

Finite-Point Compactifications

Let (X, τ) be a $T_{3\frac{1}{2}}$ space. A T_2 compactification of X , say (Y, f) , is a finite-point compactification provided $|Y - f[X]|$ is finite. If such a compactification exists, clearly $f[X]$ would be open in Y , a fact which is equivalent to the local compactness of X . (See, for example, Wilansky [4].) Consequently, unless explicitly stated otherwise, (X, τ) is also assumed to be locally compact throughout this section. Notation and facts from [5] will be used freely. Only T_2 compactifications are considered.

General Topological Facts

In addition to the one-point compactification, which is described in most introductory topology books, arbitrary finite-point compactifications have been studied by Magill [2]. (Also see [3].) The general result, which assumes only a T_2 space, is as follows.

Theorem R5.1.1[Magill] Let (X, τ) be a Hausdorff space.

The following are equivalent:

- i) X has an n -point compactification for some natural number n .
- ii) X is locally compact and contains a compact subset K whose complement is the union of n pairwise disjoint open sets $\{G_i : i = 1, \dots, n\}$ such that $K \cup G_i$ is not compact for each i .

Outline of proof: To see that i) implies ii), let (Z, g) be a T_2 compactification of X with $Z - g[X] = \{z_1, \dots, z_n\}$. Pick pairwise disjoint open subsets of Z , O_1, \dots, O_n , with $z_i \in O_i$. Then ii) can be verified for $G_i = g^{-1}[O_i]$ and $K = g^{-1}[Z - \cup_{i=1}^n O_i]$. The fact that z_i is in the Z -closure of $g[X] \cap O_i$ leads quickly to the non-compactness of $K \cup G_i$. For the converse, let p_1, \dots, p_n be n distinct objects not in X , let $Y = X \cup \{p_1, \dots, p_n\}$, and let $f : X \rightarrow Y$ by $f(x) = x$. Let $\sigma = \{O \subseteq Y : O \cap X \text{ is open in } X \text{ and } p_i \in O \Rightarrow (X - O) \cap G_i \text{ has compact closure in } X\}$. Then σ is a topology for Y and (Y, f) is an n -point compactification of X .

As in [2] a pairwise disjoint family $\{G_i : i = 1, \dots, n\}$ of open sets whose union has a compact complement K such that $K \cup G_i$ is not compact for each i will be called an n -star of X . Given an n -star of X , the T_2 compactification constructed in the proof above will be called the n -point compactification determined by the n -star. The next proposition shows that such compactifications provide representatives of every finite-point compactification class.

Proposition R5.1.2 Let (Z, g) be an n -point compactification of X . Then there is an n -star for X such that the n -point compactification of X determined by this n -star is equivalent to (Z, g) .

Outline of proof: As in the proof of R5.1.1, let $Z - g[X] = \{z_1, \dots, z_n\}$ and pick pairwise disjoint open subsets of Z , O_1, \dots, O_n , with $z_i \in O_i$. Then $\{G_i = g^{-1}[O_i] : i = 1, \dots, n\}$ is an n -star for X . Let (Y, f) be the n -point compactification determined by this n -star. Define $h : Z \rightarrow Y$ by $h(g(x)) = f(x) = x$ on $g[X]$ and $h(z_i) = p_i$ for $i = 1, \dots, n$. Then $h \circ g = f$ by definition and h is easily seen to be a bijection. For continuity, let O be open in Y and let $z \in h^{-1}[O]$. If $z = g(x)$ for some x , then $z \in g[f^{-1}[O \cap X]]$, which is an open subset of $h^{-1}[O]$. If $z = z_i$, then $p_i \in O$ so that $c_X(G_i \cap (X - O))$ is compact. Then $z_i \in O_i - g[c_X(G_i \cap (X - O))]$, which is an open subset of $h^{-1}[O]$. Thus h is continuous and, since Z is compact and Y is T_2 , a homeomorphism.

Corollary R5.1.3 Let (Z, g) be an n -point compactification of X . For each natural number m with $m \leq n$, X has an m -point compactification.

Proof: Let $\{G_i : i = 1, \dots, n\}$ be an n -star determined by (Z, g) . Let $G_m^* = \cup_{i=m}^n G_i$. Then $\{G_1, \dots, G_{m-1}, G_m^*\}$ is an m -star, which determines an m -point compactification of X .

Proposition R5.1.4 Let (Z_1, g_1) be an n -point compactification of X , and let (Z_2, g_2) be an m -point compactification of X . If (Z_1, g_1) is equivalent to (Z_2, g_2) , then $n = m$.

