Finite-Point Compactifications

Let (X, τ) be a $T_{3\frac{1}{2}}$ space. A T_2 compactification of X, say (Y, f), is a finite-point compactification provided $|Y - f[X]|$ is finite. If such a compactification exists, clearly $f[X]$ would be open in Y, a fact which is equivalent to the local compactness of X. (See, for example, Wilansky [4].) Consequently, unless explicitly stated otherwise, (X, τ) is also assumed to be locally compact throughout this section. Notation and facts from [5] will be used freely. Only T_2 compactifications are considered.

General Topological Facts

In addition to the one-point compactification, which is described in most introductory topology books, arbitrary finite-point compactifications have been studied by Magill [2]. (Also see [3].) The general result, which assumes only a T_2 space, is as follows.

Theorem R5.1.1[Magill] Let (X, τ) be a Hausdorff space.
The following are equivalent:

i) X has an n-point compactification for some natural number n.

ii) X is locally compact and contains a compact subset K whose complement is the union of n pairwise disjoint open sets $\{G_i : i = 1, \ldots, n\}$ such that $K \cup G_i$ is not compact for each i.

Outline of proof: To see that i) implies ii), let (Z, g) be a T_2 compactification of X with $Z - g[X] = \{z_1, \ldots, z_n\}$. Pick pairwise disjoint open subsets of Z, $O_1, \ldots, O_n$, with $z_i \in O_i$. Then ii) can be verified for $G_i = g^{-1}[O_i]$ and $K = g^{-1}[Z - \cup_{i=1}^{n} O_i]$. The fact that z_i is in the Z-closure of $g[X] \cap O_i$ leads quickly to the non-compactness of $K \cup G_i$. For the converse, let $p_1, \ldots, p_n$ be n distinct objects not in X, let $Y = X \cup \{p_1, \ldots, p_n\}$, and let $f : X \to Y$ by $f(x) = x$. Let $\sigma = \{O \subseteq Y : O \cap X$ is open in X and $p_i \in O \Rightarrow (X - O) \cap G_i$ has compact closure in $X\}$. Then σ is a topology for Y and (Y, f) is an n-point compactification of X.

As in [2] a pairwise disjoint family $\{G_i : i = 1, \ldots, n\}$ of open sets whose union has a compact complement K such that $K \cup G_i$ is not compact for each i will be called an n-star of X. Given an n-star of X, the T_2 compactification constructed in the proof above will be called the n-point compactification determined by the n-star. The next proposition shows that such compactifications provide representatives of every finite-point compactification class.

Proposition R5.1.2 Let (Z, g) be an n-point compactification of X. Then there is an n-star for X such that the n-point compactification of X determined by this n-star is equivalent to (Z, g).

Outline of proof: As in the proof of R5.1.1, let $Z - g[X] = \{z_1, \ldots, z_n\}$ and pick pairwise disjoint open subsets of Z, $O_1, \ldots, O_n$, with $z_i \in O_i$. Then $\{G_i = g^{-1}[O_i] : i = 1, \ldots, n\}$ is an n-star for X. Let (Y, f) be the n-point compactification determined by this n-star. Define $h : Z \to Y$ by $h(g(x)) = f(x) = x$ on $g[X]$ and $h(z_i) = p_i$ for $i = 1, \ldots, n$. Then $h \circ g = f$ by definition and h is easily seen to be a bijection. For continuity, let O be open in Y and let $z \in h^{-1}[O]$. If $z = g(x)$ for some x, then $z \in g[f^{-1}[O \cap X]]$, which is an open subset of $h^{-1}[O]$. If $z = z_i$, then $p_i \in O$ so that $c_X(G_i \cap (X - O))$ is compact. Then $z_i \in O_i - g[c_X(G_i \cap (X - O))]$, which is an open subset of $h^{-1}[O]$. Thus h is continuous and, since Z is compact and Y is T_2, a homeomorphism.

Corollary R5.1.3 Let (Z, g) be an n-point compactification of X. For each natural number m with $m \leq n$, X has an m-point compactification.

Proof: Let $\{G_i : i = 1, \dots, n\}$ be an n-star determined by (Z, g). Let $G_m^* = \cup_{i=m}^n G_i$. Then $\{G_1, \dots, G_{m-1}, G_m^*\}$ is an m-star, which determines an m-point compactification of X.

Proposition R5.1.4 Let (Z_1, g_1) be an n-point compactification of X, and let (Z_2, g_2) be an m-point compactification of X. If (Z_1, g_1) is equivalent to (Z_2, g_2), then $n = m$.

Proof: Let $h : Z_1 \to Z_2$ be the homeomorphism with $h \circ g_1 = g_2$. That equation implies that h induces a bijection between the finite sets $Z_1 - g_1[X]$ and $Z_2 - g_2[X]$. Thus $n = m$.

Theorem R5.1.5 [Magill] Let $\{G_i : i = 1, \dots, n\}$ and $\{O_i : i = 1, \dots, n\}$ be n-stars for the space X. Let $K_1 = X - \cup\{G_i : i = 1, \dots, n\}$, and let (Y_1, f_1) and (Y_2, f_2) be the n-point compactifications determined by the n-stars. Then (Y_1, f_1) is equivalent to (Y_2, f_2) if and only if there exists a permutation σ of $\{1, \dots, n\}$ such that $(K_1 \cup G_i) \cap (X - O_{\sigma(i)})$ is compact for each i.

