Uniform Continuity and Extension of Maps

For a $T_{3\frac{1}{2}}$ space (X, τ), the set of totally bounded uniformities for X that generate τ will be denoted $\mathcal{TB}(X)$, as in [4]. This section examines connections between the extendability of a continuous map relative to a given compactification and its uniform continuity relative to the element of $\mathcal{TB}(X)$ associated with the compactification. Certain classes of bounded, real-valued continuous functions on X are shown to generate elements of $\mathcal{TB}(X)$.

Main Result

For the first two results in this subsection, the following will be assumed: Let (Y, f) be a T_2 compactification of the $T_{3\frac{1}{2}}$ space (X, τ). With Ψ_0 defined as in [4], let $\mathcal{U}$ be the element of $\mathcal{TB}(X)$ with $\Psi_0(\mathcal{U}) = [(Y, f)]$. Let Z be a compact T_2 space with $\mathcal{W}$ the unique uniformity which generates the topology of Z.

The following result, which characterizes continuous maps that are extendible to Y, is probably known, but I have no reference.

Theorem R7.1.1 Let $h : X \to Z$ be continuous. Then there exists a unique continuous map $H : Y \to Z$ with $H \circ f = h$ if and only if $h : (X, \mathcal{U}) \to (Z, \mathcal{W})$ is uniformly continuous.

Proof: By R1.6a, $f : (X, \mathcal{U}) \to f[X]$ is a unimorphism, where $f[X]$ has the subspace uniformity from Y. If H exists, since Y is compact, H must be uniformly continuous. Thus $h = H \circ f$ is the composition of two uniformly continuous maps and so uniformly continuous. For the converse, assume h is uniformly continuous. Then $h \circ f^{-1}$ is a uniformly continuous map from $f[X]$, a dense subset of Y, to the complete and separated space $(Z, \mathcal{W})$. By R1.1 there exists a unique uniformly continuous map $H : Y \to Z$ which extends $h \circ f^{-1}$, i.e. $H \circ f = h$ as required.

As in definition R3.3.1, the map H in the above result will be described as the continuous extension of h to Y.

Corollary R7.1.2 Let $\mathcal{U}_M$ be the largest element of $\mathcal{TB}(X)$. Then every continuous map from X to Z is uniformly continuous from $(X, \mathcal{U}_M)$ to $(Z, \mathcal{W})$.

Proof: By R1.8 in [4] $\Psi_0(\mathcal{U}_M) = [(\beta X, \iota)]$ and every continuous map from X to Z extends to βX.

Corollary R7.1.3 Let $(X_1, \mathcal{U}_1)$ and $(X_2, \mathcal{U}_2)$ be separated, totally bounded uniform spaces with $\Psi_0(\mathcal{U}_i) = [(Y_i, f_i)]$. Let $g : (X_1, \mathcal{U}_1) \to (X_2, \mathcal{U}_2)$ be uniformly continuous. Then there is a unique continuous map $G : Y_1 \to Y_2$ such that $G \circ f_1 = f_2 \circ g$.

Proof: As noted above, the map f_2 is uniformly continuous and so $f_2 \circ g$ is a uniformly continuous map into the compact, T_2 space Y_2. G exists by the theorem.

The last corollary can be used to introduce a covariant functor from the category of separated, totally bounded uniform spaces with uniformly continuous maps to the category of compact Hausdorff spaces with continuous maps by associating $(X, \mathcal{U})$ with a representative of $\Psi_0(\mathcal{U})$ and a uniformly continuous map with the extension guaranteed by R7.1.3.

These results also provide an alternate proof of R3.3.5 in [5], as follows.

Lemma R7.1.4 Let $(X_1, \mathcal{U}_1)$ and $(X_2, \mathcal{U}_2)$ be separated, totally bounded uniform spaces with $\Psi_0(\mathcal{U}_i) = [(Y_i, f_i)]$. Let $g : X_1 \to X_2$. If g has a continuous extension from Y_1

to Y_2, i.e. there is a unique continuous map $G : Y_1 \to Y_2$ such that $G \circ f_1 = f_2 \circ g$, then $g : (X_1, \mathcal{U}_1) \to (X_2, \mathcal{U}_2)$ is uniformly continuous.