Proof: Let $h : Z_1 \rightarrow Z_2$ be the homeomorphism with $h \circ g_1 = g_2$. That equation implies that h induces a bijection between the finite sets $Z_1 - g_1[X]$ and $Z_2 - g_2[X]$. Thus $n = m$.

Theorem R5.1.5 [Magill] Let $\{G_i : i = 1, \dots, n\}$ and $\{O_i : i = 1, \dots, n\}$ be n -stars for the space X . Let $K_1 = X - \cup\{G_i : i = 1, \dots, n\}$, and let (Y_1, f_1) and (Y_2, f_2) be the n -point compactifications determined by the n -stars. Then (Y_1, f_1) is equivalent to (Y_2, f_2) if and only if there exists a permutation σ of $\{1, \dots, n\}$ such that $(K_1 \cup G_i) \cap (X - O_{\sigma(i)})$ is compact for each i .

Outline of proof: Let $Y_1 = X \cup \{p_1, \dots, p_n\}$ and $Y_2 = X \cup \{q_1, \dots, q_n\}$ with topologies and embeddings as described above. First assume the two compactifications are equivalent, so that there is a homeomorphism $h : Y_1 \rightarrow Y_2$ with $h \circ f_1 = f_2$, i.e., $h|_X$ is the identity map. h induces a permutation σ , where $\sigma(i) = j$ when $h(p_i) = q_j$. Fix i . Since $\{q_{\sigma(i)}\} \cup O_{\sigma(i)}$ is open in Y_2 , its inverse image under h , $\{p_i\} \cup O_{\sigma(i)}$, is open in Y_1 , so that $(X - O_{\sigma(i)}) \cap G_i$ has compact closure in X . It follows easily that the X -closed set $(K_1 \cup G_i) \cap (X - O_{\sigma(i)})$ is contained in a compact set and so is compact itself. For the converse, define $h : Y_1 \rightarrow Y_2$ by $h(x) = x$ for $x \in X$ and $h(p_i) = q_{\sigma(i)}$. For O open in Y_2 , $h^{-1}[O] \cap X = O \cap X$, which is open in X . If $p_i \in h^{-1}[O]$, then $(X - O) \cap G_i$ is contained in $[(K_1 \cup G_i) \cap (X - O_{\sigma(i)})] \cup [(X - O) \cap O_{\sigma(i)}]$ and so has compact closure. Thus h is continuous. Clearly h is the homeomorphism required to show that the two compactifications are equivalent.

Let \mathbf{R} denote the reals, \mathbf{C} the complex plane, and \mathbf{R}^m m -dimensional space, all with the usual topologies. Magill presents the following examples.

Corollary R5.1.6 \mathbf{C} and \mathbf{R}^m with $m \geq 2$ do not have n -point compactifications for $n \geq 2$.

Proof: By R5.1.3 it is sufficient to show the non-existence of 2-point compactifications. Deny and let $\{G_1, G_2\}$ be a 2-star. Let B be a ball containing the complement of $G_1 \cup G_2$. For these spaces, the complement of B must be connected but $\{G_1, G_2\}$ would induce a separation. Contradiction.

Corollary R5.1.7 \mathbf{R} has a 2-point compactification but does not have an n -point compactification for $n \geq 3$.

Proof: $\{(-\infty, 0), (0, \infty)\}$ is a 2-star for \mathbf{R} . Now suppose $\{G_1, G_2, G_3\}$ is a 3-star for \mathbf{R} , and let $\mathbf{R} - (G_1 \cup G_2 \cup G_3)$ be contained in $[a, b]$. Since $(-\infty, a)$ and (b, ∞) are connected, each has non-empty intersection with at most one G_i . The leftover G_i would have to be contained in $[a, b]$, which leads to a contradiction.

Corollary R5.1.8 All 2-point compactifications of \mathbf{R} are equivalent.

Proof: Let $\{G_1, G_2\}$ and $\{O_1, O_2\}$ be 2-stars of \mathbf{R} , let $K_1 = \mathbf{R} - (G_1 \cup G_2)$, and suppose $K_1 \subseteq [a, b]$. Since $(-\infty, a)$ and (b, ∞) are both connected, each must be entirely contained in one G_i and one O_i . Use that fact to define σ . Without loss of generality,

assume $(-\infty, a)$ is a subset of G_1 and $O_{\sigma(1)}$, while (b, ∞) is contained in G_2 and $O_{\sigma(2)}$. Then, for $i \in \{1, 2\}$, the closed set $(K_1 \cup G_i) \cap (\mathbf{R} - O_{\sigma(i)})$ is contained in $[a, b]$, and so the compactifications are equivalent by R5.1.5.

