T) \cap G_i$ does also. For $i \notin \Delta \cup \Gamma$, both $(X - S) \cap G_i$ and $(X - T) \cap G_i$ have compact closure in X so that $(X - (S \cap T)) \cap G_i = ((X - S) \cup (X - T)) \cap G_i$ does as well. Thus the second claim holds.

Definition R33.1.6 Let $\mathcal{S} = \{G_i : i = 1, \ldots, n\}$ be an n-star for the non-compact T_2 topological space (X, τ). $\mathcal{Z}(\mathcal{S})$ is defined to be

$$\{Z \subseteq X : Z \text{ is } \tau - \text{closed and } Z \text{ is associated with some } \Delta \subseteq \{1, \ldots, n\}\}.$$

The next lemma shows that $\mathcal{Z}(\mathcal{S})$ has the first three properties in P3.1, the definition of a normal basis. By R33.1.4i $\mathcal{Z}(\mathcal{S})$ contains all finite subsets of X and so is a non-empty family of closed sets.

Lemma R33.1.7 Let $\mathcal{S} = \{G_i : i = 1, \ldots, n\}$ be an n-star for the non-compact T_2 topological space (X, τ). Then

i) $\mathcal{Z}(\mathcal{S})$ is a base for the closed sets of (X, τ).

ii) $\mathcal{Z}(\mathcal{S})$ is closed under finite unions and intersections.

iii) If E is closed in (X, τ) and $x \notin E$, then there is $Z \in \mathcal{Z}(\mathcal{S})$ such that $x \in Z$ and $Z \cap E = \emptyset$.

Proof: Because the closed sets are closed under finite unions and intersections, part ii) is immediate from R33.1.5. Let E be closed and $x \notin E$. By R33.1.4i $\{x\} \in \mathcal{Z}(\mathcal{S})$ and so iii) holds. As noted above (X, τ) is locally compact and so there is O open with $x \in O \subseteq \overline{O} \subseteq X - E$, where $\overline{O}$ is compact. By the first two parts of R33.1.4 $X - O$ is associated with $\emptyset$ and so is in $\mathcal{Z}(\mathcal{S})$. Since $x \notin X - O$ and $E \subseteq X - O$, part i) holds.

Verifying the fourth requirement is done in the next 2 lemmas.

Lemma R33.1.8 Let $\mathcal{S} = \{G_i : i = 1, \ldots, n\}$ be an n-star for the non-compact T_2 topological space (X, τ). Let Z_1, Z_2 be in $\mathcal{Z}(\mathcal{S})$ with Z_i associated with Δ_i. Assume $Z_1 \cap \mathcal{Z}_2 = \emptyset$. Then $\Delta_1 \cup \Delta_2 = \{1, \ldots, n\}$.

Proof: By R33.1.4i $\emptyset$ is associated with $\{1,\ldots,n\}$ and by R33.1.5 $Z_1 \cap Z_2$ is associated with $\Delta_1 \cup \Delta_2$. The conclusion follows from R33.1.2.

Lemma R33.1.9 Let $\mathcal{S} = \{G_i : i = 1,\ldots,n\}$ be an n-star for the non-compact T_2 topological space (X,τ). Let Z_1, Z_2 be in $\mathcal{Z}(\mathcal{S})$ with Z_i associated with Δ_i. Assume $Z_1 \cap \mathcal{Z}_2 = \emptyset$. Then there are $C, D \in \mathcal{Z}(\mathcal{S})$ such that $C \cup D = X$, $Z_1 \subseteq X - C$, and $Z_2 \subseteq X - D$.

Proof: Let K be the complement in X of $\cup_{i=1}^n G_i$ so that $X = K \cup G_1 \cup \cdots \cup G_n$. $K \cap Z_1$ is a compact subset of the open set $X - Z_2$. As noted above, (X,τ) is locally compact and so there is $O_0 \in \tau$ with $\overline{O_0}$ compact such that $K \cap Z_1 \subseteq O_0 \subseteq \overline{O_0} \subseteq X - Z_2$. For each $i \in \Delta_1$, $\overline{Z_1 \cap G_i}$ is a compact subset of Z_1, which is contained in the open set $X - Z_2$. Again by local compactness, there is $O_i \in \tau$ with $\overline{O_i}$ compact such that $\overline{Z_1 \cap G_i} \subseteq O_i \subseteq \overline{O_i} \subseteq X - Z_2$. For $i \notin \Delta_1$, by the previous lemma $i \in \Delta_2$ and so $\overline{Z_2 \cap G_i}$ is compact. By local compactness, since $\overline{Z_2 \cap G_i} \subseteq Z_2 \subseteq X - Z_1$, there is O_i^* open with $\overline{O_i^*}$ compact such that $\overline{Z_2 \cap G_i} \subseteq O_i^* \subseteq \overline{O_i^*} \subseteq X - Z_1$. Similarly, for the compact $Z_2 \cap K$, there is O_0^* open with $\overline{O_0^*}$ compact such that $Z_2 \cap K \subseteq O_0^* \subseteq \overline{O_0^*} \subseteq X - Z_1$. Now for $i \notin \Delta_1$, let $O_i = (X - \overline{O_i^*}) \cap G_i \cap (X - \overline{O_0^*})$.

Next define $C = X - \cup_{i=0}^n O_i$ and $D = \cup_{i=0}^n \overline{O_i}$. Clearly C and D are closed sets with $C \cup D = X$. Note that, for $i \notin \Delta_1$, $Z_1 \subseteq (X - \overline{O_0^*}) \cap (X - \overline{O_i^*})$ so that $Z_1 \cap G_i \subseteq O_i$. By construction $Z_1 \cap K \subseteq O_0$ and, for $i \in \Delta_1$, $Z_1 \cap G_i \subseteq O_i$. $Z_1 = (Z_1 \cap K) \cup (\cup_{i=1}^n (Z_1 \cap G_i))$ and so $Z_1 \subseteq \cup_{i=0}^n O_i = X - C$. To see that $Z_2 \subseteq X - D$, first note that by construction $Z_2 \subseteq X - \overline{O_0}$ and, for $i \in \Delta_1$, $Z_2 \subseteq X - \overline{O_i}$. For $i \notin \Delta_1$, it is claimed that $Z_2 \subseteq X - \overline{O_i}$ as well. Deny that and let $x \in Z_2$ with $x \in \overline{O_i}$. Since $O_i \subseteq X - \overline{O_0^*} \subseteq X - O_0^*$ which is closed, $\overline{O_i} \subseteq X - O_0^* \subseteq X - (Z_2 \cap K)$. Since $x \notin Z_2 \cap K$ and $x \in Z_2$, $x \notin K$. Also $O_i \subseteq G_i \subseteq K \cup G_i$ which is closed. Thus $\overline{O_i} \subseteq K \cup G_i$. Because $x \notin K$, $x \in G_i$. Since $x \in Z_2$, $x \in Z_2 \cap G_i \subseteq O_i^*$. But $O_i \subseteq X - \overline{O_i^*} \subseteq X - O_i^*$ which is closed, so that $\overline{O_i} \subseteq X - O_i^*$ and $x \notin O_i^*$, a contradiction. In summary, $Z_2 \subseteq \cap_{i=0}^n (X - \overline{O_i}) = X - D$.