Outline of proof: Let $Y_1 = X \cup \{p_1, \dots, p_n\}$ and $Y_2 = X \cup \{q_1, \dots, q_n\}$ with topologies and embeddings as described above. First assume the two compactifications are equivalent, so that there is a homeomorphism $h : Y_1 \to Y_2$ with $h \circ f_1 = f_2$, i.e., $h|_X$ is the identity map. h induces a permutation σ, where $\sigma(i) = j$ when $h(p_i) = q_j$. Fix i. Since $\{q_{\sigma(i)}\} \cup O_{\sigma(i)}$ is open in Y_2, its inverse image under h, $\{p_i\} \cup O_{\sigma(i)}$, is open in Y_1, so that $(X - O_{\sigma(i)}) \cap G_i$ has compact closure in X. It follows easily that the X-closed set $(K_1 \cup G_i) \cap (X - O_{\sigma(i)})$ is contained in a compact set and so is compact itself. For the converse, define $h : Y_1 \to Y_2$ by $h(x) = x$ for $x \in X$ and $h(p_i) = q_{\sigma(i)}$. For O open in Y_2, $h^{-1}[O] \cap X = O \cap X$, which is open in X. If $p_i \in h^{-1}[O]$, then $(X - O) \cap G_i$ is contained in $[(K_1 \cup G_i) \cap (X - O_{\sigma(i)})] \cup [(X - O) \cap O_{\sigma(i)}]$ and so has compact closure. Thus h is continuous. Clearly h is the homeomorphism required to show that the two compactifications are equivalent.

Let $\mathbb{R}$ denote the reals, $\mathbb{C}$ the complex plane, and $\mathbb{R}^m$ m-dimensional space, all with the usual topologies. Magill presents the following examples.

Corollary R5.1.6 $\mathbb{C}$ and $\mathbb{R}^m$ with $m \geq 2$ do not have n-point compactifications for $n \geq 2$.

Proof: By R5.1.3 it is sufficient to show the non-existence of 2-point compactifications. Deny and let $\{G_1, G_2\}$ be a 2-star. Let B be a ball containing the complement of $G_1 \cup G_2$. For these spaces, the complement of B must be connected but $\{G_1, G_2\}$ would induce a separation. Contradiction.

Corollary R5.1.7 $\mathbb{R}$ has a 2-point compactification but does not have an n-point compactification for $n \geq 3$.

Proof: $\{(-\infty, 0), (0, \infty)\}$ is a 2-star for $\mathbb{R}$. Now suppose $\{G_1, G_2, G_3\}$ is a 3-star for $\mathbb{R}$, and let $\mathbb{R} - (G_1 \cup G_2 \cup G_3)$ be contained in $[a, b]$. Since $(-\infty, a)$ and (b, ∞) are connected, each has non-empty intersection with at most one G_i. The leftover G_i would have to be contained in $[a, b]$, which leads to a contradiction.

Corollary R5.1.8 All 2-point compactifications of $\mathbb{R}$ are equivalent.

Proof: Let $\{G_1, G_2\}$ and $\{O_1, O_2\}$ be 2-stars of $\mathbb{R}$, let $K_1 = \mathbb{R} - (G_1 \cup G_2)$, and suppose $K_1 \subseteq [a, b]$. Since $(-\infty, a)$ and (b, ∞) are both connected, each must be entirely contained in one G_i and one O_i. Use that fact to define σ. Without loss of generality, assume $(-\infty, a)$ is a subset of G_1 and $O_{\sigma(1)}$, while (b, ∞) is contained in G_2 and $O_{\sigma(2)}$. Then, for $i \in \{1,2\}$, the closed set $(K_1 \cup G_i) \cap (\mathbb{R} - O_{\sigma(i)})$ is contained in $[a,b]$, and so the compactifications are equivalent by R5.1.5.

For what follows certain equivalence relations closely related to n-stars will be used. As is clear from the following definition, each n-compatible equivalence relation on X determines one n-star, while an n-star determines at least one n-compatible equivalence relation.

Definition R5.1.9 Let (X, τ) be a T_2 space. An equivalence relation E on X is n-compatible provided E has finitely many distinct equivalence classes, exactly n of which form an n-star of X.

If E is an n-compatible equivalence relation on X, (Y, ι_E) will denote the n-point compactification determined as above by the associated n-star, and $\tau(E)$ will denote the topology for Y. The following facts show that these notions simplify in the discrete case.

Proposition R5.1.10 Let X be an infinite discrete space and let E be an equivalence relation on X. Then E is n-compatible if and only if E has finitely many distinct equivalence classes, exactly n of which are infinite.

Proof: This follows easily because distinct equivalence classes must be disjoint and compactness is equivalent to finiteness in a discrete space.

Proposition R5.1.11 Let X be an infinite discrete space and E an n-compatible equivalence relation on X. Let $\{e_1, \ldots, e_n\}$ be the distinct infinite equivalence classes of E. Then $\tau(E) = \{O \subseteq Y : p_i \in O \Rightarrow (X - O) \cap e_i \text{ is finite } \}$.

Proof: This follows easiliy since every $X \cap O$ is open and having compact closure in X is equivalent to finiteness.

Proposition R5.1.12 Let X be an infinite discrete space and let D and E be n-compatible equivalence relations on X. Let $\{d_1, \ldots, d_n\}$ and $\{e_1, \ldots, e_n\}$ be the distinct infinite equivalence classes of D and E respectively. Then (Y, ι_D) and (Y, ι_E) are equivalent compactifications if and only if there is a permutation σ of $\{1, \ldots, n\}$ with the property that $d_i \cap (X - e_{\sigma(i)})$ is finite for each i.

Proof: This merely restates R5.1.5 in the present context.

