Suprema of Two-point Compactifications

In this section suprema of finite-point compactifications are investigated, a main result being that the supremum of all 2-point compactifications of an infinite discrete space is its Stone-Čech compactification. The results will be approached in two ways, first using uniform space theory for certain zero-dimensional spaces and then using the fundamental extension property of βX for suitable extremely disconnected spaces. $\mathbb{R}$ provides an example showing that the result for discrete spaces does not extend to one-dimensional spaces.

For a non-compact, $T_{3\frac{1}{2}}$ space (X, τ), $\mathcal{TB}(X)$ denotes the set of totally bounded uniformities which generate the topology of X, $\mathcal{U}_M$ the largest element in $\mathcal{TB}(X)$ [3], and $\mathcal{E}_2(X)$ the set of 2-compatible equivalence relations on X [5].

For $E \in \mathcal{E}_2(X)$ with distinct equivalence classes $\{C_1(E), C_2(E)\}$ forming the associated 2-star, let p_1 and p_2 be distinct objects not in X, let $Y = X \cup \{p_1, p_2\}$, and let $\tau(E) = \{G \subseteq Y : G \cap X \text{is open and } p_i \in G \text{ implies } C_i(E) \cap (X - G) \text{ has compact closure in X }\}$. Let ι_E denote the inclusion map from X to $(Y, \tau(E))$. As shown in [5], (Y, ι_E) is a two-point T_2 compactification of X.

If X is also locally compact, $\mathcal{U}_m$ denotes the smallest element in $\mathcal{TB}(X)$. For $E \in \mathcal{E}_2(X)$, if all the E equivalence classes are open in X, $\Psi_0(\mathcal{U}_m \vee \mathcal{U}_E) = [(Y, \iota_E)]$, where $\mathcal{U}_E$ consists of all supersets of E in $X \times X$.

Uniform Space Approach

Definition R6.1.1 Let (X, τ) be a topological space. $\mathcal{U}_{zd}$ denotes $\{U \subseteq X \times X$: there is a finite, clopen cover of X, say $G_1, \ldots, G_k$, such that $\cup_{i=1}^k G_i \times G_i \subseteq U\}$.

Lemma R6.1.2 Let (X, τ) be a topological space. Then $\mathcal{U}_{zd}$ is a totally bounded uniformity for X.

Proof: Most of the requirements for a uniformity are easily verified. For the 'triangle inequality' (P2.1.v), let $U \in \mathcal{U}_{zd}$ be given with the associated clopen cover $G_1, \ldots, G_k$ such that $\cup_{i=1}^k G_i \times G_i \subseteq U$. Let $H_1 = G_1$ and, for $2 \leq i \leq k$, let $H_i = G_i - \cup_{t=1}^{i-1} H_t$. Then $H_1, \ldots, H_k$ form a finite clopen cover of X so that $V = \cup_{i=1}^k H_i \times H_i$ is in $\mathcal{U}_{zd}$. It is easy to check that $V \circ V = V \subseteq U$. For total-boundedness, given U, simply pick one element from each of the finitely many covering clopen sets.

Lemma R6.1.3 Let (X, τ) be zero-dimensional. Then $\tau(\mathcal{U}_{zd}) = \tau$.

Proof: In general $\tau(\mathcal{U}_{zd}) \subseteq \tau$. For the converse, let $x \in O \in \tau$. Since X is zero-dimensional, there is a clopen G with $x \in G \subseteq O$. For $U = G \times G \cup (X - G) \times (X - G)$, clearly $U \in \mathcal{U}_{zd}$ and $U[x] = G \subseteq O$. Thus $\tau \subseteq \tau(\mathcal{U}_{zd})$.

Proposition R6.1.4 Let (X, τ) be non-compact, zero-dimensional and T_0. Then $\mathcal{U}_{zd} = \mathcal{U}_M$ if and only if $\Psi_0(\mathcal{U}_{zd}) = [(\beta X, \iota)]$.

Proof: From above, $\mathcal{U}_{zd} \in \mathcal{TB}(X)$. This proposition is immediate from R1.5 and R1.8 in [3].

Definition R6.1.5 Let (X, τ) be non-compact, locally compact and T_2. $\mathcal{E}_2^o(X)$ denotes the set $\{E \in \mathcal{E}_2(X) : E[x] \text{ is open for all } x \in X\}$.

It is shown in R5.2.3 that, if $E \in \mathcal{E}_2^o(X)$, then $\mathcal{U}_m \vee \mathcal{U}_E \in \mathcal{TB}(X)$ and $\Psi_0(\mathcal{U}_m \vee \mathcal{U}_E) = [(Y, \iota_E)]$.