To finish, it is necessary to show that both C and D are in $\mathcal{Z}(\mathcal{S})$. First note that, for $i \in \Delta_1 \cup \{0\}$, $\overline{O_i}$ is compact and so by R33.1.4i O_i and $\overline{O_i}$ are associated with $\{1, 2, \ldots, n\}$. By R33.1.4ii $X - O_i$ is associated with $\emptyset$. Thus $X - O_i$ and $\overline{O_i}$ are both in $\mathcal{Z}(\mathcal{S})$. Now suppose $i \notin \Delta_1$. $O_i \subseteq G_i$ and so, for $j \neq i$, $O_i \cap G_j = \emptyset$, which is compact. $(X - O_i) \cap G_i = (\overline{O_i^*} \cup (X - G_i) \cup \overline{O_0^*}) \cap G_i \subseteq \overline{O_i^*} \cup \overline{O_0^*}$, which is the union of two compact sets. Thus O_i is associated with $\{1, 2, \ldots, n\} - \{i\}$. By R33.1.4iii $\overline{O_i}$ is also associated with $\{1, 2, \ldots, n\} - \{i\}$ so that $\overline{O_i}$ is in $\mathcal{Z}(\mathcal{S})$. By R33.1.4ii $X - O_i$ is associated with $\{i\}$ and so is in $\mathcal{Z}(\mathcal{S})$. Since $\mathcal{Z}(\mathcal{S})$ is closed under finite unions and intersections, $D = \cup_{i=0}^n \overline{O_i}$ and $C = \cap_{i=0}^n (X - O_i)$ are both in $\mathcal{Z}(\mathcal{S})$, as required.

Corollary R33.1.10 Let $\mathcal{S} = \{G_i : i = 1,\ldots,n\}$ be an n-star for the non-compact T_2 topological space (X,τ). Then $\mathcal{Z}(\mathcal{S})$ is a normal basis for (X,τ).

Proof: R33.1.7 and R33.1.9 show that $\mathcal{Z}(\mathcal{S})$ has the properties in P3.1, the definition of a normal basis.

When (X,τ) is discrete, R33.1.10 reduces to R5.3.3.

The Compactification Generated by $\mathcal{Z}(\mathcal{S})$

Given $\{G_i : i = 1,\ldots,n\}$ an n-star for the non-compact T_2 topological space (X,τ), let $p_1,\ldots,p_n$ be n distinct objects not in X, let $Y = X \cup \{p_1,\ldots,p_n\}$, let σ be the set $\{O \subseteq Y : O \cap X \in \tau$ and $p_i \in O \Rightarrow (X - O) \cap G_i$ has compact closure in $X\}$, and let

$f : X \to Y$ by $f(x) = x$. In [4] it is noted that σ is a topology and (Y, f) is an n-point T_2 compactification of (X, τ). (Y, f) will be called the n-point compactification determined by the n-star. The argument that (Y, σ) is compact and T_2 uses only the T_2 property of X, although that is not emphasized in [4].

Definition R33.2.1 Let $\mathcal{S} = \{G_i : i = 1, \dots, n\}$ be an n-star for the non-compact T_2 topological space (X, τ) and let $1 \le i \le n$. The set $\mathcal{F}_i$ is defined by

$$\mathcal{F}_i = \{Z \in \mathcal{Z}(\mathcal{S}) : Z \text{ is associated with some } \Delta \subseteq \{1, \dots, n\} - \{i\}\}.$$

To avoid subscript ambiguity, in what follows the point-filter of x in a space will be denoted $\mathcal{F}(x)$. The next lemma is a generality used in what follows.

Lemma R33.2.2 Let (X, τ) be a $T_{3\frac{1}{2}}$ space with normal basis $\mathcal{Z}$ and let $\mathcal{H}$ be a $\mathcal{Z}$-filter. Assume C is a compact subset of X and $C \in \mathcal{H}$. Then there is $x \in X$ such that $\mathcal{H} \subseteq \mathcal{F}(x)$.

Proof: The set $\{C \cap Z : Z \in \mathcal{H}\}$ is contained in $\mathcal{H}$ and is a family of closed subsets of C. Because $\mathcal{H}$ is closed under finite intersections and $\emptyset \notin \mathcal{H}$, $\{C \cap Z : Z \in \mathcal{H}\}$ has the finite intersection property. Since C is compact, there is x in $\cap\{C \cap Z : Z \in \mathcal{H}\}$. For any $Z \in \mathcal{H}$, $x \in Z \cap C$ and so $Z \in \mathcal{F}(x)$, i.e., $\mathcal{H} \subseteq \mathcal{F}(x)$.

Lemma R33.2.3 Let $\mathcal{S} = \{G_i : i = 1, \dots, n\}$ be an n-star for the non-compact T_2 topological space (X, τ) and let $1 \le i \le n$. Then

i) $\mathcal{F}_i$ is a $\mathcal{Z}(\mathcal{S})$-filter.

ii) $\mathcal{F}_i$ is a $\mathcal{Z}(\mathcal{S})$-ultrafilter.

iii) $\mathcal{F}_i$ is not a $\mathcal{Z}(\mathcal{S})$ point-filter.

iii) For $1 \le j \le n$ with $i \ne j$, $\mathcal{F}_i \ne \mathcal{F}_j$.

Proof: X is associated with $\emptyset$ and so is in $\mathcal{F}_i$. Also $\emptyset$ is associated with $\{1, \dots, n\}$ and so is not in $\mathcal{F}_i$. Thus $\mathcal{F}_i$ is a non-empty collection of non-empty $\mathcal{Z}(\mathcal{S})$-sets. Let Z_1, Z_2 be in $\mathcal{F}_i$ with Z_1 associated with Δ_1 and Z_2 associated with Δ_2. By definition $i \notin \Delta_1 \cup \Delta_2$. By R33.1.5 $Z_1 \cap Z_2$ is associated with $\Delta_1 \cup \Delta_2$ and so $Z_1 \cap Z_2 \in \mathcal{F}_i$. Next let $Z \subseteq W$, where $Z \in \mathcal{F}_i$ and $W \in \mathcal{Z}(\mathcal{S})$ is associated with Γ. If $i \in \Gamma$, $\overline{Z \cap G_i} \subseteq \overline{W \cap G_i}$, which is compact, so that i is in the set associated with Z, a contradiction. Thus $i \notin \Gamma$ so that $W \in \mathcal{F}_i$ and the first assertion holds. For part ii), let $\mathcal{G}$ be a $\mathcal{Z}(\mathcal{S})$-filter with $\mathcal{F}_i \subseteq \mathcal{G}$. Let $Z \in \mathcal{G}$ be associated with Δ and suppose $Z \notin \mathcal{F}_i$, i.e., $i \in \Delta$. Note that G_i is associated with $\{1, \dots, n\} - \{i\}$, as is $\overline{G_i}$ by R33.1.4iii, so that $\overline{G_i}$ is in $\mathcal{F}_i$ and so in $\mathcal{G}$. $Z \cap \overline{G_i}$ is in $\mathcal{G}$ and by R33.4.5 it is associated with $\{1, \dots, n\}$. By R33.1.4i $Z \cap \overline{G_i}$ is compact. By R33.2.2 there is $x \in X$ such that $\mathcal{G} \subseteq \mathcal{F}(x)$. By local compactness there is $O \in \tau$ with $x \in O$ and $\overline{O}$ compact. By parts i) and ii) of R33.1.4, $X - O$ is associated with $\emptyset$ so that $X - O \in \mathcal{F}_i$. But $X - O \notin \mathcal{F}(x)$, which contradicts $\mathcal{G} \subseteq \mathcal{F}(x)$. Thus $Z \in \mathcal{F}_i$ and ii) holds. For iii), let $x \in X$. Since $\{x\}$ is associated with $\{1, \dots, n\}$, $\{x\} \in \mathcal{Z}(\mathcal{S})$, $\{x\} \in \mathcal{F}(x)$, and $\{x\} \notin \mathcal{F}_i$. Thus $\mathcal{F}_i \ne \mathcal{F}(x)$. Finally, let $j \ne i$ with $1 \le j \le n$. As above, $\overline{G_i}$ is in $\mathcal{F}_i$ and is associated with $\{1, \dots, n\} - \{i\}$. By definition $\overline{G_i} \notin \mathcal{F}_j$ and so $\mathcal{F}_i \ne \mathcal{F}_j$.