Magill uses infinite discrete spaces as examples which have infinitely many non-equivalent n-compactifications for $n \geq 2$. Such examples are implicit in R5.1.12. It also provides simple examples of non-equivalent compactifications which are homeomorphic. With X discrete, it can be shown that, in the notation of R5.1.12, $(Y, \tau(D))$ and $(Y, \tau(E))$ are homeomorphic if $|X| = \aleph_0$ or if a σ exists such that $|d_i| = |e_{\sigma(i)}|$ for each i.

Uniform Space Constructions

Basic facts and notation for uniform spaces, which will be used in this subsection, can be found in [6].

Definition R5.2.1 [1] Let E be an equivalence relation on set X. $\mathcal{U}_E$ denotes $\{U : X \times X \supseteq U \supseteq E\}$.

Lemma R5.2.2 [1] Let E be an equivalence relation on X. Then $\mathcal{U}_E$ is a uniformity for X, and $\mathcal{U}_E$ is totally bounded if and only if E has finitely many distinct equivalence classes.

Proof: The key to the first assertion is that $E \circ E = E$; the second follows easily since total boundedness is, in this case, equivalent to the equation $X = \bigcup_{i=1}^{n} E[x_i]$ for some finite set $x_1 \ldots x_n$.

Recall the following notation from [8]: For (X, τ) a non-compact locally compact Hausdorff space, $\mathcal{U}_m$ denotes $\{U : U \supseteq \bigcup_{i=1}^n O_i \times O_i$ where $O_1, \ldots, O_n$ are an open cover of X and at least one O_i has a compact complement$\}$. It is shown in [8] that $\mathcal{U}_m$ is a totally bounded uniformity with $\tau(\mathcal{U}_m) = \tau$ and that a separated completion of $(X,\mathcal{U}_m)$ determines the compactification class of the one-point compactification for X, i.e., in the notation of [8], $\Psi_0(\mathcal{U}_m) = [(X^+, \iota^+)]$. Also recall that $\mathcal{TB}(X)$ denotes the set of totally bounded uniformities on X that generate τ.

Proposition R5.2.3 Let (X, τ) be a non-compact, locally compact T_2 space. Let E be an n-compatible equivalence relation on X with each E equivalence class open in τ. Then $\mathcal{U}_m \vee \mathcal{U}_E \in \mathcal{TB}(X)$ and $\Psi_0(\mathcal{U}_m \vee \mathcal{U}_E) = [(Y, \iota_E)]$.

Proof: By P2.13 $\mathcal{U}_m \vee \mathcal{U}_E$ is totally bounded and and by P2.14 $\tau(\mathcal{U}_m \vee \mathcal{U}_E) = \tau \vee \tau(\mathcal{U}_E)$. Since $E[x] \in \tau$ for all x, $\tau(\mathcal{U}_E) \subseteq \tau$ and so $\mathcal{U}_m \vee \mathcal{U}_E \in \mathcal{TB}(X)$.

Now let $O_1, \ldots, O_n$ denote the equivalence classes of E which form the n-star, and let $\mathcal{V} \in \mathcal{TB}(X)$ be such that $\Psi_0(\mathcal{V}) = [(Y, \iota_E)]$. Note that $\mathcal{V}$ is simply the subspace uniformity on X induced from the unique uniformity for Y, i.e., the collection of all neighborhoods of the diagonal in $Y \times Y$. (See P2.4 in [6].) One such neighborhood is $N = \bigcup_{i=1}^{n+j} G_i \times G_i$ where $G_i = O_i \cup \{p_i\}$ for $i = 1, \ldots, n$ and $G_{n+1}, \ldots, G_{n+j}$ are the remaining equivalence classes of E. Clearly $N \cap (X \times X) = E$ and so $\mathcal{U}_E \subseteq \mathcal{V}$. Since $[(X^+, \iota^+)] \leq [Y, \iota_E)]$, by R1.5 $\mathcal{U}_m \subseteq \mathcal{V}$. Thus $\mathcal{U}_m \vee \mathcal{U}_E \subseteq \mathcal{V}$.

To verify the reverse containment, let $V \in \mathcal{V}$, and let M be a neighborhood of the diagonal in Y such that $V = (X \times X) \cap M$. For each $x \in X$, there exists $O_x \in \tau$ with $O_x \times O_x \subseteq M$. Also there exist $H_1, \ldots, H_n$ open in Y with $p_i \in H_i$ and $H_i \times H_i \subseteq M$. For $S = \cup_{i=1}^n O_i$, the complement is compact and so there is a finite set Δ_0 such that $X - S \subseteq \cup\{O_x : x \in \Delta_0\}$. Let $U_0 = (\cup\{O_x \times O_x : x \in \Delta_0\}) \cup (S \times S)$. Clearly $U_0 \in \mathcal{U}_m$. Since $p_i \in H_i$ and each O_i is clopen, $T_i = (X - H_i) \cap O_i$ is compact, and so there is Δ_i such that $T_i \subseteq \cup\{O_x \times O_x : x \in \Delta_i\}$. Let $U_i = (\cup\{O_x \times O_x : x \in \Delta_i\}) \cup (X - T_i) \times (X - T_i)$. Then $U_1, \ldots, U_n$ are also in $\mathcal{U}_m$. To finish it is sufficient to show that $([\cap_{i=0}^n U_i] \cap E) \subseteq V$. Let (x, y) be in the intersection with $x \neq y$. If (x, y) is in $\cup\{O_x \times O_x : x \in \Delta_i\}$ for any i, clearly (x, y) is in V. Thus assume (x, y) is in $S \times S$ and $(X - T_i) \times (X - T_i)$ for every i. Since (x, y) is in both E and $S \times S$, $x, y \in O_j$ for some j. Then $x, y \notin T_j$ implies $x, y \in H_j$. Thus $(x, y) \in H_j \times H_j$, which yields $(x, y) \in V$.