For what follows certain equivalence relations closely related to n -stars will be used. As is clear from the following definition, each n -compatible equivalence relation on X determines one n -star, while an n -star determines at least one n -compatible equivalence relation.

Definition R5.1.9 Let (X, τ) be a T_2 space. An equivalence relation E on X is n -compatible provided E has finitely many distinct equivalence classes, exactly n of which form an n -star of X .

If E is an n -compatible equivalence relation on X , (Y, ι_E) will denote the n -point compactification determined as above by the associated n -star, and $\tau(E)$ will denote the topology for Y . The following facts show that these notions simplify in the discrete case.

Proposition R5.1.10 Let X be an infinite discrete space and let E be an equivalence relation on X . Then E is n -compatible if and only if E has finitely many distinct equivalence classes, exactly n of which are infinite.

Proof: This follows easily because distinct equivalence classes must be disjoint and compactness is equivalent to finiteness in a discrete space.

Proposition R5.1.11 Let X be an infinite discrete space and E an n -compatible equivalence relation on X . Let $\{e_1, \dots, e_n\}$ be the distinct infinite equivalence classes of E . Then $\tau(E) = \{O \subseteq Y : p_i \in O \Rightarrow (X - O) \cap e_i \text{ is finite}\}$.

Proof: This follows easily since every $X \cap O$ is open and having compact closure in X is equivalent to finiteness.

Proposition R5.1.12 Let X be an infinite discrete space and let D and E be n -compatible equivalence relations on X . Let $\{d_1, \dots, d_n\}$ and $\{e_1, \dots, e_n\}$ be the distinct infinite equivalence classes of D and E respectively. Then (Y, ι_D) and (Y, ι_E) are equivalent compactifications if and only if there is a permutation σ of $\{1, \dots, n\}$ with the property that $d_i \cap (X - e_{\sigma(i)})$ is finite for each i .

Proof: This merely restates R5.1.5 in the present context.

Magill uses infinite discrete spaces as examples which have infinitely many non-equivalent n -compactifications for $n \geq 2$. Such examples are implicit in R5.1.12. It also provides simple examples of non-equivalent compactifications which are homeomorphic. With X discrete, it can be shown that, in the notation of R5.1.12, $(Y, \tau(D))$ and $(Y, \tau(E))$ are homeomorphic if $|X| = \aleph_0$ or if a σ exists such that $|d_i| = |e_{\sigma(i)}|$ for each i .

Uniform Space Constructions

Basic facts and notation for uniform spaces, which will be used in this subsection, can be found in [6].

Definition R5.2.1 [1] Let E be an equivalence relation on set X . \mathcal{U}_E denotes $\{U : X \times X \supseteq U \supseteq E\}$.

Lemma R5.2.2 [1] Let E be an equivalence relation on X . Then \mathcal{U}_E is a uniformity for X , and \mathcal{U}_E is totally bounded if and only if E has finitely many distinct equivalence classes.

Proof: The key to the first assertion is that $E \circ E = E$; the second follows easily since total boundedness is, in this case, equivalent to the equation $X = \bigcup_{i=1}^n E[x_i]$ for some

finite set $x_1 \dots x_n$.

Recall the following notation from [8]: For (X, τ) a non-compact locally compact Hausdorff space, \mathcal{U}_m denotes $\{U : U \supseteq \bigcup_{i=1}^n O_i \times O_i \text{ where } O_1, \dots, O_n \text{ are an open cover of } X \text{ and at least one } O_i \text{ has a compact complement}\}$. It is shown in [8] that \mathcal{U}_m is a totally bounded uniformity with $\tau(\mathcal{U}_m) = \tau$ and that a separated completion of (X, \mathcal{U}_m) determines the compactification class of the one-point compactification for X , i.e., in the notation of [8], $\Psi_0(\mathcal{U}_m) = [(X^+, \iota^+)]$. Also recall that $\mathcal{TB}(X)$ denotes the set of totally bounded uniformities on X that generate τ .

Proposition R5.2.3 Let (X, τ) be a non-compact, locally compact T_2 space. Let E be an n -compatible equivalence relation on X with each E equivalence class open in τ . Then $\mathcal{U}_m \vee \mathcal{U}_E \in \mathcal{TB}(X)$ and $\Psi_0(\mathcal{U}_m \vee \mathcal{U}_E) = [(Y, \iota_E)]$.