Proposition R6.1.6 Let (X, τ) be non-compact, locally compact, T_2, and zero-dimensional. Then $\mathcal{U}_{zd} = \bigvee\{\mathcal{U}_m \vee \mathcal{U}_E : E \in \mathcal{E}_2^o(X)\}$.

Proof: Let $U \in \mathcal{U}_{zd}$ contain $\cup_{i=1}^{k}(G_i \times G_i)$, where $G_1, \ldots, G_k$ form a finite, clopen cover of X. Let $E_i = G_i \times G_i \cup (X - G_i) \times (X - G_i)$. If both G_i and $X - G_i$ are non-compact, then $E_i \in \mathcal{E}_2^o(X)$. Otherwise, $E_i \in \mathcal{U}_m$. In any case, $V = \cap_{i=1}^{k} E_i$ must be in $\bigvee\{\mathcal{U}_m \vee \mathcal{U}_E : E \in \mathcal{E}_2^o(X)\}$. Since $\cap_{i=1}^{k}(X - G_i) = \emptyset$, it follows easily that $V \subseteq \cup_{i=1}^{k}(G_i \times G_i)$, which shows $\mathcal{U}_{zd} \subseteq \bigvee\{\mathcal{U}_m \vee \mathcal{U}_E : E \in \mathcal{E}_2^o(X)\}$.

For the reverse containment, first note that $\mathcal{U}_m \subseteq \mathcal{U}_{zd}$ since $\mathcal{U}_m$ is the smallest element of $\mathcal{TB}(X)$. For any $E \in \mathcal{E}_2^o(X)$, each E equivalence class is clopen since its complement is the union of the other classes, all of which are open. Thus $E \in \mathcal{U}_{zd}$ so that $\mathcal{U}_E \subseteq \mathcal{U}_{zd}$.The desired containment is now clear, since the supremum must be contained in any upper bound.

Proposition R6.1.7 Let (X, τ) be non-compact, locally compact, T_2, and zero-dimensional. Assume that every finite open cover of X has a finite clopen refinement, which also covers X. Then the Stone-Čech compactification of X is the supremum of the two-point compactifications of X.

Proof: Let $U \in \mathcal{U}_M$. Since $\mathcal{U}_M$ is totally bounded, there is an open cover $O_1, \ldots, O_n$ such that $\cup_{i=1}^{n} O_i \times O_i \subseteq U$. By hypothesis, there is a finite, clopen cover $G_1, \ldots, G_j$ such that, for each t with $1 \le t \le j$, $G_t \subseteq O_i$ for some i. As a result $\cup_{i=1}^{j} G_i \times G_i \subseteq U$ so that $U \in \mathcal{U}_{zd}$. Since $\mathcal{U}_{zd}$ is in $\mathcal{TB}(X)$, which has $\mathcal{U}_M$ as its largest element, one can use R6.1.6 to obtain $\mathcal{U}_{zd} = \bigvee\{\mathcal{U}_m \vee \mathcal{U}_E : E \in \mathcal{E}_2^o(X)\} = \mathcal{U}_M$. As mentioned above, for $E \in \mathcal{E}_2^o(X), \Psi_0(\mathcal{U}_m \vee \mathcal{U}_E)$ is the compactification class of a two-point compactification. Combining R6.1.4 with R1.5 [3], we see that $[(\beta X, \iota)]$ has the properties of the least upper bound relative to the two-point compactifications associated with uniformities of the form $\mathcal{U}_m \vee \mathcal{U}_E$ for $E \in \mathcal{E}_2^o(X)$ and so, as the largest compactification class, relative to all two-point compactifications.

Corollary R6.1.8 Let X be an infinite discrete space. Then the Stone-Čech compactification of X is the supremum of the two-point compactifications of X.

Proof: An infinite discrete space satisfies the hypothesis of R6.1.7.

It is easy to check that a two-point compactification of $\mathbb{N}$ must be second countable and so is metrizable by Urysohn's theorem. This observation combined with R6.1.8 shows that, in general, the supremum operation for compactifications does not preserve metrizability, second countability, weight, or first countability. See R3.2.8 and R3.2.9 in [4] for the countable case.

Because $\mathbb{R}$ has a unique two-point compactification up to equivalence (R5.1.8 in [5]), the analog of R6.1.8 fails for $\mathbb{R}$ and so cannot hold generally for one dimensional spaces.