Proposition R5.2.4 Let X be an infinite discrete space. Let E be an n-compatible equivalence relation on X. Then $\mathcal{U}_m \vee \mathcal{U}_E \in \mathcal{TB}(X)$ and $\Psi_0(\mathcal{U}_m \vee \mathcal{U}_E) = [(Y, \iota_E)]$.

Proof: In the discrete case the assumption that every E-equivalence class is open is automatically satisfied. This is a special case of R5.2.3.

Note that for two n-compatible equivalence relations on a discrete X, E and F, R5.1.12 and R1.5 can be combined to characterize $\mathcal{U}_m \vee \mathcal{U}_E = \mathcal{U}_m \vee \mathcal{U}_F$. It can also be shown that $(X,\mathcal{U}_m \vee \mathcal{U}_E)$ and $(X,\mathcal{U}_m \vee \mathcal{U}_F)$ are unimorphic if there ia a one-to-one correspondence between the infinite equivalence classes of E and F such that corresponding classes have the same cardinality. This leads to examples of unimorphic spaces which determine non-equivalent compactifications.

Normal Basis Constructions for Discrete Spaces

Basic facts and notation used here can be found in [7]. Throughout this subsection

X will denote an infinite discrete space and E an n-compatible equivalence relation on X with distinct infinite equivalence classes $C_1, \ldots, C_n$.

Definition R5.3.1 Let $S \subseteq X$ and let $\Delta \subseteq \{1, \ldots, n\}$. S is associated with Δ if and only if $S \cap C_i$ is finite for all $i \in \Delta$ and $(X - S) \cap C_i$ is finite for all $i \notin \Delta$.

Definition R5.3.2
$\mathcal{Z}(E) = \{S \subseteq X : S$ is associated with Δ for some $\Delta \subseteq \{1, \ldots, n\}\}$.

Proposition R5.3.3 $\mathcal{Z}(E)$ is a normal basis for X.

Proof: Note that finite subsets of X are associated with $\{1, \ldots, n\}$ and so are in $\mathcal{Z}(E)$. Also, if $Z \in \mathcal{Z}(E)$ is associated with Δ, then $X - Z$ is associated with $\{1, \ldots, n\} - \Delta$ and thus is also in $\mathcal{Z}(E)$. Since $x \notin S$ means $S \subseteq X - \{x\}$, $\mathcal{Z}(E)$ is a base for the closed sets. For $Z_1, Z_2 \in \mathcal{Z}(E)$ associated with Δ_1, Δ_2 respectively, $Z_1 \cup Z_2$ is associated with $\Delta_1 \cap \Delta_2$ and $Z_1 \cap Z_2$ is associated with $\Delta_1 \cup \Delta_2$. Thus $\mathcal{Z}(E)$ is closed under finite unions and intersections. The third requirement of definition P3.1 is satisfied because, for $x \notin S$, $\{x\} \in \mathcal{Z}(E)$ and $S \cap \{x\} = \emptyset$. The fourth is equally straightforward: for $Z_1, Z_2 \in \mathcal{Z}(E)$ with $Z_1 \cap Z_2 = \emptyset$, $X - Z_1$ and $X - Z_2$ are in $\mathcal{Z}(E)$ and form the needed cover.

Definition R5.3.4 Let $i \in \{1, \ldots, n\}$.
$\mathcal{G}_i = \{S \in \mathcal{Z}(E) : S$ is associated with some Δ contained in $\{1, \ldots, n\} - \{i\}\}$.

Lemma R5.3.5 An element of $\mathcal{Z}(E)$ is associated with a unique subset of $\{1, \ldots, n\}$.

Proof: Deny and pick Z in $\mathcal{Z}(E)$ associated with both Δ_1 and Δ_2. For any i in $(\Delta_1 - \Delta_2) \cup (\Delta_2 - \Delta_1)$, both $Z \cap C_i$ and $(X - Z) \cap C_i$ must be finite, which contradicts the assumption that C_i is infinite.

Proposition R5.3.6 For $i \in \{1, \ldots, n\}$, $\mathcal{G}_i$ is a $\mathcal{Z}(E)$-ultrafilter.

Proof: The co-finite subsets of X, being associated with $\emptyset$, are in $\mathcal{G}_i$, while $\emptyset$ is not since it is associated with $\{1, \ldots, n\}$. Let S_1, S_2 in $\mathcal{Z}(E)$ be associated with Δ_1 and Δ_2 respectively. Since $S_1 \cap S_2$ is associated with $\Delta_1 \cup \Delta_2$, clearly $\mathcal{G}_i$ is closed under finite intersections. If $S_1 \in \mathcal{G}_i$ and $S_1 \subseteq S_2$, then $\Delta_2 \subseteq \Delta_1$ so that $S_2 \in \mathcal{G}_i$. Thus $\mathcal{G}_i$ is a $\mathcal{Z}(E)$-filter. Now suppose $\mathcal{F}$ is a $\mathcal{Z}(E)$-filter with $\mathcal{G}_i \subseteq \mathcal{F}$. If $Z \in \mathcal{F}$ is associated with Δ and $i \in \Delta$, then $X - Z$, which is associated with $\{1, \ldots, n\} - \Delta$, must be in $\mathcal{G}_i$. That implies $Z \cap (X - Z) \in \mathcal{F}$, a contradiction. Thus $\mathcal{G}_i$ is a $\mathcal{Z}(E)$-ultrafilter.