Proof: By R1.6a, $f_1 : (X_1, \mathcal{U}_1) \to f_1[X_1]$ and $f_2 : (X_2, \mathcal{U}_2) \to f_2[X_2]$ are unimorphisms, where $f_i[X_i]$ has the subspace uniformity from the unique uniformity for Y_i. By compactness, G is uniformly continuous and, because the image of $G \circ f_1$ is contained in $f_2[X_2]$, the extension equation can be written $g = f_2^{-1} \circ (G \circ f_1)$. Thus g is a composition of uniformly continuous maps and so uniformly continuous.

Proposition R7.1.5 Let (X, τ) and (W, σ) be a $T_{3\frac{1}{2}}$ spaces. Let Δ be a non-empty set. Let $\{(Y_\alpha, f_\alpha) : \alpha \in \Delta\}$ and $\{(Z_\alpha, g_\alpha) : \alpha \in \Delta\}$ be collections of T_2 compactifications of X and W respectively. Let $h : X \to W$ be continuous and assume, for every $\alpha \in \Delta$, h has an extension $h_\alpha : Y_\alpha \to Z_\alpha$. Then there exists $H : \vee Y_\alpha \to \vee Z_\alpha$, which is an extension of h.

Proof: For each $\alpha \in \Delta$ let $\mathcal{U}_\alpha$ in $\mathcal{TB}(X)$ and $\mathcal{V}_\alpha$ in $\mathcal{TB}(W)$ be the uniformities such that $\Psi_0(\mathcal{U}_\alpha) = [(Y_\alpha, f_\alpha)]$ and $\Psi_0(\mathcal{V}_\alpha) = [(Z_\alpha, g_\alpha)]$. For each $\alpha \in \Delta$ by R7.1.4 $h : (X, \mathcal{U}_\alpha) \to (W, \mathcal{V}_\alpha)$ is uniformly continuous. Since $(h \times h)^{-1}(\cap_{i=1}^n V_i) = \cap_{i=1}^n (h \times h)^{-1}[V_i]$ where $V_i \in \cup\{\mathcal{V}_\alpha : \alpha \in \Delta\}$, it follows easily that $h : (X, \vee\mathcal{U}_\alpha) \to (W, \vee\mathcal{V}_\alpha)$ is uniformly continuous. By R1.5 $\Psi_0(\vee\mathcal{U}_\alpha)$ is the compactification class of $\vee Y_\alpha$ and $\Psi_0(\vee\mathcal{V}_\alpha)$ is the compactification class of $\vee Z_\alpha$. By R7.1.3 the required extension H exists.

Generating $\mathcal{U}$ from Families of Maps

For this subsection some background material on uniform spaces not included in [3] is needed. The first few results describe weak uniformities induced by a map and the relation to the corresponding weak topologies. The straightforward proofs can be found in [2].

Definition R7.2.1 Let $(Y, \mathcal{W})$ be a uniform space and let $f : X \to Y$. $f^{-1}\mathcal{W}$ is defined to be $\{U \subseteq X \times X : \text{there is } W \in \mathcal{W} \text{ with } (f \times f)^{-1}[W] \subseteq U\}$.

Proposition R7.2.2 Let $(Y, \mathcal{W})$ be a uniform space and let $f : X \to Y$. Then $f^{-1}\mathcal{W}$ is the smallest uniformity for X which makes f uniformly continuous.

$f^{-1}\mathcal{W}$ is sometimes called the weak uniformity induced by f and $\mathcal{W}$ or, if there is no ambiguity about the uniformity on Y, simply the weak uniformity induced by f.

Definition R7.2.3 Let (Y, σ) be a topological space and let $f : X \to Y$. $f^{-1}\sigma$ is defined to be $\{f^{-1}[O] : O \in \sigma\}$.

Proposition R7.2.4 Let (Y, σ) be a topological space and let $f : X \to Y$. $f^{-1}\sigma$ is the smallest topology for X which makes f continuous.

As in the uniform space case, $f^{-1}\sigma$ is sometimes called the weak topology induced by f and σ or simply the weak topology induced by f.

Proposition R7.2.5 Let $(Y, \mathcal{W})$ be a uniform space and let $f : X \to Y$. Then $\tau(f^{-1}\mathcal{W}) = f^{-1}\tau(\mathcal{W})$.

Proposition R7.2.6 Let $(Y, \mathcal{W})$ be a uniform space and let $f : X \to Y$. If $f[X]$ with the subspace uniformity from $\mathcal{W}$ is totally bounded, then $(X, f^{-1}\mathcal{W})$ is also totally bounded.