Lemma R33.2.4 Let $\mathcal{S} = \{G_i : i = 1, \dots, n\}$ be an n-star for the non-compact T_2 topological space (X, τ). Let $\mathcal{H}$ be a $\mathcal{Z}(\mathcal{S})$-ultrafilter. Then either there is $1 \le i \le n$ such that $\mathcal{H} = \mathcal{F}_i$ or $\mathcal{H}$ is a $\mathcal{Z}(\mathcal{S})$ point-filter.

Proof: Assume $\mathcal{H} \ne \mathcal{F}_i$ for all $1 \le i \le n$. For each i, since $\mathcal{H}$ is a $\mathcal{Z}(\mathcal{S})$-ultrafilter, $\mathcal{H}$ cannot be a proper subset of $\mathcal{F}_i$ and so there is Z_i associated with Δ_i with $Z_i \in \mathcal{H}$

and $Z_i \notin \mathcal{F}_i$, i.e., $i \in \Delta_i$. Let $Z = \cap_{i=1}^n Z_i$. Then Z is in $\mathcal{H}$ and Z is associated with $\cup_{i=1}^n \Delta_i = \{1, \ldots, n\}$ by R33.1.5. By R33.1.4i Z is compact and by R33.2.2 there is $x \in X$ such that $\mathcal{H} \subseteq \mathcal{F}(x)$. Since $\mathcal{H}$ is a $\mathcal{Z}(\mathcal{S})$-ultrafilter, $\mathcal{H} = \mathcal{F}(x)$.

The next result generalizes R5.3.8.

Proposition R33.2.5 Let $\mathcal{S} = \{G_i : i = 1, \ldots, n\}$ be an n-star for the non-compact T_2 topological space (X, τ) and let (Y, f) be the n-point compactification determined by $\mathcal{S}$. Then (Y, f) is equivalent to $(\omega(\mathcal{Z}(\mathcal{S})), \iota_{\mathcal{Z}(\mathcal{S})})$.

Proof: Define $\phi : Y \to \omega(\mathcal{Z}(\mathcal{S}))$ by $\phi(x) = \mathcal{F}(x)$ for $x \in X$ and $\phi(p_i) = \mathcal{F}_i$. It follows easily from R33.2.3 and R33.2.4 that ϕ is a bijection. By definition $\phi \circ f = \iota_{\mathcal{Z}(\mathcal{S})}$. Thus it remains to show that ϕ is continuous. For that, because $\{Z^\omega : Z \in \mathcal{Z}(\mathcal{S})\}$ is a base for the closed sets of $\omega(\mathcal{Z}(\mathcal{S}))$, it is sufficient to show that $\phi^{-1}[Z^\omega]$ is closed in Y for every $Z \in \mathcal{Z}(\mathcal{S})$. Let Z be in $\mathcal{Z}(\mathcal{S})$ be associated with Δ and let $A = Y - \phi^{-1}[Z^\omega]$. First, $x \in X - Z$ if and only if $Z \notin \mathcal{F}(x)$, i.e., $\phi(x) \notin Z^\omega$, i.e., $x \notin \phi^{-1}[Z^\omega]$. Thus $A \cap X = X - Z$, which is open in X, and $X - A = Z$. Next $p_j \in A$ if and only if $\phi(p_j) \notin Z^\omega$, i.e., $Z \notin \mathcal{F}_j$, i.e., $j \in \Delta$. Thus $p_j \in A$ implies $(X - A) \cap G_j = Z \cap G_j$ has compact closure in X. By the definition of the topology for Y, A is open in Y so that $X - A = \phi^{-1}[Z^\omega]$ is closed as required.

The next corollary could also be expressed by saying every finite-point compactification of a $T_{3\frac{1}{2}}$ space is a Wallman compactification.

Corollary R33.2.6 Let (X, τ) be a non-compact $T_{3\frac{1}{2}}$ space with a finite-point compactification (S, g). Then (S, g) is equivalent to a compactification generated from a normal basis.

Proof: The given compactification is an n-point compactification for some n. By R5.1.2 there is $\mathcal{S} = \{G_i : i = 1, \ldots, n\}$ an n-star for X such that (Y, f), the n-point compactification determined by $\mathcal{S}$, is equivalent to (S, g). By the previous proposition and transitivity, (S, g) is equivalent to $(\omega(\mathcal{Z}(\mathcal{S})), \iota_{\mathcal{Z}(\mathcal{S})})$.

Ordering of Finite-Point Compactifications

The first proposition on equivalence is essentially a corollary of a result of Magill [1]. It will subsequently be refined for ordered but non-equivalent cases.

Proposition R33.3.1 Let $\mathcal{S} = \{G_i : i = 1, \ldots, n\}$ and $\mathcal{R} = \{O_i : i = 1, \ldots, n\}$ be n-stars for the non-compact T_2 topological space (X, τ). Let $K_1 = X - \cup_{i=1}^n G_i$. Then $(\omega(\mathcal{Z}(\mathcal{S})), \iota_{\mathcal{Z}(\mathcal{S})})$ is equivalent to $(\omega(\mathcal{Z}(\mathcal{R})), \iota_{\mathcal{Z}(\mathcal{R})})$ if and only if there is σ, a permutation of $\{1, \ldots, n\}$, such that $(K_1 \cup G_i) \cap (X - O_{\sigma(i)})$ is compact for every $1 \leq i \leq n$.

Proof: Let $(Y_\mathcal{S}, f_\mathcal{S})$ and $(Y_\mathcal{R}, f_\mathcal{R})$ be the n-point compactifications determined by $\mathcal{S}$, $\mathcal{R}$ respectively. By R5.1.5 $(Y_\mathcal{S}, f_\mathcal{S})$ is equivalent to $(Y_\mathcal{R}, f_\mathcal{R})$ if and only if there is σ, a permutation of $\{1, \ldots, n\}$, such that $(K_1 \cup G_i) \cap (X - O_{\sigma(i)})$ is compact for every $1 \leq i \leq n$. By transitivity and R33.2.5 the conclusion holds.

Comment: The proposition is expressed asymmetrically with regard to $\mathcal{S}$ and $\mathcal{R}$. Reversing their roles would produce a permutation μ such that $(J \cup O_i) \cap (X - G_{\mu(i)})$ is compact for every $1 \leq i \leq n$, where $J = X - \cup_{i=1}^n O_i$. By using the uniqueness of the connecting map and details of its construction, it can be shown that $\mu = \sigma^{-1}$. That will not be needed in what follows.

The next lemma applies to any T_2 compactification, not just finite-point examples. It is undoubtedly known but is recorded here for completeness and ease of reference.

Lemma R33.3.2 Let (X, τ) be a $T_{3\frac{1}{2}}$ space with T_2 compactifications (Y, f) and (Z, g). Assume $\phi : Z \to Y$ is continuous with $\phi \circ g = f$. Then $\phi[Z - g[X]] = Y - f[X]$.