Proof: By P2.13 $\mathcal{U}_m \vee \mathcal{U}_E$ is totally bounded and by P2.14 $\tau(\mathcal{U}_m \vee \mathcal{U}_E) = \tau \vee \tau(\mathcal{U}_E)$. Since $E[x] \in \tau$ for all x , $\tau(\mathcal{U}_E) \subseteq \tau$ and so $\mathcal{U}_m \vee \mathcal{U}_E \in \mathcal{TB}(X)$.

Now let O_1, \dots, O_n denote the equivalence classes of E which form the n -star, and let $\mathcal{V} \in \mathcal{TB}(X)$ be such that $\Psi_0(\mathcal{V}) = [(Y, \iota_E)]$. Note that \mathcal{V} is simply the subspace uniformity on X induced from the unique uniformity for Y , i.e., the collection of all neighborhoods of the diagonal in $Y \times Y$. (See P2.4 in [6].) One such neighborhood is $N = \bigcup_{i=1}^{n+j} G_i \times G_i$ where $G_i = O_i \cup \{p_i\}$ for $i = 1, \dots, n$ and G_{n+1}, \dots, G_{n+j} are the remaining equivalence classes of E . Clearly $N \cap (X \times X) = E$ and so $\mathcal{U}_E \subseteq \mathcal{V}$. Since $[(X^+, \iota^+)] \leq [(Y, \iota_E)]$, by R1.5 $\mathcal{U}_m \subseteq \mathcal{V}$. Thus $\mathcal{U}_m \vee \mathcal{U}_E \subseteq \mathcal{V}$.

To verify the reverse containment, let $V \in \mathcal{V}$, and let M be a neighborhood of the diagonal in Y such that $V = (X \times X) \cap M$. For each $x \in X$, there exists $O_x \in \tau$ with $O_x \times O_x \subseteq M$. Also there exist H_1, \dots, H_n open in Y with $p_i \in H_i$ and $H_i \times H_i \subseteq M$. For $S = \bigcup_{i=1}^n O_i$, the complement is compact and so there is a finite set Δ_0 such that $X - S \subseteq \bigcup \{O_x : x \in \Delta_0\}$. Let $U_0 = (\bigcup \{O_x \times O_x : x \in \Delta_0\}) \cup (S \times S)$. Clearly $U_0 \in \mathcal{U}_m$. Since $p_i \in H_i$ and each O_i is clopen, $T_i = (X - H_i) \cap O_i$ is compact, and so there is Δ_i such that $T_i \subseteq \bigcup \{O_x \times O_x : x \in \Delta_i\}$. Let $U_i = (\bigcup \{O_x \times O_x : x \in \Delta_i\}) \cup (X - T_i) \times (X - T_i)$. Then U_1, \dots, U_n are also in \mathcal{U}_m . To finish it is sufficient to show that $([\bigcap_{i=0}^n U_i] \cap E) \subseteq V$. Let (x, y) be in the intersection with $x \neq y$. If (x, y) is in $\bigcup \{O_x \times O_x : x \in \Delta_i\}$ for any i , clearly (x, y) is in V . Thus assume (x, y) is in $S \times S$ and $(X - T_i) \times (X - T_i)$ for every i . Since (x, y) is in both E and $S \times S$, $x, y \in O_j$ for some j . Then $x, y \notin T_j$ implies $x, y \in H_j$. Thus $(x, y) \in H_j \times H_j$, which yields $(x, y) \in V$.

Proposition R5.2.4 Let X be an infinite discrete space. Let E be an n -compatible equivalence relation on X . Then $\mathcal{U}_m \vee \mathcal{U}_E \in \mathcal{TB}(X)$ and $\Psi_0(\mathcal{U}_m \vee \mathcal{U}_E) = [(Y, \iota_E)]$.

Proof: In the discrete case the assumption that every E -equivalence class is open is automatically satisfied. This is a special case of R5.2.3.

Note that for two n -compatible equivalence relations on a discrete X , E and F , R5.1.12 and R1.5 can be combined to characterize $\mathcal{U}_m \vee \mathcal{U}_E = \mathcal{U}_m \vee \mathcal{U}_F$. It can also be shown that $(X, \mathcal{U}_m \vee \mathcal{U}_E)$ and $(X, \mathcal{U}_m \vee \mathcal{U}_F)$ are unimorphic if there is a one-to-one correspondence between the infinite equivalence classes of E and F such that corresponding classes have the same cardinality. This leads to examples of unimorphic spaces which determine non-equivalent compactifications.