Extension Approach

The proof in this subsection was outlined by an anonymous reader of the uniform space approach for infinite discrete spaces. For any map g from X into a compact T_2 space, the unique continuous extension of g to βX will be denoted βg.

R6.2.1 Lemma Let (X, τ) be non-compact, locally compact, and T_2. Suppose βX is zero-dimensional, and let $p \neq q$ be in $\beta X - \iota[X]$. Then there is $E \in \mathcal{E}_2(X)$ such that $\beta\iota_E(p) \neq \beta\iota_E(q)$.

Proof: By hypothesis there is G clopen in βX such that $p \in G$ and $q \notin G$. Let $O = \iota^{-1}[G \cap \iota[X]]$, which is clopen in X. Note that O is non-compact: if not, $\iota[O]$ would be compact and so closed in βX. Then $G - \iota[O]$ is open and contains p, but

$(G - \iota[O]) \cap \iota[X] = \emptyset$, which contradicts the density of $\iota[X]$. Similarly, $X - O$ is non-compact. Thus $\{O, X - O\}$ is a 2-star for X and so determines a 2-compatible $E \in \mathcal{E}_2(X)$, with its associated two-point compactification $[(Y, \iota_E)]$.

Since $\beta X = \overline{\iota[O]} \cup \overline{\iota[X - O]}$, $\overline{\iota[O]} \subseteq G$, and $\overline{\iota[X - O]} \subseteq (\beta X - G)$, it follows easily that $\overline{\iota[O]} = G$ and $\overline{\iota[X - O]} = \beta X - G$, where the closures are in βX. Because $\beta\iota_E \circ \iota = \iota_E$, the closed sets $\beta\iota_E[G]$ and $\beta\iota_E[X - G]$ contain $c_Y(O)$ and $c_Y(X - O)$ respectively, where c_Y denotes the closure in Y. By the continuity of $\beta\iota_E$, $\beta\iota_E[\overline{\iota[O]}] \subseteq c_Y(O)$ and $\beta\iota_E[\overline{\iota[X - O]}] \subseteq c_Y(X - O)$. Combining these facts we have $\beta\iota_E[G] = c_Y(O)$ and $\beta\iota_E[\beta X - G] = c_Y(X - O)$.

Next note that $\{p_1\} \cup O$ is clopen in Y so that $\beta\iota_E[G] = \{p_1\} \cup O$ and $\beta\iota_E[\beta X - G] = \{p_2\} \cup (X - O)$. Thus $\beta\iota_E(p) \neq \beta\iota_E(q)$.

Lemma R6.2.2 Let (X, τ) be a $T_{3\frac{1}{2}}$ space. Let (Z, f) be a T_2 compactification of X, and let $t \in \beta X - \iota[X]$. Then $\beta f(t) \notin f[X]$.

Proof: Let $\{x_\alpha\}$ be a net in X with $\iota(x_\alpha) \to t$. Since $\beta f \circ \iota = f$, $f(x_\alpha) \to \beta f(t)$. Now suppose $\beta f(t) \in f[X]$. Then $f(x_\alpha) \to f(x)$ for some $x \in X$. Since $f : X \to f[X]$ is a homeomorphism, $x_\alpha \to x$ and so $\iota(x_\alpha) \to \iota(x)$. By uniqueness of limits in a T_2 space, $\iota(x) = t$, which contradicts $t \notin \iota[X]$.

Theorem R6.2.3 Let (X, τ) be non-compact, locally compact, T_2, and extremely disconnected. The Stone-Čech compactification of X is the supremum of the two-point compactifications of X.

Proof: Let (Z, f) be a T_2 compactification whose equivalence class acts as the least upper bound for the two-point compactifications of X. From general condiderations, $\beta f : \beta X \to Z$ is continuous, closed and onto. Therefore, to show that (Z, f) is equivalent to $(\beta X, \iota)$, it is sufficient to show that βf is one-to-one. Thus let $p \neq q$ be in βX. Because f is one-to-one, $\beta f \circ \iota = f$, and R6.2.2 holds, we need only consider the case where p and q are both in $\beta X - \iota[X]$.

Since X is extremely disconnected, so is βX. Since an extremely disconnected, regular space must be zero-dimensional, R6.2.1 applies. Thus there is $E \in \mathcal{E}_2(X)$ with $\beta\iota_E(p) \neq \beta\iota_E(q)$. Since $[(Z, f)] \geq [(Y, \iota_E)]$, there is $g : Z \to (Y, \tau(E))$ continuous and onto with $g \circ f = \iota_E$. Then $g \circ \beta f \circ \iota = \iota_E = \beta\iota_E \circ \iota$. By the uniqueness of the Stone-Čech extension, $g \circ \beta f = \beta\iota_E$. Since $\beta\iota_E(p) \neq \beta\iota_E(q)$, $\beta f(p) \neq \beta f(q)$. Thus βf is one-to-one.