Proof: As usual ϕ is onto from general considerations and by hypothesis $\phi(g(x))$ is in $f[X]$ for every $x \in X$. Thus $Y - f[X] \subseteq \phi[Z - g[X]]$. Now let $z \in Z - g[X]$ and suppose $\phi(z) = f(x)$ for some $x \in X$. By the density of $g[X]$ in Z, there is a net $\{x_\alpha\}$ in X such that $\{g(x_\alpha)\}$ converges to z. Since ϕ is continuous and $\phi(g(x_\alpha)) = f(x_\alpha)$, the net $\{f(x_\alpha)\}$ converges to $\phi(z) = f(x)$. Because $f : X \to f[X]$ is a homeomorphism, $\{x_\alpha\}$ converges to x in X. By the continuity of g, $\{g(x_\alpha)\}$ converges to $g(x)$. Since limits are unique in T_2 spaces, $z = g(x)$, which contradicts the choice of z. Thus $\phi[Z - g[X]] \subseteq Y - f[X]$.

Proposition R33.3.3 Let $\mathcal{S} = \{G_i : i = 1, \ldots, n\}$ be an n-star for the non-compact T_2 topological space (X, τ) and let $\mathcal{R} = \{O_i : i = 1, \ldots, m\}$ be an m-star for the same space. Let $K = X - \cup_{i=1}^m O_i$. Then $(\omega(\mathcal{Z}(\mathcal{S})), \iota_{\mathcal{Z}(\mathcal{S})}) \leq (\omega(\mathcal{Z}(\mathcal{R})), \iota_{\mathcal{Z}(\mathcal{R})})$ if and only if there is an onto map $r : \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that, for every $1 \leq i \leq n$ and $j \in r^{-1}[\{i\}]$, $(X - G_i) \cap (K \cup O_j)$ is compact in X.

Proof: Let $(Y_\mathcal{S}, f_\mathcal{S})$ and $(Y_\mathcal{R}, f_\mathcal{R})$ be the n-point, respectively m-point, compactifications determined by $\mathcal{S}$ and $\mathcal{R}$. Notationally, assume $Y_\mathcal{S} = X \cup \{p_1, \ldots, p_n\}$ and $Y_\mathcal{R} = X \cup \{q_1, \ldots, q_m\}$. By transitivity and R33.2.5, the proposition holds if it can be verified for these representatives. First assume r exists. Define $\phi : Y_\mathcal{R} \to Y_\mathcal{S}$ by $\phi(x) = x$ and $\phi(q_t) = p_{r(t)}$. By definition $\phi \circ f_\mathcal{R} = f_\mathcal{S}$ and, since r is onto, ϕ is onto. It remains to check that ϕ is continuous. Let O be open in $Y_\mathcal{S}$. By definition of ϕ, $\phi^{-1}[O] \cap X = O \cap X$, which is in τ. Let $q_j \in \phi^{-1}[O]$. To see that $(X - \phi^{-1}[O]) \cap O_j$ has compact closure in X, first note that $\phi(q_j) = p_{r(j)}$ is in O and so $(X - O) \cap G_{r(j)}$ has compact closure in X. By hypothesis for this part, $(X - G_{r(j)}) \cap (K \cup O_j)$ is compact in X. It is sufficient to verify that $(X - \phi^{-1}[O]) \cap O_j \subseteq ((X - O) \cap G_{r(j)}) \cup ((X - G_{r(j)}) \cap (K \cup O_j))$. Let $x \in (X - \phi^{-1}[O]) \cap O_j$. If $x \in G_{r(j)}$, by definition of ϕ, $x \in (X - O) \cap G_{r(j)}$. If $x \notin G_{r(j)}$, then, since $x \in O_j \subseteq (K \cup O_j)$, $x \in (X - G_{r(j)}) \cap (K \cup O_j)$. Thus the needed containment holds. Conversely, assume $\psi : Y_\mathcal{R} \to Y_\mathcal{S}$ is continuous with $\psi \circ f_\mathcal{R} = f_\mathcal{S}$. In this situation ψ is onto and, by R33.3.2, maps $\{q_1, \ldots, q_m\}$ onto $\{p_1, \ldots, p_n\}$. Define $r : \{1, \ldots, m\} \to \{1, \ldots, n\}$ by $r(j) = i$ where $\psi(q_j) = p_i$. Clearly r is onto. Let $1 \leq i \leq n$ and $j \in r^{-1}[\{i\}]$ so that $\psi(q_j) = p_i$. By definition of the topology on $Y_\mathcal{S}$, $G_i \cup \{p_i\}$ is open in $Y_\mathcal{S}$ and so $\psi^{-1}[G_i \cup \{p_i\}]$ is open in $Y_\mathcal{R}$. By definition of $f_\mathcal{R}$ and $g_\mathcal{S}$, $\psi(x) = x$ and so $X - \psi^{-1}[G_i \cup \{p_i\}] = X - G_i$. Since $q_j \in \psi^{-1}[G_i \cup \{p_i\}]$, $(X - G_i) \cap O_j$ has compact closure in X. Because $K \cup O_j$ is closed in X and K is compact, $(X - G_i) \cap (K \cup O_j)$ is compact in X.

In the last proposition the proof that the existence of r is sufficient does not make clear the role of its surjectivity. It guarantees that the defined ϕ is onto, of course, but that would follow from general considerations if ϕ could be shown continuous without using the fact that r is onto. The next lemma and proposition, which are a bit of a digression, show that the hypothesis cannot be true if r is not onto.

Lemma R33.3.4 Let $\mathcal{R} = \{O_i : i = 1, \ldots, m\}$ be an m-star for the non-compact T_2 space (X, τ). Let $(Y_\mathcal{R}, f_\mathcal{R})$ m-point compactification determined $\mathcal{R}$. Notationally, assume $Y_\mathcal{R} = X \cup \{q_1, \ldots, q_m\}$. Let $O \in \tau$ and assume that $O \cup \{q_1, \ldots, q_n\}$ is open in $Y_\mathcal{R}$. Then $X - O$ is compact in X.

Proof: $X - O = \overline{X - O} = (K \cap (X - O) \cup (\cup_{i=1}^m \overline{O_i \cap (X - O)}$, where $K = X - \cup_{i=1}^m O_i$.

By hypothesis each term in that finite union is compact and so $X - O$ is compact.

Proposition R33.3.5 Let $\mathcal{S} = \{G_i : i = 1, \ldots, n\}$ be an n-star for the non-compact T_2 topological space (X, τ) and let $\mathcal{R} = \{O_i : i = 1, \ldots, m\}$ be an m-star for the same space. Let $K = X - \cup_{i=1}^m O_i$. Assume $r : \{1, \ldots, m\} \to \{1, \ldots, n\}$ is not onto. Then there is $1 \le i \le n$ and $j \in r^{-1}[\{i\}]$ such that $(X - G_i) \cap (K \cup O_j)$ is not compact in X.

Proof: Let $G = \cup\{G_i : r^{-1}[\{i\}] \neq \emptyset\}$, an open set in X. If one assumes the conclusion is false, then, for every $1 \le j \le m$, $(X - G) \cap O_j$ has compact closure in X because it is a subset of $(X - G_{r(j)}) \cap O_j$. Thus $G \cup \{q_1, \ldots, q_m\}$ is open in $Y_{\mathcal{R}}$. By the lemma $X - G$ is compact. Because r is not onto, there is $1 \le k \le n$ such that $r^{-1}[\{k\}] = \emptyset$. By definition of G and the disjointness property of the n-star, $G_k \subseteq X - G$. For $J = X - \cup_{i=1}^n G_i$, the closed non-compact set $J \cup G_k$ is contained in the compact set $J \cup (X - G)$, a contradiction.

This proposition shows that R33.3.3 can be simplified as follows.