Normal Basis Constructions for Discrete Spaces

Basic facts and notation used here can be found in [7]. Throughout this subsection

X will denote an infinite discrete space and E an n -compatible equivalence relation on X with distinct infinite equivalence classes C_1, \dots, C_n .

Definition R5.3.1 Let $S \subseteq X$ and let $\Delta \subseteq \{1, \dots, n\}$. S is associated with Δ if and only if $S \cap C_i$ is finite for all $i \in \Delta$ and $(X - S) \cap C_i$ is finite for all $i \notin \Delta$.

Definition R5.3.2

$\mathcal{Z}(E) = \{S \subseteq X : S \text{ is associated with } \Delta \text{ for some } \Delta \subseteq \{1, \dots, n\}\}$.

Proposition R5.3.3 $\mathcal{Z}(E)$ is a normal basis for X .

Proof: Note that finite subsets of X are associated with $\{1, \dots, n\}$ and so are in $\mathcal{Z}(E)$. Also, if $Z \in \mathcal{Z}(E)$ is associated with Δ , then $X - Z$ is associated with $\{1, \dots, n\} - \Delta$ and thus is also in $\mathcal{Z}(E)$. Since $x \notin S$ means $S \subseteq X - \{x\}$, $\mathcal{Z}(E)$ is a base for the closed sets. For $Z_1, Z_2 \in \mathcal{Z}(E)$ associated with Δ_1, Δ_2 respectively, $Z_1 \cup Z_2$ is associated with $\Delta_1 \cap \Delta_2$ and $Z_1 \cap Z_2$ is associated with $\Delta_1 \cup \Delta_2$. Thus $\mathcal{Z}(E)$ is closed under finite unions and intersections. The third requirement of definition P3.1 is satisfied because, for $x \notin S$, $\{x\} \in \mathcal{Z}(E)$ and $S \cap \{x\} = \emptyset$. The fourth is equally straightforward: for $Z_1, Z_2 \in \mathcal{Z}(E)$ with $Z_1 \cap Z_2 = \emptyset$, $X - Z_1$ and $X - Z_2$ are in $\mathcal{Z}(E)$ and form the needed cover.

Definition R5.3.4 Let $i \in \{1, \dots, n\}$.

$\mathcal{G}_i = \{S \in \mathcal{Z}(E) : S \text{ is associated with some } \Delta \text{ contained in } \{1, \dots, n\} - \{i\}\}$.

Lemma R5.3.5 An element of $\mathcal{Z}(E)$ is associated with a unique subset of $\{1, \dots, n\}$.

Proof: Deny and pick Z in $\mathcal{Z}(E)$ associated with both Δ_1 and Δ_2 . For any i in $(\Delta_1 - \Delta_2) \cup (\Delta_2 - \Delta_1)$, both $Z \cap C_i$ and $(X - Z) \cap C_i$ must be finite, which contradicts the assumption that C_i is infinite.

Proposition R5.3.6 For $i \in \{1, \dots, n\}$, \mathcal{G}_i is a $\mathcal{Z}(E)$ -ultrafilter.

Proof: The co-finite subsets of X , being associated with \emptyset , are in \mathcal{G}_i , while \emptyset is not since it is associated with $\{1, \dots, n\}$. Let S_1, S_2 in $\mathcal{Z}(E)$ be associated with Δ_1 and Δ_2 respectively. Since $S_1 \cap S_2$ is associated with $\Delta_1 \cup \Delta_2$, clearly \mathcal{G}_i is closed under finite intersections. If $S_1 \in \mathcal{G}_i$ and $S_1 \subseteq S_2$, then $\Delta_2 \subseteq \Delta_1$ so that $S_2 \in \mathcal{G}_i$. Thus \mathcal{G}_i is a $\mathcal{Z}(E)$ -filter. Now suppose \mathcal{F} is a $\mathcal{Z}(E)$ -filter with $\mathcal{G}_i \subseteq \mathcal{F}$. If $Z \in \mathcal{F}$ is associated with Δ and $i \in \Delta$, then $X - Z$, which is associated with $\{1, \dots, n\} - \Delta$, must be in \mathcal{G}_i . That implies $Z \cap (X - Z) \in \mathcal{F}$, a contradiction. Thus \mathcal{G}_i is a $\mathcal{Z}(E)$ -ultrafilter.

Proposition R5.3.7 The distinct, non-point ultrafilters in $\omega(\mathcal{Z}(E))$ are $\mathcal{G}_1, \dots, \mathcal{G}_n$.