Corollary R6.2.4 Let X be an infinite discrete space. The Stone-Čech compactification of X is the supremum of the two-point compactifications of X.

Discussion

The first topic of this subsection is the relationship between R6.1.7 and R6.2.3. Of course, in the context of regular spaces, an extremely disconnected space must be zero-dimensional. In what follows the abbreviation FCCR is used for the refinement hypothesis of R6.1.7: every finite open cover of the space has a finite clopen refinement, which is also a cover. The following proposition shows the relation of FCCR to separation properties.

Proposition R6.3.1 Let (X, τ) be FCCR. Then (X, τ) is normal.

Proof: This follows easily from this equivalent formulation of normality: Given open sets O_1 and O_2 with $O_1 \cup O_2 = X$, there exist closed sets F_1 and F_2 with $F_i \subseteq O_i$ and $F_1 \cup F_2 = X$.

A careful reading of the proof of R6.1.7 reveals that FCCR is stronger than actually needed. The result can be improved as follows.

Definition R6.3.2 Let (X, τ) be $T_{3\frac{1}{2}}$. An open cover of X, $\{O_\alpha : \alpha \in \Delta\}$, is M-uniform provided there is $U \in \mathcal{U}_M$ such that $\{U[x] : x \in X\}$ refines $\{O_\alpha : \alpha \in \Delta\}$.

Definition R6.3.3 Let (X, τ) be $T_{3\frac{1}{2}}$. (X, τ) is MU-FCCR provided every M-uniform finite open cover has a finite clopen refinement, which is also a cover.

The following three results, which are a digression, show that, for T_4 spaces, MU-FCCR is equivalent to FCCR.

Lemma R6.3.4 Let (X, τ) be T_4. Then $\mathcal{U}_M = \{U \subseteq X \times X :$ there is a finite open cover $O_1, \ldots, O_n$ with $\cup_{i=1}^n O_i \times O_i \subseteq U\}$.

Outline of proof: Let $\mathcal{V}$ denote the set on the right side of the equation. Total boundeness and general facts about uniformities imply $\mathcal{U}_M \subseteq \mathcal{V}$. In verifying that $\mathcal{V}$ is a uniformity, only the triangle inequality is non-routine. Given an open cover, $O_1, \ldots, O_n$, use normality to construct inductively an open cover $G_1, \ldots, G_n$ with $\overline{G_i} \subseteq O_i$. Let $W_i = (O_i \times O_i) \cup (X - \overline{G_i}) \times (X - \overline{G_i})$ and let $W = \cap_{i=1}^n W_i$. Then $W \in \mathcal{V}$ and $W \circ W \subseteq \cup_{i=1}^n O_i \times O_i$. The uniformity $\mathcal{V}$ is clearly totally bounded, and regularity implies $\tau(\mathcal{V}) = \tau$. Since $\mathcal{V} \in \mathcal{TB}(X)$, $\mathcal{V} \subseteq \mathcal{U}_M$.

Lemma R6.3.5 Let (X, τ) be T_4, and let $O_1, \ldots, O_n$ be an open cover of X. Then $\{O_1, \ldots, O_n\}$ is M-uniform.

Proof: Let $G_1, \ldots, G_n, W_1, \ldots, W_n$ and W be as in the proof of R6.3.4. For $x \in X$ there is k with $x \in G_k$. Then $W[x] \subseteq W_k[x] = O_k$.

Proposition R6.3.6 Let (X, τ) be T_4. Then (X, τ) is MU-FCCR if and only if it is FCCR.

Proof: This is immediate from the preceding lemma since, in a T_4 space, every finite open cover is M-uniform.

Next is the improved version of R6.1.7.

Proposition R6.3.7 Let (X, τ) be non-compact, locally compact, T_2, and zero-dimensional. Assume that MU-FCCR holds in X. Then the Stone-Čech compactification of X is the supremum of the two-point compactifications of X.