Corollary R33.3.6 Let $\mathcal{S} = \{G_i : i = 1, \ldots, n\}$ be an n-star for the non-compact T_2 topological space (X, τ) and let $\mathcal{R} = \{O_i : i = 1, \ldots, m\}$ be an m-star for the same space. Let $K = X - \cup_{i=1}^m O_i$. Then $(\omega(\mathcal{Z}(\mathcal{S})), \iota_{\mathcal{Z}(\mathcal{S})}) \le (\omega(\mathcal{Z}(\mathcal{R})), \iota_{\mathcal{Z}(\mathcal{R})})$ if and only if there is a map $r : \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that, for every $1 \le i \le n$ and $j \in r^{-1}[\{i\}]$, $(X - G_i) \cap (K \cup O_j)$ is compact in X.

Proof: The necessity of the condition is immediate from R33.3.3. The sufficiency follows from R33.3.5 and R33.3.3.

Lemma R33.3.7 Let $\mathcal{S} = \{G_i : i = 1, \ldots, n\}$ be an n-star for the non-compact T_2 topological space (X, τ) and let $\mathcal{R} = \{O_i : i = 1, \ldots, m\}$ be an m-star for the same space. Assume there is a map $r : \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that, for every $1 \le i \le n$ and $j \in r^{-1}[\{i\}]$, $(X - G_i) \cap (K \cup O_j)$ is compact in X, where $K = X - \cup_{i=1}^m O_i$. Let Z be associated with Δ relative to $\mathcal{S}$. Then Z is associated with $r^{-1}[\Delta]$ relative to $\mathcal{R}$.

Proof: Let $j \in \{1, \ldots, m\}$ and let $i = r(j)$. If $j \in r^{-1}[\Delta]$, $i \in \Delta$ so that $Z \cap G_i$ has compact closure in X. $Z \cap O_j = (Z \cap O_j \cap G_i) \cup (Z \cap O_j \cap (X - G_i))$. The first term of that union is contained in $Z \cap G_i$ and the second is contained in the compact $(X - G_i) \cap (K \cup O_j)$. Thus $Z \cap O_j$ has compact closure in X. If $j \notin r^{-1}[\Delta]$, $i \notin \Delta$ and so $(X - Z) \cap G_i$ has compact closure in X. Now proceed exactly as before: $(X - Z) \cap O_j = ((X - Z) \cap O_j \cap G_i) \cup ((X - Z) \cap O_j \cap (X - G_i))$. The first term of that union is contained in $(X - Z) \cap G_i$ and the second is contained in the compact $(X - G_i) \cap (K \cup O_j)$.

Corollary R33.3.8 Let $\mathcal{S} = \{G_i : i = 1, \ldots, n\}$ be an n-star for the non-compact T_2 topological space (X, τ) and let $\mathcal{R} = \{O_i : i = 1, \ldots, m\}$ be an m-star for the same space. Assume $(\omega(\mathcal{Z}(\mathcal{S})), \iota_{\mathcal{Z}(\mathcal{S})}) \le (\omega(\mathcal{Z}(\mathcal{R})), \iota_{\mathcal{Z}(\mathcal{R})})$. Then $\mathcal{Z}(\mathcal{S}) \subseteq \mathcal{Z}(\mathcal{R})$.

Proof: By R33.3.6 the hypothesis of R33.3.7 holds. The conclusion follows from the definition of the normal basis determined by a k-star and R33.3.7.

In the next proposition superscript notation will be used to distinguish filters in the two normal bases, $\mathcal{Z}(\mathcal{S})$ and $\mathcal{Z}(\mathcal{R})$.

Proposition R33.3.9 Let $\mathcal{S} = \{G_i : i = 1, \ldots, n\}$ be an n-star for the non-compact T_2 topological space (X, τ) and let $\mathcal{R} = \{O_i : i = 1, \ldots, m\}$ be an m-star for the same space. Assume $(\omega(\mathcal{Z}(\mathcal{S})), \iota_{\mathcal{Z}(\mathcal{S})}) \le (\omega(\mathcal{Z}(\mathcal{R})), \iota_{\mathcal{Z}(\mathcal{R})})$. Then, for every $\mathcal{F}^{\mathcal{R}}$ in $\omega(\mathcal{Z}(\mathcal{R}))$, $\mathcal{F}^{\mathcal{R}} \cap \mathcal{Z}(\mathcal{S})$ is in $\omega(\mathcal{Z}(\mathcal{S}))$.

Proof: Let $\mathcal{F}^{\mathcal{R}}(x)$ be the $\mathcal{Z}(\mathcal{R})$ point-filter of x. By R33.3.8 $\mathcal{Z}(\mathcal{S}) \subseteq \mathcal{Z}(\mathcal{R})$ and so by R9.1.1ii $\mathcal{F}^{\mathcal{R}}(x) \cap \mathcal{Z}(\mathcal{S}) = \mathcal{F}^{\mathcal{S}}(x)$, the $\mathcal{Z}(\mathcal{S})$ point-filter of x. Now assume $\mathcal{F}^{\mathcal{R}}$ is a non-point

$\mathcal{Z}(\mathcal{R})$-ultrafilter. By R33.2.4 $\mathcal{F}^{\mathcal{R}} = \mathcal{F}_j^{\mathcal{R}}$ for some $j \in \{1, \ldots, m\}$. By R33.3.6 there is a map r so that the hypothesis of R33.3.7 holds. Let $i = r(j)$. Let Z in $\mathcal{F}_i^{\mathcal{S}}$ be associated with Δ. By definition R33.2.1 $\Delta \subseteq \{1, \ldots, n\} - \{i\}$ and $Z \in \mathcal{Z}(\mathcal{S})$. By R33.3.7 Z is associated with $r^{-1}[\Delta]$ relative to $\mathcal{Z}(\mathcal{R})$ and so $Z \in \mathcal{Z}^{\mathcal{R}})$ by definition. Note that $j \notin r^{-1}[\Delta]$ because $r(j) = i$ is not in Δ. By R33.2.1 $Z \in \mathcal{F}_j^{\mathcal{R}} \cap \mathcal{Z}(\mathcal{S})$. Thus $\mathcal{F}_i^{\mathcal{S}} \subseteq \mathcal{F}_j^{\mathcal{R}} \cap \mathcal{Z}(\mathcal{S})$. The latter is a $\mathcal{Z}(\mathcal{S})$-filter by R9.1.1i and $\mathcal{F}_i^{\mathcal{S}}$ is a $\mathcal{Z}(\mathcal{S})$-ultrafilter by R32.2.3ii. Thus $\mathcal{F}_i^{\mathcal{S}} = \mathcal{F}_j^{\mathcal{R}} \cap \mathcal{Z}(\mathcal{S})$.

Suprema of Finite-Point Compactifications

Proposition R33.4.1 Let $\mathcal{S} = \{G_i : i = 1, \ldots, n\}$ be an n-star for the non-compact T_2 topological space (X, τ) and let $\mathcal{R} = \{O_i : i = 1, \ldots, m\}$ be an m-star for the same space. Let $\mathcal{P} = \{G_i \cap O_j : \overline{G_i \cap O_j}$ is not compact in $X\}$. Then $\mathcal{P}$ is a k-star for (X, τ) for some k, with $\max\{m, n\} \leq k \leq mn$.