Proof: For all i , C_i is associated with $\{1, \dots, n\} - \{i\}$ and so $C_i \in \mathcal{G}_i$ and $\mathcal{G}_i \neq \mathcal{G}_j$ if $j \neq i$. Since all finite sets are associated with $\{1, \dots, n\}$, the ultrafilter \mathcal{G}_i does not contain any finite set. Thus $\mathcal{G}_1, \dots, \mathcal{G}_n$ are distinct, non-point $\mathcal{Z}(E)$ -ultrafilters. Now let \mathcal{F} be a $\mathcal{Z}(E)$ -ultrafilter with $\mathcal{F} \neq \mathcal{G}_i$ for all i . Pick F_i associated with Δ_i such that $F_i \in \mathcal{F}$ but $F_i \notin \mathcal{G}_i$. Then $i \in \Delta_i$ for each i . Let $F = \bigcap_{i=1}^n F_i$. F is in \mathcal{F} and is associated with $\bigcup_{i=1}^n \Delta_i = \{1, \dots, n\}$. That means F is finite. Since only point-ultrafilters contain any finite sets, \mathcal{F} must be \mathcal{F}_x for some x .

Proposition R5.3.8 $(\omega(\mathcal{Z}(E)), \iota_{\mathcal{Z}(E)})$ is equivalent to (Y, ι_E) .

Proof: Define $h : \omega(\mathcal{Z}(E)) \rightarrow Y$ by $h(\mathcal{F}_x) = x$ and $h(\mathcal{G}_i) = p_i$. Clearly h is one-to-one and onto, and $h \circ \iota_{\mathcal{Z}(E)} = \iota_E$. Since the spaces are compact and T_2 , continuity of h is sufficient to show that h is the homeomorphism required for equivalence. Let F be closed in Y and suppose $\mathcal{F} \notin h^{-1}[F]$. If $\mathcal{F} = \mathcal{F}_x$ for some x , then $h^{-1}[F] \subseteq (X - \{x\})^\omega$ and $\mathcal{F} \notin (X - \{x\})^\omega$. Now suppose $\mathcal{F} = \mathcal{G}_i$ for some i . Then $p_i \in Y - F$ so that $(X - (Y - F)) \cap C_i$ is finite. Let $Z = (X - (Y - F)) \cup (\bigcup \{C_j : j \neq i\})$. Then Z is associated

with $\{i\}$ so that $Z \in \mathcal{Z}(E)$, $Z \notin \mathcal{G}_i$, and $Z \in \mathcal{G}_j$ for $j \neq i$. Since $X \cap F \subseteq Z$, it follows that $h^{-1}[F] \subseteq Z^\omega$ and $\mathcal{G}_i \notin Z^\omega$. From the description of the closed sets in $\omega(\mathcal{Z}(E))$ (P3.6 in [7]), $h^{-1}[F]$ is closed and so h is continuous as required.

Note that R5.1.2 and R5.3.8 show that every finite-point compactification of a discrete space can be constructed from a normal basis.

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An asterisk indicates a reference not seen by me.

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Added 2013

This addendum points out that not every finite point compactification class corresponds to a uniformity of the form $\mathcal{U}_m \vee \mathcal{U}_E$.

Lemma R5.Add.1 Let X be a set and let E be an equivalence relation. Then for every $x \in X$, $E[x]$ is $\tau(\mathcal{U}_E)$ -clopen.

Proof: Let x be in X and let $t \in E[x]$. $E[t]$ is a $\tau(\mathcal{U}_E)$ -neighborhood of t and $E[t] = E[x]$ since E -sections are equivalence classes and xEt . Thus $E[x]$ is a $\tau(\mathcal{U}_E)$ -neighborhood of each of its points and so $E[x]$ is open. The complement of $E[x]$ is the union of the other equivalence classes and so open. Thus $E[x]$ is also $\tau(\mathcal{U}_E)$ -closed.

Example R5.Add.2 Let $X = (0, 1)$ and \mathcal{U} be the usual uniformity on X . Let $Y = [0, 1]$ and let $f : X \rightarrow Y$ be the inclusion map. Since \mathcal{U} is the subspace uniformity of the usual uniformity on Y , \mathcal{U} corresponds to the compactification class of (Y, f) . Let \mathcal{U}_m be the uniformity for X corresponding to the one-point compactification. Since (Y, f) is a two-point compactification, \mathcal{U}_m is a proper subset of \mathcal{U} . Suppose there is an equivalence relation E on X such that $\mathcal{U}_m \vee \mathcal{U}_E = \mathcal{U}$. Since X is connected and $E[x]$ is clopen in $\tau(\mathcal{U}_E) \subseteq \tau(\mathcal{U}_m \vee \mathcal{U}_E) = \tau(\mathcal{U})$, $E = X \times X$. But then $\mathcal{U}_m \vee \mathcal{U}_E = \mathcal{U}_m$, a contradiction.