Proposition R5.3.7 The distinct, non-point ultrafilters in $\omega(\mathcal{Z}(E))$ are $\mathcal{G}_1, \ldots, \mathcal{G}_n$.

Proof: For all i, C_i is associated with $\{1, \ldots, n\} - \{i\}$ and so $C_i \in \mathcal{G}_i$ and $\mathcal{G}_i \neq \mathcal{G}_j$ if $j \neq i$. Since all finite sets are associated with $\{1, \ldots, n\}$, the ultrafilter $\mathcal{G}_i$ does not contain any finite set. Thus $\mathcal{G}_1, \ldots, \mathcal{G}_n$ are distinct, non-point $\mathcal{Z}(E)$-ultrafilters. Now let $\mathcal{F}$ be a $\mathcal{Z}(E)$-ultrafilter with $\mathcal{F} \neq \mathcal{G}_i$ for all i. Pick F_i associated with Δ_i such that $F_i \in \mathcal{F}$ but $F_i \notin \mathcal{G}_i$. Then $i \in \Delta_i$ for each i. Let $F = \cap_{i=1}^n F_i$. F is in $\mathcal{F}$ and is associated with $\cup_{i=1}^n \Delta_i = \{1, \ldots, n\}$. That means F is finite. Since only point-ultrafilters contain any finite sets, $\mathcal{F}$ must be $\mathcal{F}_x$ for some x.

Proposition R5.3.8 $(\omega(\mathcal{Z}(E)), \iota_{\mathcal{Z}(E)})$ is equivalent to (Y, ι_E).

Proof: Define $h : \omega(\mathcal{Z}(E)) \to Y$ by $h(\mathcal{F}_x) = x$ and $h(\mathcal{G}_i) = p_i$. Clearly h is one-to-one and onto, and $h \circ \iota_{\mathcal{Z}(E)} = \iota_E$. Since the spaces are compact and T_2, continuity of h is sufficient to show that h is the homeomorphism required for equivalence. Let F be closed in Y and suppose $\mathcal{F} \notin h^{-1}[F]$. If $\mathcal{F} = \mathcal{F}_x$ for some x, then $h^{-1}[F] \subseteq (X - \{x\})^\omega$ and $\mathcal{F} \notin (X - \{x\})^\omega$. Now suppose $\mathcal{F} = \mathcal{G}_i$ for some i. Then $p_i \in Y - F$ so that $(X - (Y - F)) \cap C_i$ is finite. Let $Z = (X - (Y - F)) \cup (\cup\{C_j : j \neq i\}$. Then Z is associated with $\{i\}$ so that $Z \in \mathcal{Z}(E)$, $Z \notin \mathcal{G}_i$, and $Z \in \mathcal{G}_j$ for $j \neq i$. Since $X \cap F \subseteq Z$, it follows that $h^{-1}[F] \subseteq Z^\omega$ and $\mathcal{G}_i \notin Z^\omega$. From the description of the closed sets in $\omega(\mathcal{Z}(E))$ (P3.6 in [7]), $h^{-1}[F]$ is closed and so h is continuous as required.

Note that R5.1.2 and R5.3.8 show that every finite-point compactification of a discrete space can be constructed from a normal basis.

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Added 2013

This addendum points out that not every finite point compactification class corresponds to a uniformity of the form $\mathcal{U}_m \vee \mathcal{U}_E$.

Lemma R5.Add.1 Let X be a set and let E be an equivalence relation. Then for every $x \in X$, $E[x]$ is $\tau(\mathcal{U}_E)$-clopen.

Proof: Let x be in X and let $t \in E[x]$. $E[t]$ is a $\tau(\mathcal{U}_E)$-neighborhood of t and $E[t] = E[x]$ since E-sections are equivalence classes and xEt. Thus $E[x]$ is a $\tau(\mathcal{U}_E)$-neighborhood of each of its points and so $E[x]$ is open. The complement of $E[x]$ is the union of the other equivalence classes and so open. Thus $E[x]$ is also $\tau(\mathcal{U}_E)$-closed.

Example R5.Add.2 Let $X = (0, 1)$ and $\mathcal{U}$ be the usual uniformity on X. Let $Y = [0, 1]$ and let $f : X \to Y$ be the inclusion map. Since $\mathcal{U}$ is the subspace uniformity of the usual uniformity on Y, $\mathcal{U}$ corresponds to the compactification class of (Y, f). Let $\mathcal{U}_m$ be the uniformity for X corresponding to the one-point compactification. Since (Y, f) is a two-point compactification, $\mathcal{U}_m$ is a proper subset of $\mathcal{U}$. Suppose there is an equivalence relation E on X such that $\mathcal{U}_m \vee \mathcal{U}_E = \mathcal{U}$. Since X is connected and $E[x]$ is clopen in $\tau(\mathcal{U}_E) \subseteq \tau(\mathcal{U}_m \vee \mathcal{U}_E) = \tau(\mathcal{U})$, $E = X \times X$. But then $\mathcal{U}_m \vee \mathcal{U}_E = \mathcal{U}_m$, a contradiction.

Added 2018

Much of this section focused on finite point compactifications of discrete spaces. This note points out a way to construct non-discrete spaces with finite point compactifications and characterizes spaces whose Stone-Čech compactification is a finite point compactification.

Lemma R5.Add.3 Let (X, τ) be a $T_{3\frac{1}{2}}$ space, let (Y, f) be a T_2 compactification of (X, τ), and let $S \subseteq Y - f[X]$. Let $Z = Y - S$ have the relative topology from Y. Then (Y, s) is a T_2 compactification of Z, where $s : Z \to Y$ is the inclusion map.

