Suprema of countable families

A supremum of a countably infinite family of compactifications will be shown to be the inverse limit of a suitable inverse spectrum. Notation consistent with that in Appendix II of Dugundji [1] is used, but the presentation does not refer to Dugundji's general results. The countable case is a special case of the main result in [2] but is presented separately to make a slightly simpler construction available.

Given $\{(Y, f_\alpha) : \alpha \in \Delta\}$, a non-empty family of T_2 compactifications of some space X, the notation $(Z, g) = \bigvee\{(Y, f_\alpha) : \alpha \in \Delta\}$ will be used instead of more cumbersome but more careful expressions that the class $[(Z, g)]$ has the properties of a supremum.

Assumptions for this section: Let (X, τ) be a non-compact $T_{3\frac{1}{2}}$ space and let $\{(Y_i, f_i) : i = 1 \ldots \infty\}$ be a countably infinite set of T_2 compactifications of X. Let $W_1 = Y_1$ and, for each $j \geq 2$, let $W_j = \overline{\{(f_1(x), f_2(x), \ldots, f_j(x)) : x \in X\}}$, where the closure is in $Y_1 \times Y_2 \times \cdots \times Y_j$, and let $F_j : X \to W_j$ be defined by $F_j(x) = (f_1(x), \ldots, f_j(x))$. In R3.1.2 of [2] it is shown that $(W_j, F_j) = \bigvee_{i=1}^{j}(Y_i, f_i)$. For positive integers k and m with $k \leq m$, let π_{mk} be the restiction to W_m of the continuous projection described by $\pi_{mk}(y_1, \ldots, y_k, \ldots, y_m) = (y_1, \ldots, y_k)$.

Lemma R4.1 π_{mk} maps W_m onto W_k.

Proof: Clearly $\pi_{mk} \circ F_m = F_k$ so that $\pi_{mk}[F_m[X]] = F_k[X]$. By continuity $\pi_{mk}[W_m] = \pi_{mk}[\overline{F_m[X]}] \subseteq \overline{F_k[X]} = W_k$. Surjectivity of π_{mk} follows immediately because $\pi_{mk}[W_m]$ is closed and contains the dense $F_k[X]$.

Lemma R4.2 Let k,l,m be positive integers with $k \leq l \leq m$. Then $\pi_{lk} \circ \pi_{ml} = \pi_{mk}$.

Proof: Clear from the definition.

Additional assumptions for this section: In $\prod_{i=1}^{\infty} W_i$, let $S = \{y : k \leq m \Rightarrow y(k) = \pi_{mk}(y(m))\}$. For $x \in X$, y_x in $\prod_{i=1}^{\infty} W_i$ is defined by $y_x(k) = F_k(x)$, i.e., $y_x(k) = (f_1(x), f_2(x), \ldots, f_k(x))$.

Lemma R4.3 S is closed in $\prod_{i=1}^{\infty} W_i$ and, for all $x \in X$, $y_x \in S$.

Proof: Let $\{x_t\}$ be a net in S converging to $y \in \prod_{i=1}^{\infty} W_i$, and let $k \leq m$. By definition of S, for every t, $x_t(k) = \pi_{mk}(x_t(m))$. Since the product has the topology of pointwise convergence, $\{x_t(k)\}$ and $\{x_t(m)\}$ converge to $y(k)$ and $y(m)$ respectively. Since π_{mk} is continuous and limits are unique in a T_2 space, $y(k) = \pi_{mk}(y(m))$. Thus $y \in S$ and S is closed. The second assertion is clear from the definitions of y_x, π_{mk}, and S.

In Dugundji's terminology [1], $\{W_j; \pi_{mk}\}$ is an inverse spectrum over the natural numbers with spaces W_j and connecting maps π_{mk}. S is the inverse limit space of the spectrum.

Additional assumptions for this section: Let $f : X \to S$ be defined by $f(x) = y_x$. Let ρ_j denote the projection from $\prod_{i=1}^{\infty} W_i$ onto W_j.

Lemma R4.4 f is an embedding.

Proof: For $x \neq t$ in X, $y_x(1) = f_1(x) \neq f_1(t) = y_t(1)$ since f_1 is one-to-one, so that $f(x) \neq f(t)$. Also, $\rho_k \circ f = F_k$ is continuous for every k. Thus f is continuous, one-to-one, and onto $f[X]$. Now let O be open in X with $x \in O$. Since f_1 is an embedding, there exists U open in Y_1 with $f_1[O] = U \cap f_1[X]$. Then $\rho_1^{-1}[U]$ is open in $\prod_{i=1}^{\infty} W_i$ and it is easy to check that $f(x) \in \rho_1^{-1}[U] \cap f[X] \subseteq f[O] \cap f[X]$. Thus f is also relatively open and so an embedding.

Lemma R4.5 Let B be a non-empty open subset of S and let $y \in B$. Then there exist a natural number m and $O_1, O_2, \dots O_m$ with O_i open in Y_i such that $y \in \rho_m^{-1}[(O_1 \times O_2 \times \cdots \times O_m) \cap W_m] \bigcap S \subseteq B$.