Added 2018

Much of this section focused on finite point compactifications of discrete spaces. This note points out a way to construct non-discrete spaces with finite point compactifications and characterizes spaces whose Stone-Ćech compactification is a finite point compactification.

Lemma R5.Add.3 Let (X, τ) be a $T_{3\frac{1}{2}}$ space, let (Y, f) be a T_2 compactification of (X, τ) , and let $S \subseteq Y - f[X]$. Let $Z = Y - S$ have the relative topology from Y . Then (Y, s) is a T_2 compactification of Z , where $s : Z \rightarrow Y$ is the inclusion map.

Proof: Since the dense $f[X]$ is contained in Z , Z is dense in Y . Since Z has the relative topology, s is an embedding.

In the last lemma $Y - s[Z] = S$ and so, if S is finite, Z has a finite point compactification.

Lemma R5.Add.4 Let (X, τ) be a $T_{3\frac{1}{2}}$ space, let $(\beta X, \iota)$ be the Stone-Ćech compactification of (X, τ) , and let $S \subseteq \beta X - \iota[X]$. Let $Z = \beta X - S$ have the relative topology from βX . Then $(\beta X, s)$ is the Stone-Ćech compactification of Z , where $s : Z \rightarrow \beta X$ is the inclusion map.

Proof: It is sufficient to show that every continuous map from Z to a compact T_2 space has a continuous extension to βX . Let $h : Z \rightarrow K$ be continuous, where K is compact and T_2 . Let g be h restricted to $\iota[X]$. Then g has a continuous extension G to βX . Then $G|_{\iota[X]} = g = h|_{\iota[X]}$. Since $\iota[X]$ is dense in Z and K is T_2 , $G|_Z = h$, i.e., G is a continuous extension of h .

By choosing S finite, one obtains a space whose Stone-Ćech compactification is a finite point compactification. The rest of this added note characterizes such spaces.

Lemma R5.Add.5 Let (X, τ) be a $T_{3\frac{1}{2}}$ space, let (Y, f) be a T_2 compactification of (X, τ) , let A be a dense subset of X , and let τ_A be the relative topology on A from X . Then $(Y, f|_A)$ is a T_2 compactification of (A, τ_A) .

Proof: Since f is an embedding and A has the relative topology, $f|_A$ is also an embedding. Clearly its image is $f[A]$. Since A is dense in X , $f[A]$ is dense in $f[X]$, which is dense in Y . Thus $f[A]$ is dense in Y .

Lemma R5.Add.6 Let (X, τ) be a $T_{3\frac{1}{2}}$ space, let (Y, f) and (Z, g) be a T_2 compactifications of (X, τ) , and let A be a dense subset of X . If $(Y, f|_A)$ is equivalent to $(Z, g|_A)$, then (Y, f) is equivalent to (Z, g) .

Proof: Assume $\phi : Y \rightarrow Z$ is a homeomorphism with $\phi \circ f|_A = (\phi \circ f)|_A = g|_A$. Since these are continuous maps into the T_2 space Z which agree on a dense subset of the domain, $\phi \circ f = g$. By definition (Y, f) is equivalent to (Z, g) .

Proposition Add.5.7 Let (X, τ) be a non-compact $T_{3\frac{1}{2}}$ space which has exactly M distinct compactification classes, where M is a positive integer. Then its Stone-Ćech compactification, $(\beta X, \iota)$, is a finite point compactification with $|\beta X - \iota[X]| \leq M$.

Proof: Assume $|\beta X - \iota[X]| \geq M + 1$. Let $Y = \beta X - \{t_1, \dots, t_{M+1}\}$, where t_1, \dots, t_{M+1} are distinct elements of $\beta X - \iota[X]$. By R5.Add.3 βX with the inclusion map is a compactification of Y with $|\beta X - Y| = M + 1$. For each $1 \leq k \leq M + 1$, by R5.1.3, there is a k -point compactification (Z_k, f_k) of Y . By R5.1.4, if $k \neq l$, (Z_k, f_k) is not equivalent to (Z_l, f_l) . Since $\iota[X]$ is dense in Y , by R5.Add.6, these $M + 1$ compactifications of Y induce $M + 1$ non-equivalent compactifications of $\iota[X]$. But $\iota[X]$, a homeomorph of X , has exactly M distinct compactification classes, a contradiction.