Proof: Let $P = \{(i, j) : G_i \cap O_j : \overline{G_i \cap O_j}$ is not compact in $X\}$ and let $k = |P|$. Since $P \subseteq \{1, \ldots, n\} \times \{1, \ldots, m\}$, $k \leq mn$. Clearly $k = |\mathcal{P}|$. By definition each $G_i \cap O_j$ is open and, if $(i, j) \neq (r, s)$, $(G_i \cap O_j) \cap (G_r \cap O_s)$ is empty. Thus $\mathcal{P}$ is a pairwise disjoint collection of open sets. Now let $L = X - \cup\{G_i \cap O_j : (i, j) \in P\}$, a closed set. For $K = X - \cup_{i=1}^n G_i$ and $J = X - \cup_{j=1}^m O_j$, it will be shown that $L \subseteq K \cup J \cup (\cup\{\overline{G_i \cap O_j} : \overline{G_i \cap O_j}$ is compact in $X\})$, a finite union of compact sets. Let $x \in L$ and suppose $x \notin K \cup J$. Then there exist i, j such that $x \in G_i \cap O_j$. Since $x \in L$, $(i, j) \notin P$ so that $\overline{G_i \cap O_j}$ is compact. Thus the claim is verified so that L is compact. Now let $(i, j) \in P$. $L \cup (G_i \cap O_j)$ is closed because its complement is open by pairwise disjointness. Thus the non-compact $\overline{G_i \cap O_j}$ is a subset of $L \cup (G_i \cap O_j)$ and so $L \cup (G_i \cap O_j)$ must also be non-compact. By definition $\mathcal{P}$ is a k-star. Finally let $1 \leq i \leq n$ and suppose, for every $1 \leq j \leq m$, $\overline{G_i \cap O_j}$ is compact. Because $G_i = (J \cap G_i) \cup (\cup_{j=1}^m (G_i \cap O_j))$, $\overline{G_i} = \overline{J \cap G_i} \cup (\cup_{j=1}^m \overline{G_i \cap O_j})$ so that $\overline{G_i}$ is compact. But the non-compact closed set $K \cup G_i$ is contained in the compact $K \cup \overline{G_i}$, a contradiction. Thus there is $1 \leq j \leq m$ such that $(i, j) \in P$ and so $k = |P| \geq n$. Similarly $k \geq m$.

For results through R33.4.5 the following notation will be used: Let (X, τ) be a non-compact T_2 topological space with n-star $\mathcal{S} = \{G_i : i = 1, \ldots, n\}$ and m-star $\mathcal{R} = \{O_i : i = 1, \ldots, m\}$. Let $P = \{(i, j) : \overline{G_i \cap O_j}$ is not compact in $X\}$ and let $\mathcal{P} = \{G_i \cap O_j : (i, j) \in P\}$. The compactifications determined by $\mathcal{S}, \mathcal{R}$, and $\mathcal{P}$ will be denoted $(Y_{\mathcal{S}}, f_{\mathcal{S}}), (Y_{\mathcal{R}}, f_{\mathcal{R}})$, and $(Y_{\mathcal{P}}, f_{\mathcal{P}})$ respectively with $Y_{\mathcal{S}} = X \cup \{s_1, \ldots, s_n\}$, $Y_{\mathcal{R}} = X \cup \{r_1, \ldots, r_n\}$, and $Y_{\mathcal{P}} = X \cup \{p_{(i,j)} : (i, j) \in P\}$.

Proposition R33.4.2 Let $\mathcal{S} = \{G_i : i = 1, \ldots, n\}$ be an n-star for the non-compact T_2 topological space (X, τ) and let $\mathcal{R} = \{O_i : i = 1, \ldots, m\}$ be an m-star for the same space. Let $\mathcal{P} = \{G_i \cap O_j : \overline{G_i \cap O_j}$ is not compact in $X\}$. Then $(Y_{\mathcal{S}}, f_{\mathcal{S}}) \leq (Y_{\mathcal{P}}, f_{\mathcal{P}})$ and $(Y_{\mathcal{R}}, f_{\mathcal{R}}) \leq (Y_{\mathcal{P}}, f_{\mathcal{P}})$.

Proof: Define $\sigma_{\mathcal{S}} : Y_{\mathcal{P}} \to Y_{\mathcal{S}}$ by $\sigma_{\mathcal{S}}(x) = x$ for $x \in X$ and $\sigma_{\mathcal{S}}(p_{(i,j)}) = s_i$. By definition $\sigma_{\mathcal{S}} \circ f_{\mathcal{P}} = f_{\mathcal{S}}$. Now let G be open in $Y_{\mathcal{S}}$. It is easy to check that $X \cap \sigma_{\mathcal{S}}^{-1}[G] = X \cap G$, which is open in X. Likewise, $X - \sigma_{\mathcal{S}}^{-1}[G] = X - G$. If $p_{(i,j)} \in \sigma_{\mathcal{S}}^{-1}[G]$, $s_i \in G$ and $(X - \sigma_{\mathcal{S}}^{-1}[G]) \cap (G_i \cap O_j) \subseteq (X - G) \cap G_i$, which has compact closure in X. Thus $\sigma_{\mathcal{S}}^{-1}[G]$ is open in $Y_{\mathcal{P}}$ and $\sigma_{\mathcal{S}}$ is continuous. By definition $(Y_{\mathcal{S}}, f_{\mathcal{S}}) \leq (Y_{\mathcal{P}}, f_{\mathcal{P}})$. Similarly, $(Y_{\mathcal{R}}, f_{\mathcal{R}}) \leq (Y_{\mathcal{P}}, f_{\mathcal{P}})$.

The next two lemmas simplify the proof of the subsequent proposition.

Lemma R33.4.3 Let $\mathcal{Q} = \{W_i : i = 1, \ldots, j\}$ be a j-star for the non-compact T_2

topological space (X,τ) and let $(Y_{\mathcal{Q}}, f_{\mathcal{Q}})$ be the compactification determined by $\mathcal{Q}$, where $Y_{\mathcal{Q}} = X \cup \{q_1, \ldots, q_j\}$. Let C be a compact subset of X. Then

i) For $1 \leq t \leq j$, $W_t \cup \{q_t\}$ is open in $Y_{\mathcal{S}}$.

ii) For $1 \leq t \leq j$, $(X - C) \cup \{q_t\}$ is open in $Y_{\mathcal{S}}$.

Proof: Let $1 \leq t \leq j$. $(W_t \cup \{q_t\}) \cap X = W_t$, which is in τ by definition of a j-star, and $(X - (W_t \cup \{q_t\})) \cap W_t = \emptyset$, which is compact. By definition $W_t \cup \{q_t\}$ is open in $Y_{\mathcal{Q}}$ and i) holds. Similarly $((X - C) \cup \{q_t\}) \cap X = X - C$, an open set, and $(X - ((X - C) \cup \{q_t\}) \cap W_t = C \cap W_t$, which is contained in the compact C and so has compact closure in X. By definition $(X - C) \cup \{q_t\}$ is open in $Y_{\mathcal{Q}}$ and ii) holds.

Lemma R33.4.4 Let $\mathcal{S} = \{G_i : i = 1, \ldots, n\}$ be an n-star for the non-compact T_2 topological space (X,τ) and let $\mathcal{R} = \{O_i : i = 1, \ldots, m\}$ be an m-star for the same space. Let (Z, g) be a compactification of (X,τ) with continuous maps $\psi_{\mathcal{S}} : Z \to Y_{\mathcal{S}}$ and $\psi_{\mathcal{R}} : Z \to Y_{\mathcal{R}}$ such that $\psi_{\mathcal{S}} \circ g = f_{\mathcal{S}}$ and $\psi_{\mathcal{R}} \circ g = f_{\mathcal{R}}$. Let $z \in Z - g[X]$ with $\psi_{\mathcal{S}}(z) = s_a$ and $\psi_R(z) = r_b$. Then $G_a \cap O_b$ does not have compact closure in X.