Proof: Since the dense $f[X]$ is contained in Z, Z is dense in Y. Since Z has the relative topology, s is an embedding.

In the last lemma $Y - s[Z] = S$ and so, if S is finite, Z has a finite point compactification.

Lemma R5.Add.4 Let (X, τ) be a $T_{3\frac{1}{2}}$ space, let $(\beta X, \iota)$ be the Stone-Čech compactification of (X, τ), and let $S \subseteq \beta X - \iota[X]$. Let $Z = \beta X - S$ have the relative topology from βX. Then $(\beta X, s)$ is the Stone-Čech compactification of Z, where $s : Z \to \beta X$ is the inclusion map.

Proof: It is sufficient to show that every continuous map from Z to a compact T_2 space has a contiuous extension to βX. Let $h : Z \to K$ be continuous, where K is compact and T_2. Let g be h restricted to $\iota[X]$. Then g has a continuous extension G to βX. Then $G|_{\iota[X]} = g = h|_{\iota[X]}$. Since $\iota[X]$ is dense in Z and K is T_2, $G|_Z = h$, i.e., G is a continuous extension of h.

By choosing S finite, one obtains a space whose Stone-Čech compactification is a finite point compactification. The rest of this added note characterizes such spaces.

Lemma R5.Add.5 Let (X, τ) be a $T_{3\frac{1}{2}}$ space, let (Y, f) be a T_2 compactification of (X, τ), let A be a dense subset of X, and let τ_A be the relative topology on A from X. Then $(Y, f|_A)$ is a T_2 compactification of (A, τ_A).

Proof: Since f is an embedding and A has the relative topology, $f|_A$ is also an embedding. Clearly its image is $f[A]$. Since A is dense in X, $f[A]$ is dense in $f[X]$, which is dense in Y. Thus $f[A]$ is dense in Y.

Lemma R5.Add.6 Let (X, τ) be a $T_{3\frac{1}{2}}$ space, let (Y, f) and (Z, g) be a T_2 compactifications of (X, τ), and let A be a dense subset of X. If $(Y, f|_A)$ is equivalent to $(Z, g|_A)$, then (Y, f) is equivalent to (Z, g).

Proof: Assume $\phi : Y \to Z$ is a homeomorphism with $\phi \circ f|_A = (\phi \circ f)|_A = g|_A$. Since these are continuous maps into the T_2 space Z which agree on a dense subset of the domain, $\phi \circ f = g$. By definition (Y, f) is equivalent to (Z, g).

Proposition R5.Add.7 Let (X, τ) be a non-compact $T_{3\frac{1}{2}}$ space which has exactly M distinct compactification classes, where M is a positive integer. Then its Stone-Čech compactification, $(\beta X, \iota)$, is a finite point compactification with $|\beta X - \iota[X]| \leq M$.

Proof: Assume $|\beta X - \iota[X]| \geq M+1$. Let $Y = \beta X - \{t_1, \ldots, t_{M+1}\}$, where $t_1, \ldots, t_{M+1}$ are distinct elements of $\beta X - \iota[X]$. By R5.Add.3 βX with the inclusion map is a compactification of Y with $|\beta X - Y| = M + 1$. For each $1 \leq k \leq M + 1$, by R5.1.3, there is a k-point compactification (Z_k, f_k) of Y. By R5.1.4, if $k \neq l$, (Z_k, f_k) is not equivalent to (Z_l, f_l). Since $\iota[X]$ is dense in Y, by R5.Add.6, these $M + 1$ compactifications of Y induce $M + 1$ non-equivalent compactifications of $\iota[X]$. But $\iota[X]$, a homeomorph of X, has exactly M distinct compactification classes, a contradiction.

The next few results will yield the other half of the characterization.

Lemma R5.Add.8 Let (X, τ) be a $T_{3\frac{1}{2}}$ space, let (Y, f) and (Z, g) be a T_2 compactifications of (X, τ), and let $\phi : Y \to Z$ be continuous and onto with $\phi \circ f = g$. Then, for every $x \in X$, $\phi^{-1}[\{g(x)\}] = \{f(x)\}$.

Proof: Fix $x \in X$. Since $\phi(f(x)) = g(x)$, $f(x) \in \phi^{-1}[\{g(x)\}]$. Let $y \in \phi^{-1}[\{g(x)\}]$. There is a net $S : D \to X$ such that $f \circ S$ converges to y. By continuity, $\phi \circ (f \circ S)$ converges to $\phi(y) = g(x)$. Thus $(\phi \circ f) \circ S = g \circ S$ converges to $g(x)$. Since $g : X \to g[X]$ is a homeomorphism, S converges to x and so $f \circ S$ converges to $f(x)$. Since limits are unique in a T_2 space, $y = f(x)$.

The relation $[(Z, g)] \leq [(Y, f)]$ is defined by the existence of ϕ as in the previous lemma, and such a ϕ must be unique. $\mathcal{P}(Z)$ will denote the partition of Y induced by ϕ, i.e., $\{\phi^{-1}[\{z\}] : z \in Z\}$.

Next recall some general facts: If A is compact, B is T_2, and $m : A \to B$ is continuous and onto, then m is a quotient map and so B is homeomorphic to the quotient space A/E, where E is the equivalence relation determined by the partition $\{m^{-1}[\{b\}] : b \in B\}$. The map $b \mapsto m^{-1}[\{b\}]$ is a homeomorphism.

