Lattice and Semilattice Properties

Given a $T_{3\frac{1}{2}}$ space (X, τ), $\mathcal{COM}(X)$ denotes the set described in [6], which consists of representatives of each T_2 compactification class of X. In R2.6 and R2.7 of [6], Lubben's results that $\mathcal{COM}(X)$ is a complete upper semi-lattice and that, if X is also locally compact, $\mathcal{COM}(X)$ is a complete lattice are presented. As in [5], $\mathcal{TB}(X)$ denotes the set of totally bounded uniformities for X which generate τ. The map Ψ_0 of [5], which associates $\mathcal{U} \in \mathcal{TB}(X)$ with an equivalence class of compactifications, can easily be converted into an order isomorphism as follows.

Definition R8.1 $\Psi : \mathcal{TB}(X) \to \mathcal{COM}(X)$ is defined by letting $\Psi(\mathcal{U})$ be the element of $\mathcal{COM}(X)$ which represents the equivalence class $\Psi_0(\mathcal{U})$.

Proposition R8.2 $\mathcal{TB}(X)$ and $\mathcal{COM}(X)$ are order isomorphic, with Ψ as an order isomorphism.

Proof: By R1.5 [5], for $\mathcal{U}_1, \mathcal{U}_2$ in $\mathcal{TB}(X)$, $\mathcal{U}_1 \supseteq \mathcal{U}_2$ if and only if $\Psi(\mathcal{U}_1) \geq \Psi(\mathcal{U}_2)$. Thus Ψ is one-to-one and order preserving. By R1.4 Ψ is onto.

Note that, by using R7.2.18 of [8] in a similar way, one could establish an order isomorphism between $\mathcal{COM}(X)$ and the set of $\mathcal{C}(\mathcal{U})$ for $\mathcal{U}$ in $\mathcal{TB}(X)$, where $\mathcal{C}(\mathcal{U})$ is the set of all real-valued, uniformly continuous maps from $(X, \mathcal{U})$.

In the following example $\mathcal{COM}(X)$ is a complete lattice by R2.7, but $\mathcal{TB}(X)$ (and so also $\mathcal{COM}(X)$ by R8.2) is not even finitely distributive. Because in a lattice, either ditributive law implies the other, the example only needs to verify that one fails.

Example R8.3 Let $(\mathbb{N}, \tau)$ be the positive intergers with the discrete topology, let E_4 be equivalence mod 4, and let $[x]_4$ denote the equivalence class of x mod 4. Let E_3 be the equivalence relation generated by the partition $\{[1]_4 \cup [4]_4, [2]_4 \cup [3]_4\}$, let E_2 be the equivalence relation generated by $\{[1]_4 \cup [2]_4, [3]_4 \cup [4]_4\}$, and let E_1 be equivalence mod 2. Let $\mathcal{V}_i = \mathcal{U}_m \vee \mathcal{U}_{E_i}$, where $\mathcal{U}_m$ and $\mathcal{U}_{E_i}$ are the uniformities defined in subsection 2 of [7]. By R5.2.3, $\mathcal{V}_i$ is in $\mathcal{TB}(\mathbb{N})$ for $i = 1, \ldots, 4$. For $i = 1, 2, 3$ clearly $E_4 \subseteq E_i$ and so $\mathcal{V}_i \subseteq \mathcal{V}_4$. Also $E_2 \cap E_3 = E_4$ so that $E_4 \in \mathcal{V}_2 \vee \mathcal{V}_3$. As a result $\mathcal{V}_2 \vee \mathcal{V}_3 = \mathcal{V}_4$ and $\mathcal{V}_1 \wedge (\mathcal{V}_2 \vee \mathcal{V}_3) = \mathcal{V}_1 \neq \mathcal{U}_m$. The next claim is that $\mathcal{V}_1 \wedge \mathcal{V}_2 = \mathcal{V}_1 \wedge \mathcal{V}_3 = \mathcal{U}_m$, which would show that $\wedge$ is not distributive over $\vee$. Clearly $\mathcal{U}_m \subseteq \mathcal{V}_1 \wedge \mathcal{V}_2$. For the converse let V be in $\mathcal{V}_1 \wedge \mathcal{V}_2$ and pick $W \in \mathcal{V}_1 \wedge \mathcal{V}_2$ with $W = W^{-1}$ and $W \circ W \subseteq V$. There is $U \in \mathcal{U}_m$ such that $U \cap E_1 \subseteq W$ and $U \cap E_2 \subseteq W$, and there is $S \subseteq \mathbb{N}$ such that $\mathbb{N} - S$ is finite and $S \times S \subseteq U$. Since V contains the diagonal, $S \times S \subseteq V$ would imply $V \in \mathcal{U}_m$. Let $(x, y) \in S \times S$. If $(x, y) \in E_1$ or $(x, y) \in E_2$, then $(x, y) \in W \subseteq V$. In the remaining case, pick $t \in S$ such that $(x, t) \in E_1$ but $(x, t) \notin E_4$. Then $(x, t) \in U \cap E_1 \subseteq W$ and $(t, y) \in U \cap E_2 \subseteq W$ and so $(x, y) \in W \circ W \subseteq V$. Similarly, $\mathcal{V}_1 \wedge \mathcal{V}_3 = \mathcal{U}_m$.