Proof: Deny the conclusion. Since $g[X]$ is dense in Z, there is $\{x_\alpha\}$, a net in X, such that $\{g(x_\alpha)\}$ converges to z. By continuity $\{\psi_{\mathcal{S}}(g(x_\alpha))\}$ converges to $\psi_{\mathcal{S}}(z) = s_a$ in $Y_{\mathcal{S}}$ and $\{\psi_{\mathcal{R}}(g(x_\alpha))\}$ converges to $\psi_{\mathcal{R}}(z) = r_b$ in $Y_{\mathcal{R}}$, i.e., $\{x_\alpha\}$ converges to s_a in $Y_{\mathcal{S}}$ and to r_b in $Y_{\mathcal{R}}$. By the definitions of convergence and directed set, since $G_a \cup \{s_a\}$ and $O_b \cup \{r_b\}$ are open in $Y_{\mathcal{S}}$, $Y_{\mathcal{R}}$ respectively, there is α_0 such that $\alpha \geq \alpha_0$ implies $x_\alpha \in G_a \cap O_b$. By the assumed compactness of $\overline{G_a \cap O_b}$, the net $\{x_\alpha\}_{\alpha \geq \alpha_0}$ has a subnet $\{x_{\alpha_\beta}\}$ converging to some x in $\overline{G_a \cap O_b} \subseteq X$. By continuity of g, $\{g(x_{\alpha_\beta})\}$ converges to $g(x)$ in Z. Since $\{g(x_{\alpha_\beta})\}$ is a subnet of $\{g(x_\alpha)\}$, $\{g(x_{\alpha_\beta})\}$ also converges to z. Since limits in the T_2 space Z are unique, $z = g(x)$, which contradicts the hypothesis for z.

Proposition R33.4.5 Let $\mathcal{S} = \{G_i : i = 1, \ldots, n\}$ be an n-star for the non-compact T_2 topological space (X,τ) and let $\mathcal{R} = \{O_i : i = 1, \ldots, m\}$ be an m-star for the same space. Let $\mathcal{P} = \{G_i \cap O_j : \overline{G_i \cap O_j} \text{ is not compact in } X\}$. Let (Z, g) be a compactification of (X,τ) with $(Y_{\mathcal{S}}, f_{\mathcal{S}}) \leq (Z, g)$ and $(Y_{\mathcal{R}}, f_{\mathcal{R}}) \leq (Z, g)$. Then $(Y_{\mathcal{P}}, f_{\mathcal{P}}) \leq (Z, g)$.

Proof: By hypothesis there are continuous maps $\psi_{\mathcal{S}} : Z \to Y_{\mathcal{S}}$ and $\psi_{\mathcal{R}} : Z \to Y_{\mathcal{R}}$ such that $\psi_{\mathcal{S}} \circ g = f_{\mathcal{S}}$ and $\psi_{\mathcal{R}} \circ g = f_{\mathcal{R}}$. Define $\phi : Z \to Y_{\mathcal{P}}$ as follows: For $z \in g[X]$ with $z = g(x)$, let $\phi(g(x)) = x$. For $z \in Z - g[X]$, by R33.3.2 $\psi_{\mathcal{S}}(z) = s_i$ for some $1 \leq i \leq n$ and $\psi_{\mathcal{R}}(z) = r_j$ for some $1 \leq j \leq m$. By R33.4.4 $(i, j) \in P$ and so define $\phi(z) = p_{(i,j)}$. By definition $\phi \circ g = f_{\mathcal{P}}$. To see that ϕ is continuous, let G be open in $Y_{\mathcal{P}}$. It is easy to check that $\phi^{-1}[G] \cap g[X] = g[X \cap G]$. Since X is locally compact, $g[X]$ is open in Z and so the homeomorphism $g : X \to g[X]$ is an open map into Z. By definition of the topology on $Y_{\mathcal{P}}$, $X \cap G$ is open in X. Thus $\phi^{-1}[G]$ is a neighborhood of every point in $\phi^{-1}[G] \cap g[X]$. Now let $z \in \phi^{-1}[G] - g[X]$ with $\psi_{\mathcal{S}}(z) = s_i$ and $\psi_{\mathcal{R}}(z) = r_j$. Then $\phi(z) = p_{(i,j)}$ is in G and so $\overline{(X - G) \cap (G_i \cap O_j)}$ is compact in X. By R33.4.3 $\{s_i\} \cup G_i$ and $\{s_i\} \cup (X - \overline{(X - G) \cap (G_i \cap O_j)})$ are open in $Y_{\mathcal{S}}$. Similarly $\{r_j\} \cup O_j$ and $\{r_j\} \cup (X - \overline{(X - G) \cap (G_i \cap O_j)})$ are open in $Y_{\mathcal{R}}$. Let

$$V = (\{s_i\} \cup G_i) \cap (\{s_i\} \cup (X - \overline{(X - G) \cap (G_i \cap O_j)})) \text{ and}$$
$$W = (\{r_j\} \cup O_j) \cap (\{r_j\} \cup (X - \overline{(X - G) \cap (G_i \cap O_j)}).$$

V is open in $Y_{\mathcal{S}}$ and W is open in $Y_{\mathcal{R}}$ and so by continuity $\psi_{\mathcal{S}}^{-1}[V] \cap \psi_{\mathcal{R}}^{-1}[W]$ is open in Z. Because $\psi_{\mathcal{S}}(z) = s_i$ and $\psi_{\mathcal{R}}(z) = r_j$, $z \in \psi_{\mathcal{S}}^{-1}[V] \cap \psi_{\mathcal{R}}^{-1}[W]$. Now let $w \in \psi_{\mathcal{S}}^{-1}[V] \cap \psi_{\mathcal{R}}^{-1}[W]$.

If $w \in Z - g[X]$, because the only element of $Y_{\mathcal{S}} - X$ in V is s_i, $\psi_{\mathcal{S}}(w) = s_i$. Similarly, $\psi_{\mathcal{R}}(w) = r_j$ and so by definition $\phi(w) = p_{(i,j)} \in G$. Thus $w \in \phi^{-1}[G]$. Now assume $w = g(x)$ for some $x \in X$. $\psi_{\mathcal{S}}(g(x)) = f_{\mathcal{S}}(x) = x \in V$ and similarly $\psi_{\mathcal{R}}(g(x)) = x \in W$ so that $x \in (G_i \cap O_j) \cap (X - \overline{(X - G) \cap (G_i \cap O_j)})$. Then x must be in G because otherwise x would be in $\overline{(X - G) \cap (G_i \cap O_j)}$. To summarize, $z \in \psi_{\mathcal{S}}^{-1}[V] \cap \psi_{\mathcal{R}}^{-1}[W] \subseteq \phi^{-1}[G]$ so that $\phi^{-1}[G]$ is a neighborhood of z. Since $\phi^{-1}[G]$ is a neighborhood of each of its points, it is open and ϕ is continuous as required.

Corollary R33.4.6 Let $\mathcal{S} = \{G_i : i = 1, \ldots, n\}$ be an n-star for the non-compact T_2 topological space (X, τ) and let $\mathcal{R} = \{O_i : i = 1, \ldots, m\}$ be an m-star for the same space. Let $\mathcal{P} = \{G_i \cap O_j : \overline{G_i \cap O_j} \text{ is not compact in } X\}$. Then the compactification $(Y_{\mathcal{P}}, f_{\mathcal{P}})$ acts as the supremum of $(Y_{\mathcal{S}}, f_{\mathcal{S}})$ and $(Y_{\mathcal{R}}, f_{\mathcal{R}})$.

Proof: R33.4.2 shows it is an upper bound and R33.4.5 shows it is the least upper bound.