Lemma R5.Add.9 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let (Y, f), (W, h) and (Z, g) be T_2 compactifications of (X, τ) with $[(Z, g)] \leq [(Y, f)]$ and $[(W, h)] \leq [(Y, f)]$. Assume $\mathcal{P}(Z) = \mathcal{P}(W)$. Then (Z, g) is equivalent to (W, h).

Proof: Let $\phi : Y \to Z$ and $\psi : Y \to W$ be continuous and onto with $\phi \circ f = g$ and $\psi \circ f = h$. Since $\mathcal{P}(Z) = \mathcal{P}(W)$, Z and W are homeomorphic to the same quotient space, Y/E, where E is the equivalence relation determined by $\mathcal{P}(Z) = \mathcal{P}(W)$. Let $\rho : Z \to Y/E$ and $\sigma : W \to Y/E$ be the homeomorphisms given by $\rho(z) = \phi^{-1}[\{z\}]$ and $\sigma(w) = \psi^{-1}[\{w\}]$. Then $\sigma^{-1} \circ \rho$ is a homeomorphism from Z onto W. It is sufficient to show $(\sigma^{-1} \circ \rho) \circ g = h$. Let $x \in X$. Then $\rho \circ g(x) = \phi^{-1}[\{g(x)\}] = \{f(x)\}$ by R5.Add.8. By the same lemma, $\psi^{-1}[\{h(x)\}] = \{f(x)\}$ so that $\sigma^{-1}(\{f(x)\}) = h(x)$. Thus the claim holds.

Proposition R5.Add.10 Let (X, τ) be a non-compact $T_{3\frac{1}{2}}$ space and let $(\beta X, \iota)$ be the Stone-Čech compactification of (X, τ). Then βX is a finite point compactification of X if and only if the number of distinct compactification classes of (X, τ) is finite.

Proof: The sufficiency of the condition follows from R5.Add.7. For necessity, assume βX is a finite point compactification of X. For any (Y, f), a compactification of (X, τ), since $[(Y, F)] \leq [(\beta X, \iota)]$, Y is homeomorphic to a quotient space $\beta X/E$. By R5.Add.8 the partition determining E is the union of $\{\{\iota(x)\} : x \in X\}$ and a partition of $\beta X - \iota[X]$. Since $\beta X - \iota[X]$ is finite, it has finitely many partitions. By R5.Add.9 the number of distinct compactification classes of (X, τ) is finite.

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Given (Y, f), a T_2 compactification of a $T_{3\frac{1}{2}}$ space (X, τ), the compact T_2 space Y will be called the target space of the compactification. In what follows, an example of two non-equivalent finite-point compactifications with homeomorphic target spaces is presented. The target space of the supremum is not homeomorphic to either. Finally, it is shown that all 2-point compactifications of a countably infinite discrete space have homeomorphic target spaces.

Lemma R5.Add.11 Let (X, τ) be a discrete topological space with X infinite. Let A be an infinite subset of X with an infinite complement. Then $\{A, X - A\}$ is a 2-star for X.

Proof: By hypothesis $\{A, X - A\}$ is a disjoint pair of non-compact open sets. The complement of their union is $\emptyset$, which is compact. By definition $\{A, X - A\}$ is a 2-star.

Lemma R5.Add.12 Let (X, τ) be a discrete topological space with X countably infinite. Let A, B be infinite subsets of X with both having infinite complements. Then the 2-point compactifications generated by the 2-stars $\{A, X-A\}$ and $\{B, X-B\}$ have homeomorphic target spaces.

Proof: Representations of the compactifications as in R5.1.1 will be used: $Y = X \cup \{a_1, a_2\}$ and $Z = X \cup \{b_1, b_2\}$, where $a_1 \neq a_2$, $b_1 \neq b_2$, and $X \cap \{a_1, a_2, b_1, b_2\} = \emptyset$. The topology for Y is α, where $O \subseteq Y$ is in α if and only if $a_1 \in O$ implies $(X - O) \cap A$ is finite and $a_2 \in O$ implies $(X - O) \cap (X - A)$ is finite. The topology for Z is β, where $G \subseteq Y$ is in β if and only if $b_1 \in G$ implies $(X - G) \cap B$ is finite and $b_2 \in G$ implies $(X - G) \cap (X - B)$ is finite. The embeddings are the inclusion maps, $f : X \to Y$ and $g : X \to Z$. By hypothesis A, B and their complements are all countably infinite and so there is σ, a permutation of X, such that $\sigma[A] = B$. Define $h : Y \to Z$ by $h|_X = \sigma$, $h(a_1) = b_1$, and $h(a_2) = b_2$. Clearly, h is one-to-one and onto. Now let $G \in \beta$. It is easy to check that $X - h^{-1}[G] = h^{-1}[X - G]$. Suppose $a_1 \in h^{-1}[G]$. Then $b_1 \in G$ and so $(X - G) \cap B$ is finite. Then $(X - h^{-1}[G]) \cap A = h^{-1}[X - G] \cap h^{-1}[B]$, which is $h^{-1}[(X - G) \cap B]$ and finite, since h is one-to-one. Next suppose $a_2 \in h^{-1}[G]$ so that $b_2 \in G$. Since $X - A = X - h^{-1}[B] = h^{-1}[X - B]$, a similar argument shows that $(X - h^{-1}[G]) \cap (X - A)$ is finite. By definition $h^{-1}[G] \in \alpha$ so that h is continuous. Since Y is compact and Z is T_2, h is a homeomorphism.