Proof: Let $U \in \mathcal{U}_M$. Since the interior of an entourage must also be in the uniformity, there is $V \in \mathcal{U}_M$ with $V \circ V \subseteq U$, $V = V^{-1}$, and V open in $X \times X$. Also pick $W \in \mathcal{U}_M$ with $W = W^{-1}$ and $W \circ W \subseteq V$. By total boundedness there exist $x_1, \ldots, x_n$ such that $X = \cup_{i=1}^n W[x_i]$. It is easy to check that, if $x \in W[x_t]$, then $W[x] \subseteq V[x_t]$. Letting $O_i = V[x_i]$, since $W \subseteq V$, we have a finite M-uniform cover $O_1, \ldots, O_n$ with $\cup_{i=1}^n (O_i \times O_i) \subseteq U$. From this point on, the proof is identical to that of R6.1.7.

Now it is possible to show that this sharpened version of R6.1.7 implies R6.2.3. That is immediate from R6.3.9 below, although the proof given here makes use of prior knowledge of R6.2.3.

Lemma R6.3.8 Let (X, τ) be non-compact, locally compact, T_2, and extremely disconnected. Let $\{G_1, G_2\}$ be a 2-star for X. Then $\{\overline{G_1}, \overline{G_2}\}$ is also a 2-star for X, and it generates the same compactification class.

Proof: Since X is extremely disconnected and $G_1 \cap G_2 = \emptyset$, $\overline{G_1}$ and $\overline{G_2}$ are disjoint clopen sets. Let $K = X - (\overline{G_1} \cup \overline{G_2})$. K is a closed subset of the compact set $K_1 = X - (G_1 \cup G_2)$ and so is compact. Next suppose that $K \cup \overline{G_i}$ is compact. Then $K_1 \cup K \cup \overline{G_i} = K_1 \cup \overline{G_i}$ is also compact. But $K_1 \cup G_i$, being the complement of an open set, is a closed subset of a compact set, which contradicts the assumption that $\{G_1, G_2\}$ is a 2-star. Thus $K \cup \overline{G_i}$ is non-compact and so $\{\overline{G_1}, \overline{G_2}\}$ is a 2-star for X. The second assertion follows from R5.1.5 since $(K_1 \cup G_i) \cap (X - \overline{G_i})$ is a closed subset of the compact K_1.

Proposition R6.3.9 Let (X, τ) be non-compact, locally compact, T_2, and extremely disconnected. Then MU-FCCR holds in X.

Proof: An arbitrary 2-point compactification is determined by a 2-star $\{O_1, O_2\}$, as shown in R5.1.2, and by R6.3.8 we can assume both O_1 and O_2 are clopen. Let $K = X - (O_1 \cup O_2)$, which is also clopen, and let $E = O_1 \times O_1 \cup O_2 \times O_2 \cup K \times K$. Clearly $E \in \mathcal{E}_2^o(X)$ so that $\Psi_0(\mathcal{U}_m \vee \mathcal{U}_E)$ is the equivalence class of the compactification determined by $\{O_1, O_2\}$. By definition $E \in \mathcal{U}_{zd}$ and so $\Psi_0(\mathcal{U}_m \vee \mathcal{U}_E) \leq \Psi_0(\mathcal{U}_{zd})$. It follows from R6.2.3 that $\Psi_0(\mathcal{U}_{zd})$ is the Stone-Čech compactification of X. By R6.1.4 $\mathcal{U}_{zd} = \mathcal{U}_M$, from which it easily follows that X is MU-FCCR.

The second topic in this subsection is the following question, for which I currently do not have an answer: For a non-compact, locally compact, T_2, zero-dimensional space, is the Stone-Čech compactification of X the supremum of the two-point compactifications of X? The rest of this subsection contains results and comments which seem to be related to the question. The first is a variation of R6.2.3.

Proposition R6.3.10 Let (X, τ) be non-compact, locally compact, and T_2. Assume βX is zero-dimensional. The Stone-Čech compactification of X is the supremum of the two-point compactifications of X.

Proof: In the proof of R6.2.3, the hypothesis of X being extremely disconnected was only used to guarantee that βX would be zero-dimensional. With that given, the argument applies here as well.

The following result appears in [2].

Theorem R6.3.11 [Magill and Glasenapp] Let (X, τ) be $T_{3\frac{1}{2}}$ and zero-dimensional. The supremum of any non-empty collection of zero-dimensional compactifications is also zero-dimensional.

That result raises the question of when n-point compactifications are zero-dimensional. It can be shown that, if X is non-compact, locally compact, T_2, and extremely disconnected, then every n-point compactification of X is zero-dimensional. It is unknown to me whether the same holds with zero-dimensional replacing extremely disconnected.

Finally, Gillman and Jerison have an example (16P in [1]) of a T_4 zero-dimensional space X for which βX is not zero-dimensional. It is unknown to me whether this X is locally compact.

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