Proof: There exist integers $n_1 < n_2 < \cdots < n_k$ and $U_{n_1}, \dots, U_{n_k}$ with U_{n_j} open in W_{n_j} such that

$$y \in (\cap_{i=1}^{k} \rho_{n_i}^{-1}[U_{n_i}]) \cap S \subseteq B.$$

It is sufficient to derive the desired conclusion for such a basic open set with $m = n_k$. The proof continues by strong induction on n_k. If $n_k = 1$, then it is clear that the conclusion holds with $O_1 = U_1$. Now assume the conclusion holds for all positive integers less than n_k, so that there exist $\tilde{O}_1, \tilde{O}_2, \dots, \tilde{O}_{n_{k-1}}$ with $\tilde{O}_i$ open in Y_i and

$$y \in \rho_{n_{k-1}}^{-1}[(\tilde{O}_1 \times \cdots \times \tilde{O}_{n_{k-1}}) \cap W_{n_{k-1}}] \cap S \subseteq (\cap_{i=1}^{k-1} \rho_{n_i}^{-1}[U_{n_i}]) \cap S.$$

Since $y(n_k) \in U_{n_k}$, there exist $O_1^*, O_2^*, \dots, O_{n_k}^*$ with O_i^* open in Y_i such that

$$y(n_k) \in (O_1^* \times \cdots \times O_{n_k}^*) \cap W_{n_k} \subseteq U_{n_k}.$$

Now let $O_i = \tilde{O}_i \cap O_i^*$ for $1 \le i \le n_{k-1}$ and $O_i = O_i^*$ for $n_{k-1} < i \le n_k$. Since $y \in S$, $y(n_{k-1})$ appears as the first n_{k-1} components of $y(n_k)$. It follows that $y(n_k) \in (O_1 \times \cdots \times O_{n_k}) \cap W_{n_k}$. Now let $w \in (\rho_{n_k}^{-1}[(O_1 \times \cdots \times O_{n_k}) \cap W_{n_k}]) \cap S$. Since $O_i \subseteq O_i^*$, we have $w(n_k) \in U_{n_k}$ and $w \in \rho_{n_k}^{-1}[U_{n_k}]$. Since $O_i \subseteq \tilde{O}_i$ for $1 \le i \le n_{k-1}$ and, because $w \in S$, $w(n_{k-1})$ appears as the first n_{k-1} components of $w(n_k)$, we also have $w \in (\cap_{i=1}^{n_{k-1}} \rho_{n_i}^{-1}[U_{n_i}]) \cap S$. Combining all this, we obtain the required

$$y \in \rho_{n_k}^{-1}[(O_1 \times O_2 \times \cdots \times O_{n_k}) \cap W_{n_k}] \cap S \subseteq (\cap_{i=1}^{k} \rho_{n_i}^{-1}[U_{n_i}]) \cap S.$$

Proposition R4.6 (S, f) is a T_2 compactification of X.

Proof: S is T_2 and compact by virtue of its position as a closed subspace of a compact, T_2 space. Since f is an embedding from above, only the density of $f[X]$ remains to be shown. Let B be a non-empty open subset of S. For any $y \in B$, by R4.5 there exist m and $O_1, O_2, \dots, O_m$ with O_i open in Y_i and $y \in \rho_m^{-1}[(O_1 \times \cdots \times O_m) \cap W_m] \cap S \subseteq B$. Since $F_m[X]$ is dense in W_m, there exists $x \in X$ with $F_m(x) = \rho_m(y_x) \in (O_1 \times \cdots \times O_m) \cap W_m$. Thus $y_x = f(x)$ is in B and so $f[X]$ is dense in S.

Theorem R4.7 $(S, f) = \bigvee_{i=1}^{\infty} (Y_i, f_i)$.

Proof: Let $g_j = \rho_j|_S$. Clearly $g_j : S \to W_j$ is continuous and $g_j \circ f = F_j$. Therefore the closed image of g_j must contain the dense $F_j[X]$ so that g_j is onto. Thus g_j is the map required by the definition to show that $(S, f) \ge (W_j, F_j)$. As noted above, $(W_j, F_j) = \bigvee_{i=1}^{j} (Y_i, f_i)$ and so by transitivity $(S, f) \ge (Y_j, f_j)$, i.e., (S, f) is an upper bound. Now suppose the T_2 compactification (Z, h) is also an upper bound, and, for each j, let $\phi_j : Z \to Y_j$ be the continuous surjection such that $\phi_j \circ h = f_j$. For $z \in Z$ let $\{x_t\}$ be a net in X such that $\{h(x_t)\}$ converges to z. For any j, by continuity the net $(\phi_1(h(x_t)), \dots, \phi_j(h(x_t))) = (f_1(x_t), \dots, f_j(x_t))$ converges to $(\phi_1(z), \dots, \phi_j(z))$, which consequently must be in W_j. Thus it is possible to define $\phi : Z \to \prod_{i=1}^{\infty} W_i$ by $\phi(z)(j) = (\phi_1(z), \dots, \phi_j(z))$. Since $\rho_j \circ \phi$ is continuous for all j, ϕ is continuous. For $x \in X$ and any j, $\phi(h(x))(j) =$

$(f_1(x), \ldots, f_j(x)) = y_x(j) = f(x)(j)$ and so $\phi \circ h = f$. This also shows that $f[X]$ is contained in the closed set $\phi[Z]$, i.e., $S \subseteq \phi[Z]$. Now let $w = \phi(z)$. Given positive integers $m \geq k$, $\pi_{mk}(w(m)) = (\phi_1(z), \ldots, \phi_k(z)) = w(k)$. Thus $w \in S$ and so $\phi[Z] = S$. The map ϕ shows that $(Z, h) \geq (S, f)$ and so (S, f) represents the least upper bound, as claimed.

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References

1. Dugundji, J., Topology, Allyn and Bacon, 1966.
2. This website, R3: Representation of Suprema

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The representation in this section is a corollary to R3.2.7, which can be obtained as follows. Let the index set Δ be $\{1, 2, 3, \ldots\}$ and let $\mathcal{C}$, the co-final subset of Δ^*, be $\{\{1\}, \{1, 2\}, \{1, 2, 3\}, \ldots, \{1, 2, 3, \ldots, n\}, \ldots\}$.