Proof: If $W \in \mathcal{W}$, then $V = W \cap (f[X] \times f[X])$ is the resulting element of the subspace uniformity. Since $f[X]$ is totally bounded, there exist $x_1, \ldots, x_n$ in X such that $f[X] = \cup_{i=1}^n V[f(x_i)]$. It follows easily that $X = \cup_{i=1}^n (f \times f)^{-1}[W][x_i]$, which yields the desired conclusion.

For the rest of this subsection $\mathcal{V}$ will denote the canonical uniformity on $\mathbb{R}$, i.e., the uniformity generated by the absolute value metric. The facts above combined with P2.13 and P2.14 in [3] suggest using a family of continuous, bounded, real-valued functions to generate a totally bounded uniformity and consequently, if certain separation properties are satisfied, a compactification. In Chapter 2 of [1] Chandler uses suitable families to construct the compactifications directly, i.e. without uniformities as an intermediate step.

Definition R7.2.7 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{C}$ be a non-empty set of continuous, bounded real-valued maps from X. $\mathcal{U}(\mathcal{C}) = \bigvee\{f^{-1}\mathcal{V} : f \in \mathcal{C}\}$.

Proposition R7.2.8 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{C}$ be a non-empty set of continuous, bounded real-valued maps from X. Then $\mathcal{U}(\mathcal{C})$ is a totally bounded uniformity for X and $\tau(\mathcal{U}(\mathcal{C})) \subseteq \tau$. In addition each $f \in \mathcal{C}$ is uniformly continuous from $(X, \mathcal{U}(\mathcal{C}))$.

Proof: For each $f \in \mathcal{C}$, $f[X]$ is bounded in $\mathbb{R}$ and so also totally bounded. By R7.2.6 each $f^{-1}\mathcal{V}$ is totally bounded and so by P2.13 $\mathcal{U}(\mathcal{C})$ is also totally bounded. Since each f is continuous, the weak topology induced by f, which is $\tau(f^{-1}\mathcal{V})$ by R7.2.5, must be contained in τ. By P2.14 $\tau(\mathcal{U}(\mathcal{C})) \subseteq \tau$. Finally each f is uniformly continuous from $(X, f^{-1}\mathcal{V})$ by R7.2.2 and so also uniformly continuous with any larger uniformity on X.

A family $\mathcal{C}$ of continuous real-valued functions on X is said to separate points and closed sets if, given F closed in X and $x \in X - F$, there is $f \in \mathcal{C}$ such that $\{f(x)\} \cap \overline{f[F]} = \emptyset$, where $\overline{f[F]}$ denotes the closure of $f[F]$ in $\mathbb{R}$.

Proposition R7.2.9 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{C}$ be a non-empty set of continuous, bounded real-valued maps from X. If $\mathcal{C}$ separates points and closed sets, then $\tau(\mathcal{U}(\mathcal{C})) = \tau$.

Proof: Given $x \in O \in \tau$, the hypothesis implies there is $f \in \mathcal{C}$ and G open in $\mathbb{R}$ with $x \in f^{-1}[G] \subseteq O$. Since $f^{-1}[G] \in \tau(\mathcal{U}(\mathcal{C}))$, the conclusion follows.

Proposition R7.2.10 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{C}$ be a non-empty set of continuous, bounded real-valued maps from X. Assume $\mathcal{C}$ separates points and closed sets. Then $\mathcal{U}(\mathcal{C}) \in \mathcal{TB}(X)$. In addition, if $[(Y, g)] = \Psi_0(\mathcal{U}(\mathcal{C}))$, then every f in $\mathcal{C}$ has a continuous extension to Y.

Proof: The first conclusion summarizes R7.2.9 and part of R7.2.8. The second conclusion is immediate from R7.2.8 and R7.1.1.