The next few results will yield the other half of the characterization.

Lemma Add.5.8 Let (X, τ) be a $T_{3\frac{1}{2}}$ space, let (Y, f) and (Z, g) be a T_2 compactifications of (X, τ) , and let $\phi : Y \rightarrow Z$ be continuous and onto with $\phi \circ f = g$. Then, for every $x \in X$, $\phi^{-1}[\{g(x)\}] = \{f(x)\}$.

Proof: Fix $x \in X$. Since $\phi(f(x)) = g(x)$, $f(x) \in \phi^{-1}[\{g(x)\}]$. Let $y \in \phi^{-1}[\{g(x)\}]$. There is a net $S : D \rightarrow X$ such that $f \circ S$ converges to y . By continuity, $\phi \circ (f \circ S)$ converges to $\phi(y) = g(x)$. Thus $(\phi \circ f) \circ S = g \circ S$ converges to $g(x)$. Since $g : X \rightarrow g[X]$ is a homeomorphism, S converges to x and so $f \circ S$ converges to $f(x)$. Since limits are unique in a T_2 space, $y = f(x)$.

The relation $[(Z, g)] \leq [(Y, f)]$ is defined by the existence of ϕ as in the previous lemma, and such a ϕ must be unique. $\mathcal{P}(Z)$ will denote the partition of Y induced by ϕ , i.e., $\{\phi^{-1}[\{z\}] : z \in Z\}$.

Next recall some general facts: If A is compact, B is T_2 , and $m : A \rightarrow B$ is continuous and onto, then m is a quotient map and so B is homeomorphic to the quotient space A/E , where E is the equivalence relation determined by the partition $\{m^{-1}[\{b\}] : b \in B\}$. The map $b \mapsto m^{-1}[\{b\}]$ is a homeomorphism.

Lemma Add.5.9 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let (Y, f) , (W, h) and (Z, g) be a T_2 compactifications of (X, τ) with $[(Z, g)] \leq [(Y, f)]$ and $[(W, h)] \leq [(Y, f)]$. Assume $\mathcal{P}(Z) = \mathcal{P}(W)$. Then (Z, g) is equivalent to (W, h) .

Proof: Let $\phi : Y \rightarrow Z$ and $\psi : Y \rightarrow W$ be continuous and onto with $\phi \circ f = g$ and $\psi \circ f = h$. Since $\mathcal{P}(Z) = \mathcal{P}(W)$, Z and W are homeomorphic to the same quotient space, Y/E , where E is the equivalence relation determined by $\mathcal{P}(Z) = \mathcal{P}(W)$. Let $\rho : Z \rightarrow Y/E$ and $\sigma : W \rightarrow Y/E$ be the homeomorphisms given by $\rho(z) = \phi^{-1}[\{z\}]$ and $\sigma(w) = \psi^{-1}[\{w\}]$. Then $\sigma^{-1} \circ \rho$ is a homeomorphism from Z onto W . It is sufficient to show $(\sigma^{-1} \circ \rho) \circ g = h$. Let $x \in X$. Then $\rho \circ g(x) = \phi^{-1}[\{g(x)\}] = \{f(x)\}$ by Add.5.8. By the same lemma, $\psi^{-1}[\{h(x)\}] = \{f(x)\}$ so that $\sigma^{-1}(\{f(x)\}) = h(x)$. Thus the claim holds.

Proposition Add.5.10 Let (X, τ) be a non-compact $T_{3\frac{1}{2}}$ space and let $(\beta X, \iota)$ be the Stone-Ćech compactification of (X, τ) . Then βX is a finite point compactification of X if and only if the number of distinct compactification classes of (X, τ) is finite.

Proof: The sufficiency of the condition follows from Add.5.7. For necessity, assume βX is a finite point compactification of X . For any (Y, f) , a compactification of (X, τ) , since $[(Y, F)] \leq [(\beta X, \iota)]$, Y is homeomorphic to a quotient space $\beta X/E$. By Add.5.8 the partition determining E is the union of $\{\{\iota(x)\} : x \in X\}$ and a partition of $\beta X - \iota[X]$. Since $\beta X - \iota[X]$ is finite, it has finitely many partitions. By Add.5.9 the number of distinct compactification classes of (X, τ) is finite.