Chandler [1] presents a number of results focusing on this question: When is $\mathcal{COM}(X)$ a complete lattice or a lattice ? The next two lemmas lead to a sharpened version of R2.7. For both lemmas the following will be assumed: (Y, f) will denote a T_2 compactification of X with $y_1 \neq y_2$ in $Y - f[X]$. Let $\mathcal{U} \in \mathcal{TB}(X)$ with $\Psi_0(\mathcal{U}) = [(Y, f)]$, and, making use of R7.1.1, let $\mathcal{C} = \{g \in \mathcal{C}(\mathcal{U}) : G(y_1) = G(y_2)$ where G is the unique continuous extension of g to $Y\}$. Finally let $\mathcal{U}_1$ denote $\mathcal{U}(\mathcal{C})$, which by R7.2.7 and [4] is the smallest uniformity for X making every element of $\mathcal{C}$ uniformly continuous.

Lemma R8.4 $\mathcal{U}_1 \in \mathcal{TB}(X)$.

Proof: By R7.2.10 it is sufficient to show that $\mathcal{C}$ separates points and closed sets of X in the sense of [8]. Let F be closed in X and let $x \in X - F$. Since f is an embedding, $f(x) \notin c_Y(f[F])$. Let $S = c_Y(f[F]) \cup \{y_1, y_2\}$, and pick $W \in \mathcal{W}$, the unique uniformity for Y, such that $W[f(x)] \subseteq Y - S$. There is a pseudo-metric d for Y such that $W \in \mathcal{U}_d \subseteq \mathcal{W}$. Then $G(y) = \inf\{d(y,t) : t \in S\}$ is uniformly continuous by R7.2.14 and $G[S] = \{0\}$ so that $g = G \circ f$ is in $\mathcal{C}$ and $g[F] \subseteq \{0\}$. Since $W[f(x)]$ is a d-neighborhood of $f(x)$, $g(x) = G(f(x)) \neq 0$ and so $\{g(x)\} \cap \overline{g[F]} = \emptyset$, i.e., $\mathcal{C}$ separates points and closed sets of X.

Lemma R8.5 $\mathcal{U}_1$ is a proper subset of $\mathcal{U}$.

Proof: Since every element of $\mathcal{C}$ is uniformly continuous from $(X,\mathcal{U})$, the smallest such uniformity, $\mathcal{U}_1$, is clearly contained in $\mathcal{U}$. Let $\mathcal{W}$ be the unique uniformity for Y, let $W_1 \in \mathcal{W}$ with $W_1[y_1] \cap W_1[y_2] = \emptyset$, and pick $W \in \mathcal{W}$ with $W = W^{-1}$ and $W \circ W \circ W \subseteq W_1$. Let $(f \times f)^{-1}[W \cap f[X] \times f[X]] = U^*$. Clearly $U^* \in \mathcal{U}$ and it is sufficient to show $U^* \notin \mathcal{U}_1$. Deny the latter. Then there exist $g_1, \ldots, g_n$ in $\mathcal{C}$ and $\epsilon > 0$ such that

$$\bigcap_{i=1}^{n} (g_i \times g_i)^{-1}[V_\epsilon] \subseteq U^*.$$

Let $\{a_t\}$ and $\{b_r\}$ be nets in X such that $f(a_t) \to y_1$ and $f(b_r) \to y_2$. With G_i denoting the continuous extension of g_i to Y, we have $g_i(a_t) = G_i(f(a_t)) \to G_i(y_1) = G_i(y_2)$ and also $g_i(b_r) = G_i(f(b_r)) \to G_i(y_2)$. Using the definition of net convergence and the fact that $W[y_i]$ is a neighborhood of y_i, we can find t and r such that $f(a_t) \in W[y_1]$, $f(b_r) \in W[y_2]$, and $|g_i(a_t) - g_i(b_r)| < \epsilon$ for all i. Then $(a_t, b_r) \in U^*$ and so $(f(a_t), f(b_r)) \in W$. But then $(y_1, y_2) \in W \circ W \circ W \subseteq W_1$, a contradiction.