Examples R7.2.11 Let $(X, \mathcal{U})$ be $[0, 1)$ with the subspace uniformity from $(\mathbb{R}\ , \mathcal{V})$. It can be shown that $\mathcal{U}$ is the smallest element of $\mathcal{TB}(X)$, i.e. the uniformity $\mathcal{U}_m$ associated with the one point compactification. Let $I : X \to \mathbb{R}$ by $I(x) = x$. $\{I\}$ separates points and closed sets as defined above and so $\mathcal{U}(\{I\}) \in \mathcal{TB}(X)$. Thus $\mathcal{U}_m \subseteq \mathcal{U}(\{I\})$, and the reverse containment holds since I is uniformly continuous from $(X, \mathcal{U}_m)$. In this example the set of uniformly continuous maps from $(X, \mathcal{U}(\{I\}))$ is strictly larger than the generating family $\{I\}$. As a second example, let $\mathcal{C} = \{f, g\}$, where $f(x) = 2x$ for $0 \leq x \leq \frac{1}{2}$, $f(x) = 1$ for $\frac{1}{2} < x < 1$, $g(x) = 1$ for $0 \leq x \leq \frac{1}{2}$, and $g(x) = 2 - 2x$ for $\frac{1}{2} < x < 1$. It can be shown that $f^{-1}\mathcal{V} \vee g^{-1}\mathcal{V}$ is $\mathcal{U}_m$, but $\mathcal{C}$ can not separate $\frac{1}{2}$ and $[0, \frac{1}{3}] \cup [\frac{2}{3}, \frac{3}{4}]$. Thus the conclusions of R7.2.10 may hold even if $\mathcal{C}$ does not separate points and closed sets.

Definition R7.2.12 Let (X, τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{U} \in \mathcal{TB}(X)$. $\mathcal{C}(\mathcal{U})$ is defined to be $\{f : (X, \mathcal{U}) \to (\mathbb{R}\ , \mathcal{V}) : f \text{ is uniformly continuous}\}$.

Since $(X, \mathcal{U})$ is totally bounded, each element of $\mathcal{C}(\mathcal{U})$ is a bounded continuous map. The following well-known lemmas will lead to the conclusion that $\mathcal{C}(\mathcal{U})$ separates points and closed sets. Notation intoduced in [3] for a uniformity generated by a pseudometric will be used. The fact that a uniformity is pseudo-metrizable if and only if it has a countable base (Theorem 11.5.1 in [2]) is also needed.

Lemma R7.2.13 Let $(X,\mathcal{U})$ be a uniform space and let $U \in \mathcal{U}$. Then there exists a pseudo-metric d for X such that $U \in \mathcal{U}_d \subseteq \mathcal{U}$.

Proof: By the definition of a uniformity a sequence of entourages $\{V_n\}$ can be found in $\mathcal{U}$ with $V_n = V_n^{-1}$, $V_n \circ V_n \subseteq V_{n-1}$ for $n \geq 2$, and $V_1 \circ V_1 \subseteq U$. It is easy to check that $\mathcal{W} = \{W \subseteq X \times X : V_n \subseteq W \text{ for some } n\}$ is a uniformity for X. By the theorem just mentioned, there is a pseudo-metric for X with $\mathcal{U}_d = \mathcal{W}$. Clearly $U \in \mathcal{W} \subseteq \mathcal{U}$.

Lemma R7.2.14 Let d be a pseudo-metric for X and let S be a non-empty subset of X. Let $f : X \to \mathbb{R}$ be defined by $f(x) = \inf\{d(x,t) : t \in S\}$.
Then $f : (X,\mathcal{U}_d) \to (\mathbb{R}\ ,\mathcal{V})$ is uniformly continuous.

Proof: $|f(a) - f(b)| \leq d(a,b)$ for any $a, b \in X$.

Proposition R7.2.15 Let (X,τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{U} \in \mathcal{TB}(X)$. Then $\mathcal{C}(\mathcal{U})$ separates points and closed sets.

Proof: Let F be closed relative to τ and let $x \in X - F$. There is $U \in \mathcal{U}$ with $U[x] \subseteq X - F$. By R7.2.13 there is a pseudo-metric d for X such that $U \in \mathcal{U}_d \subseteq \mathcal{U}$. There is $\epsilon > 0$ with $V_\epsilon \subseteq U$. For $\delta = \epsilon/2$, let $f : X \to \mathbb{R}$ by $f(t) = \inf\{d(t,y) : y \in V_\delta[x]\}$. By R7.2.14 f is uniformly continuous from $(X,\mathcal{U}_d)$ and so also from $(X,\mathcal{U})$. Clearly $f(x) = 0$ and $f(t) \geq \delta > 0$ for all $t \in F$ so that $\{f(x)\} \cap \overline{f[F]} = \emptyset$.

Proposition R7.2.16 Let (X,τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{U} \in \mathcal{TB}(X)$.
Then $\mathcal{U}(\mathcal{C}(\mathcal{U})) = \mathcal{U}$.