Corollary R33.4.7 Let $\{\mathcal{S}_j : 1 \leq j \leq m\}$ be a finite collection of finite stars for the non-compact T_2 topological space (X, τ). Then there is a k-star $\mathcal{P}$ for (X, τ) such that the compactification $(Y_{\mathcal{P}}, f_{\mathcal{P}})$ acts as the supremum of the collection $\{(Y_{\mathcal{S}_j}, f_{\mathcal{S}_j}) : 1 \leq j \leq m\}$.

Proof: By induction: The claim is trivial for $m = 1$ and true for $m = 2$ by R33.4.6. If it holds for any collection of size m, let a collection of size $m + 1$ be given. Apply the induction hypothesis to obtain $\mathcal{P}^*$ such that compactification $(Y_{\mathcal{P}^*}, f_{\mathcal{P}^*})$ acts as the supremum of $\{(Y_{\mathcal{S}_j}, f_{\mathcal{S}_j}) : 1 \leq j \leq m\}$. Apply R33.4.6 to obtain $\mathcal{P}$ such that $(Y_{\mathcal{P}}, f_{\mathcal{P}})$ acts as the supremum of $(Y_{\mathcal{S}_{m+1}}, f_{\mathcal{S}_{m+1}})$ and $(Y_{\mathcal{P}^*}, f_{\mathcal{P}^*})$. Then $(Y_{\mathcal{P}}, f_{\mathcal{P}})$ acts as the supremum of $\{(Y_{\mathcal{S}_j}, f_{\mathcal{S}_j}) : 1 \leq j \leq m + 1\}$.

Comment: If needed, the k-star $\mathcal{P}$ could be described explicitly as in R33.4.1.

Corollary R33.4.8 Let $\{(Y_j, g_j) : 1 \leq j \leq m\}$ be a collection of finite-point compactifications of the non-compact T_2 space (X, τ) and let (Z, g) act as the supremum of $\{(Y_j, g_j) : 1 \leq j \leq m\}$. Then (Z, g) is a finite-point compactification of (X, τ).

Proof: By R5.1.2, for each $1 \leq j \leq m$, there is a finite star $\mathcal{S}_j$ such that $(Y_{\mathcal{S}_j}, f_{\mathcal{S}_j})$ is equivalent to (Y_j, g_j). By R33.4.7 there is a k-star $\mathcal{P}$ for (X, τ) such that the compactification $(Y_{\mathcal{P}}, f_{\mathcal{P}})$ acts as the supremum of the collection $\{(Y_{\mathcal{S}_j}, f_{\mathcal{S}_j}) : 1 \leq j \leq m\}$. By the transitivity of equivalence the k-point compactification $(Y_{\mathcal{P}}, f_{\mathcal{P}})$ is equivalent to (Z, g), which is therefore also a k-point compactification.

Comment: This could also be derived in other ways, e.g., by using the representation of a finite supremum from R3.1.2.

Lemma R33.4.9 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$ be a non-empty set of compactifications of (X, τ). Let (Z, g) be a compactification of (X, τ). Let Δ^* be the set of all non-empty finite subsets of Δ and, for each $F \in \Delta^*$, let (Y_F, f_F) be a compactification which acts as the supremum of $\{(Y_\alpha, f_\alpha) : \alpha \in F\}$. Then (Z, g) acts as the supremum of $\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$ if and only if (Z, g) acts as the supremum of $\{(Y_F, f_F) : F \in \Delta^*\}$.

Proof: First assume (Z, g) act as the supremum of $\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$. For any F in Δ^*, since $F \subseteq \Delta$, (Z, g) is an upper bound of $\{(Y_\alpha, f_\alpha) : \alpha \in F\}$ and so $(Z, g) \geq (Y_F, f_F)$. Thus (Z, g) is an upper bound of $\{(Y_F, f_F) : F \in \Delta^*\}$. Now let (W, h) be an upper bound of $\{(Y_F, f_F) : F \in \Delta^*\}$. For every $\alpha \in \Delta$, $(Y_{\{\alpha\}}, f_{\{\alpha\}})$ is equivalent to (Y_α, f_α). Thus (W, h) is an upper bound of $\{(Y_\alpha, f_\alpha) : \alpha \in F\}$ and so $(W, h) \geq (Z, g)$, i.e., (Z, g) acts as the least upper bound of $\{(Y_F, f_F) : F \in \Delta^*\}$. Now assume (Z, g) acts as the supremum of $\{(Y_F, f_F) : F \in \Delta^*\}$. Since, for every $\alpha \in \Delta$, $(Y_{\{\alpha\}}, f_{\{\alpha\}})$ is equivalent to (Y_α, f_α), (Z, g) is an upper bound of $\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$. As in the first half of this proof, an upper bound (W, h) of $\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$ is also an upper bound of $\{(Y_F, f_F) : F \in \Delta^*\}$ and so $(W, h) \geq (Z, g)$. Thus (Z, g) acts as the least upper bound of $\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$.

Proposition R33.4.10 Let (X, τ) be a non-compact, locally compact T_2 space and let $\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$ be a non-empty set of finite-point compactifications of (X, τ). Let the compactification (Z, g) act as the supremum of $\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$. Then there is a normal basis $\mathcal{Z}$ for (X, τ) such that $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$ is equivalent to (Z, g).

Proof: Let Δ^* be the set of all non-empty finite subsets of Δ and, for each $F \in \Delta^*$, let (Y_F, f_F) be a compactification which acts as the supremum of $\{(Y_\alpha, f_\alpha) : \alpha \in F\}$. By R33.4.8 each (Y_F, f_F) is a finite-point compactification and so by R5.1.2 there is a finite star $\mathcal{S}_F$ for (X, τ) such that $(Y_{\mathcal{S}_F}, f_{\mathcal{S}_F})$ is equivalent to (Y_F, f_F). Next it will be shown that $\{\mathcal{Z}(\mathcal{S}_F) : F \in \Delta^*\}$ has the directed set property under containment. Let $F, H \in \Delta^*$. $F \cup H$ is also in Δ^* and $(Y_{F\cup H}, f_{F\cup H})$ is an upper bound of $\{(Y_\alpha, f_\alpha) : \alpha \in F\}$ and so $(Y_{F\cup H}, f_{F\cup H}) \geq (Y_F, f_F)$. Similarly $(Y_{F\cup H}, f_{F\cup H}) \geq (Y_H, f_H)$ and by equivalence the same relationships hold for the compactifications determined by $\mathcal{S}_{F\cup H}$, $\mathcal{S}_F$, and $\mathcal{S}_H$. By R33.2.5 and R33.3.8 $\mathcal{Z}(\mathcal{S}_{F\cup H}) \supseteq \mathcal{Z}(\mathcal{S}_F) \cup \mathcal{Z}(\mathcal{S}_H)$ and the claim is verified. By R9.2.1 $\mathcal{Z} = \cup\{\mathcal{Z}(\mathcal{S}_F) : F \in \Delta^*\}$ is a normal basis for (X, τ). Because of R33.3.9 the hypothesis of R9.Add.5 holds and so $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$ acts as a supremum for $\{(\omega(\mathcal{Z}(\mathcal{S}_F)), \iota_{\mathcal{Z}(\mathcal{S}_F)}) : F \in \Delta^*\}$ and by equivalence for $\{(Y_F, f_F) : F \in \Delta^*\}$. By R33.4.9 $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$ acts as the supremum of $\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$ and so $(\omega(\mathcal{Z}), \iota_{\mathcal{Z}})$ is equivalent to (Z, g).

In other words, the previous proposition says that a supremum of finite-point compactifications must be a Wallman compactification.

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