Example R5.Add.13 Let $X = \mathbb{N}$ have the discrete topology, let A be the even positive integers, and let B be the multiples of 3. Let (Y, f) and (Z, g) be the 2-point compactifications determined as in the previous proof by the 2-stars $\{A, X - A\}$ and $\{B, X - B\}$ respectively. Since $A \cap B$ and $A \cap (X - B)$ are both infinite, by R5.1.5 (Y, f) and (Z, g) are not equivalent. By the previous lemma the target spaces, Y and Z, are homeomorphic.

Example R5.Add.14 Continue with X, (Y, f), and (Z, g) as in the previous example. Let W be the closure in $Y \times Z$ of $\{(x, x) : x \in X\}$ and let $w : X \to W$ by $w(x) = (x, x)$. By R3.1.1 and R3.1.2 (W, w) is a T_2 compactification of X and represents the supremum class for (Y, f) and (Z, g). Let $S = \{(x, x) : x \in X\} \cup (\{a_1, a_2\} \times \{b_1, b_2\})$. It is claimed that $W = S$. First, $(Y \times Z) - S$ is open in $Y \times Z$: Let $(p, q) \in (Y \times Z) - S$. If $p, q \in X$, $\{(p, q)\}$ is open in $Y \times Z$. If $p \in X$ and $q \in \{b_1, b_2\}$, (p, q) is in one of the $Y \times Z$-open sets $\{p\} \times ((B - \{p\}) \cup \{b_1\})$ and $\{p\} \times (((X - B) - \{p\}) \cup \{b_2\})$, both of which are contained in $(Y \times Z) - S$. If $p \in \{a_1, a_2\}$ and $q \in X$, proceed similarly by using the Y-open sets $(A - \{q\}) \cup \{a_1\}$ and $((X - A) - \{q\}) \cup \{a_2\}$. Thus $(Y \times Z) - S$ is open, S is closed, and $W \subseteq S$. Next it will be shown that $\{a_1, a_2\} \times \{b_1, b_2\} \subseteq W$. There are four cases. First, (a_2, b_1) will be shown to be in W. Let $(a_2, b_1) \in O \times G$, where $O \in \alpha$ and $G \in \beta$. By definition of the topologies $(X - O) \cap (X - A)$ and $(X - G) \cap B$ are both finite. By the choice of A and B, $(X - A) \cap B$ is infinite and there is t in $(X - A) \cap B$ with $t \notin ((X - O) \cap (X - A)) \cup ((X - G) \cap B)$. Then t is in both O and G, i.e., $(t, t) \in \{(x, x) : x \in X\} \cap (O \times G)$. Thus $(a_2, b_1) \in W$. The other three cases are done in a similar way by using the appropriate choice from the infinite sets $A \cap B$, $A \cap (X - B)$, and $(X - A) \cap (X - B)$. It follows that $W = S$. Finally, suppose W and Y are homeomorphic, and let $H : W \to Y$ be a homeomorphism. For any t in X, $\{(t, t)\}$ is open in W. Since the singletons $\{a_1\}$ and $\{a_2\}$ are not open in Y, $H((t, t))$ must be in X. Also, no singletons from $\{a_1, a_2\} \times \{b_1, b_2\}$ are open and so $H^{-1}(t)$ must be in $\{(x, x) : x \in X\}$ for all $t \in X$. Thus $H[\{(x, x) : x \in X\}] = X$ and so $H[\{a_1, a_2\} \times \{b_1, b_2\}] = \{a_1, a_2\}$, which contradicts the assumption that H is one-to-one. No such H exists, i.e., W and Y are not homeomorphic.

Lemma R5.Add.15 Let (X, τ) be a discrete topological space with X countably infinite and let (Y, f) be a 2-point compactification of X. Then X has an infinite subset A with $X - A$ also infinite such that (Y, f) is equivalent to the 2-point compactification generated by the 2-star $\{A, X - A\}$.

Proof: By R5.1.2 there is a 2-star $\{G_1, G_2\}$ for X such that the 2-point compactification generated by $\{G_1, G_2\}$ is equivalent to (Y, f). Let $K = X - (G_1 \cup G_2)$. By definition of a 2-star, K must be finite and both $K \cup G_1$ and $K \cup G_2$ are both infinite. Let $A = K \cup G_1$. Since $G_2 \cap G_1 = \emptyset = G_2 \cap K$, $X - A = G_2$. Since K is finite and $K \cup G_2$ is infinite, $X - A$ is infinite. R5.1.5 will be applied with $O_1 = A$ and $O_2 = X - A$. Let σ be the identity permutation of $\{1, 2\}$. $(K \cup G_1) \cap (X - O_1) = A \cap (X - A) = \emptyset$, which is finite. In addition, $(K \cup G_2) \cap (X - O_2) = (K \cup G_2) \cap A = K$, which is finite. By R5.1.5 the 2-point compactification generated by $\{G_1, G_2\}$ is equivalent to the 2-point compactification generated by $\{A, X - A\}$, and so the conclusion holds.

Corollary R5.Add.16 All 2-point compactifications of a countably infinite discrete space have homeomorphic target spaces.

Proof: Let X be a countably infinite discrete space, and let (Y, f) and (Z, g) be 2-point compactifications of X. By R5.Add.15 there exist A and B, infinite subsets of X, with both $X - A$, $X - B$ also infinite such that (Y, f) is equivalent to the 2-point compactification generated by the 2-star $\{A, X - A\}$ and (Z, g) is equivalent to to the 2-point compactification generated by the 2-star $\{B, X - B\}$. Since equivalent spaces have homeomorphic target spaces and homeomorphism is transitive, it follows from R5.Add.12 that Y is homeomorphic to Z.

Added Reference

9. This website, R3: Representation of Suprema