Theorem R8.6 [Lubben] $\mathcal{COM}(X)$ is a complete lattice if and only if X is locally compact.

Proof: The 'if' half of this result is R2.7. For the converse, it is sufficient to show that $\mathcal{COM}(X)$ does not have a smallest element if X is not locally compact. What follows is a uniformity-oriented modification of Chandler's argument for Theorem 2.19 in [1]. Let (Y, f) be a T_2 compactification of X and let $\mathcal{U} \in \mathcal{TB}(X)$ with $\Psi_0(\mathcal{U}) = [(Y, f)]$. Since X is not locally compact, $f[X]$ is not open in Y and so there exist distinct points y_1, y_2 in $Y - f[X]$. Let $\mathcal{U}_1$ be defined as in the preceding lemmas. By R8.5 and R1.5 $\Psi(\mathcal{U}_1)$ is strictly smaller than $\Psi(\mathcal{U})$, the element of $\mathcal{COM}(X)$ representing (Y, f).

The next two results provide a condition under which a non-empty family in $\mathcal{COM}(X)$ has an infimum. To do this, it is enough to focus on the isomorphic upper semi-lattice $\mathcal{TB}(X)$. Recall that any non-empty family of uniformities has an infimum, whose topology may be strictly smaller than the infimum (i.e., intersection) of the corresponding family of topologies generated by the uniformities.

Lemma R8.7 Let $\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ be a non-empty family in $\mathcal{TB}(X)$. Let $\mathcal{U} = \wedge\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. Then $\mathcal{U}$ is a totally bounded uniformity for X with $\tau(\mathcal{U}) \subseteq \tau$ and $\mathcal{C}(\mathcal{U}) \subseteq \cap\{\mathcal{C}(\mathcal{U}_\alpha) : \alpha \in \Delta\}$.

Proof: This is clear from general facts about uniformities.

If one transfers the next result to $\mathcal{COM}(X)$ via the order isomorphim Ψ, it provides a condition for the existence of an infimum of a family. This condition is a uniformity-oriented version of Chandler's Theorem 2.18 in [1].

Proposition R8.8 Let $\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ be a non-empty family in $\mathcal{TB}(X)$. Let $\mathcal{U} = \wedge\{\mathcal{U}_\alpha : \alpha \in \Delta\}$. Then $\mathcal{U} \in \mathcal{TB}(X)$ if and only if $\cap\{\mathcal{C}(\mathcal{U}_\alpha) : \alpha \in \Delta\}$ separates points and closed sets.

Proof: First, let $\mathcal{C} = \cap\{\mathcal{C}(\mathcal{U}_\alpha) : \alpha \in \Delta\}$ separate points and closed sets. By R7.2.10 $\mathcal{U}(\mathcal{C}) \in \mathcal{TB}(X)$ and, since $\mathcal{C} \subseteq \mathcal{C}(\mathcal{U}_\alpha)$, by R7.2.16 $\mathcal{U}(\mathcal{C}) \subseteq \mathcal{U}_\alpha$ for all α. Thus $\mathcal{TB}(X)$ contains a lower bound for $\{\mathcal{U}_\alpha : \alpha \in \Delta\}$ and so $\mathcal{U} \in \mathcal{TB}(X)$. For the converse, the hypothesis implies by R7.2.15 that $\mathcal{C}(\mathcal{U})$ separates points and closed sets, and clearly a larger family such as $\mathcal{C}$ does also.

The next result shows that, if X is the rationals with the usual topology, then $\mathcal{COM}(X)$ is not a lattice.

Theorem R8.9 [Shirota 2] If X is first countable and not locally compact, then $\mathcal{COM}(X)$ is not a lattice.

Proof: See Chandler [1], Theorem 5.12.

Finally, $\mathcal{COM}(X)$ can have any infinite cardinality. Chandler [1] uses ideas from the proof of the following theorem to construct an example of a non-locally compact X for which $\mathcal{COM}(X)$ is a lattice.

Theorem R8.10 [Visliseni and Flaksmaier 3] Let $\aleph$ be an infinite cardinal. Then there is a $T_{3\frac{1}{2}}$ space (X, τ) such that $|\mathcal{COM}(X)| = \aleph$.

Proof: See Chandler [1], Theorem 5.4.

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An asterisk indicates a reference not seen by me.

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