Proof: Since each $f \in \mathcal{C}(\mathcal{U})$ is uniformly continuous from $(X,\mathcal{U})$, $f^{-1}\mathcal{V} \subseteq \mathcal{U}$ and so $\mathcal{U}(\mathcal{C}(\mathcal{U})) \subseteq \mathcal{U}$. For the converse, let $U \in \mathcal{U}$ and let d be a pseudo-metric with $U \in \mathcal{U}_d \subseteq \mathcal{U}$. There is $\epsilon > 0$ with the ϵ-strip $V_\epsilon(d) = \{(x,y) : d(x,y) < \epsilon\} \subseteq U$. Let $\delta = \epsilon/3$. Since $\mathcal{U}$ is totally bounded, there exist $x_1, \ldots, x_n$ with $\cup_{i=1}^n V_\delta(d)[x_i] = X$. For $i = 1, \ldots, n$ let $f_i(x) = d(x, x_i)$. Each f_i is uniformly continuous and so $\cap_{i=1}^n (f_i \times f_i)^{-1}[V_\delta]$ is in $\mathcal{U}(\mathcal{C}(\mathcal{U}))$, where V_δ is the δ-strip determined by the absolute value metric on $\mathbb{R}$. Let $(a,b) \in \cap_{i=1}^n (f_i \times f_i)^{-1}[V_\delta]$. It is sufficient to show $(a,b) \in U$. There is j such that $a \in V_\delta(d)[x_j]$. Since $(a,b) \in (f_j \times f_j)^{-1}[V_\delta]$, $d(b,x_j) < d(a,x_j) + \delta$. Thus $d(a,b) < 3\delta = \epsilon$ and so $(a,b) \in U$.

Proposition R7.2.17 Let (X,τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{U} \in \mathcal{TB}(X)$. Then $\mathcal{C}(\mathcal{U}) = \cup\{\mathcal{C} : \mathcal{U}(\mathcal{C}) = \mathcal{U}\}$, i.e., $\mathcal{C}(\mathcal{U})$ is the largest collection of continuous, bounded, real-valued functions generating $\mathcal{U}$.

Proof: For any such collection $\mathcal{C}$, every map in $\mathcal{C}$ is uniformly continuous from $(X,\mathcal{U})$ by R7.2.8 so that $\mathcal{C} \subseteq \mathcal{C}(\mathcal{U})$. By R7.2.16 $\mathcal{C}(\mathcal{U})$ is one such collection.

Proposition R7.2.18 Let (X,τ) be a $T_{3\frac{1}{2}}$ space and let $\mathcal{U}_1$ and $\mathcal{U}_2$ be in $\mathcal{TB}(X)$. Then $\mathcal{U}_1 \subseteq \mathcal{U}_2$ if and only if $\mathcal{C}(\mathcal{U}_1) \subseteq \mathcal{C}(\mathcal{U}_2)$.

Proof: Assume $\mathcal{C}(\mathcal{U}_1) \subseteq \mathcal{C}(\mathcal{U}_2)$. Clearly from the definition $\mathcal{U}(\mathcal{C}(\mathcal{U}_1)) \subseteq \mathcal{U}(\mathcal{C}(\mathcal{U}_2))$, i.e., by R7.2.16, $\mathcal{U}_1 \subseteq \mathcal{U}_2$. The converse is immediate from the definition of uniform continuity.

The collection $\mathcal{C}(\mathcal{U})$ must contain constant maps, separate points and closed sets, be a vector space under pointwise operations, and be a closed set relative to the topology of uniform convergence determined by $(\mathbb{R}\ ,\mathcal{V})$. The following example shows that those properties are not sufficient to identify a collection as $\mathcal{C}(\mathcal{U})$ for some $\mathcal{U}$.

Example R7.2.19 Let $X = [0,1)$ with uniformity $\mathcal{U}_m$ as in R7.2.11, and let $\mathcal{C} =$

$\{l : X \rightarrow \mathbb{R} : l(x) = ax + b$ for some a, b in $\mathbb{R}\}$. Since the map I as in R7.2.11 is in $\mathcal{C}$, the collection separates points and closed sets. It clearly contains all constant maps and is a vector space. It is easy to check that $\mathcal{C}$ is closed relative to the topology of uniform convergence. Also $\mathcal{C}$ is a proper subset of $\mathcal{C}(\mathcal{U}_m)$ so that $\mathcal{U}(\mathcal{C}) \subseteq \mathcal{U}_m$. By R7.2.10 $\mathcal{U}(\mathcal{C}) \in \mathcal{TB}(X)$ and, since $\mathcal{U}_m$ is the smallest element of $\mathcal{TB}(X)$, $\mathcal{U}_m \subseteq \mathcal{U}(\mathcal{C})$.

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