Overview

This collection developed from an unasked question that occurred to me during a departmental talk given by Zbiginew Piotrowski, a colleague of mine at Youngstown State University, in 1995. Zbig mentioned the Stone-Čech compactification and I wondered whether the embedding map might be uniformly continuous. The answer, of course, depends the choice of uniformity, but the thought led to an examination of the connection between uniformities and compactifications.

This overview contains brief abstracts and (sometimes) commentary for the results sections. Such references as I have are not mentioned here but can be found in each individual section.

From an academic point of view, this collection has two glaring deficiencies. First, it has not been subjected to the normal peer-review process. Secondly, it is haphazardly sourced, partly because of the lack of convenient access to a research library and partly because of a certain scholarly laxity in the author. As a result, it may contain rediscoveries as well as (I hope) some new results.

Lastly, the website title, "Suprema of Compactifications", was chosen early on. It does reflect the focus of early sections, but later sections tend to examine properties of examples like $\mathbb{N}_k$, which is obtained as a supremum of compactifications.

R1: Existence of Suprema via Uniform Space Theory This section establishes the one-to-one correspondence between the separated, totally bounded uniformities of a $T_{3\frac{1}{2}}$ topological space and the compactification classes of that space, a fact which is used or referred to repeatedly in the following sections. The correspondence is shown to preserve order, where the uniformities are ordered by containment and the compactification classes have the usual ordering. Because a non-empty set of uniformities has a supremum so does a collection of compactification classes.

R2: Existence of Suprema via Quotients of the Stone-Čech Compactification Because the Stone-Čech compactification is the largest compactification of a given $T_{3\frac{1}{2}}$ space, every compactification class of the space has a representative in which the underlying compact space is a quotient of the Stone-Čech compactification. By using these representations for a family, a representative of the family's supremum class is constructed.

R3: Representation of Suprema This section records a method of constructing a representative of the supremum class for a finite family of compactification clsses. A representative of the supremum of an infinite family can be obtained as an inverse limit of a suitable inverse sprectrum. This construction provides a third way to show the existence of suprema. Finally, it is shown that a map which is extendible for every compactification in a family is also extendible to the family's supremum.

R4: Suprema of Countable Families This section records a special case of that considered in R3 and so is logically redundant. The countability of the family allows some notational simplification in the construction.

R5: Finite-Point Compactifications This section records some general facts about finite-point compactifications and then focuses on the case when the original space is infinite and discrete. The separated, totally bounded uniformity corresponding to a finite-point compactification of a discrete space is described. For any such compactification, a normal basis for the discrete space is constructed, the Wallman compactification of which is equivalent to the given compactification. An added subsection notes that $T_{3\frac{1}{2}}$ spaces exist whose Stone-Čech compactification is finite and characterizes such spaces as those having a finite number of distinct compactification classes.

R6: Suprema of Two-Point Compactifications The main result: The supremum of all two-point compactifications of an infinite discrete space is its Stone-Čech compactification.

This result might be misleading to some accustomed to the linear world of real analysis, where the supremum of a set can be closely approximated by elements from the set. In the non-linear ordering of compactifications, there is no notion of approximation. There are simply so many two-point compactifications that the Stone-Čech compactification is the only one that is an upper bound for them all.

Two proofs are presented. The first verifies that the uniformity corresponding to the Stone-Čech compactification is the supremum of the uniformities corresponding to the two-point compactifications. The second uses extensions of characteristic functions to obtain a more general result. The section closes with a brief examination of connections between the two approaches.

R7: Uniform Continuity and Extensions The question of which continuous maps have extensions to a given compactification is considered. The main result: Let (Y, f) be a T_2 compactification of a given $T_{3\frac{1}{2}}$ space X and let $\mathcal{U}$ be the separated, totally bounded uniformity corresponding to (Y, f). A continuous map g from X to a compact, T_2 space Z can be continuously extended to Y if an only if g is uniformly continuous from $(X,\mathcal{U})$ to Z with its unique uniformity.

One can also start with a set of bounded, continuous, real-valued functions on X and ask for a compactifcation which allows extension of every g in the set. The weak uniformity generated by the set of functions is totally bounded. If the set of functions separates points and closed sets, the weak uniformity generates the given topology on X and so is separated. In this case a compactification in the corresponding compactification class allows the desired extensions.

R8: Lattice and Semi-lattice Properties For a given $T_{3\frac{1}{2}}$ space, its separated, totally bounded uniformities form a complete upper semi-lattice with containment as ordering. Because the ordering of compactifications is equivalent to containment of the corresponding uniformities, a set-image of the compactification classes is an isomorphic complete upper semi-lattice. If the original space is also locally compact, these structures are complete lattices. An example is given in which these lattices are not finitely distributive.

R9: Directed Sets of Normal Bases In general, a Wallman compactification of a $T_{3\frac{1}{2}}$ space is defined using the set of $\mathcal{Z}$-ultrafilters, where $\mathcal{Z}$ is a normal basis for the space. It is shown that the union of a non-empty collection of normal bases is also a normal basis provided the collection is a directed set under containment. Moreover, in that circumstance, the Wallman compactification from the union represents the supremum of the Wallman compactifications generated by the collection.

For an infinite discrete space with a compactification (Y, f), it is shown that Y is zero-dimensional if and only if (Y, f) represents the supremum of a family of finite-point compactifications.

R10: Some Metrc Compactifications of $\mathbb{N}$ For a positive integer $k \geq 2$, the normal basis $\mathcal{Z}_k$ for $\mathbb{N}$ with the discrete topology is obtained as the union of the normal bases generated by equivalence mod k^n with $n = 1, 2, 3, \ldots$. $\mathbb{N}_k$ is the Wallman compactification generated by $\mathcal{Z}_k$. This compactification represents the class of the supremum of an increasing sequence of finite-point compactifications.

$\mathcal{Z}_\infty$ which is also a normal basis for $\mathbb{N}$ with the discrete topology, is the union of all $\mathcal{Z}_k$. $\mathbb{N}_\infty$ is the Wallman compactification generated by $\mathcal{Z}_\infty$. $\mathbb{N}_\infty$ represents the class of the supremum of the $\mathbb{N}_k$. The underlying topological spaces of $\mathbb{N}_k$ and $\mathbb{N}_\infty$ share several topological properties: They are second countable, metrizable, zero-dimensional, and not extremely disconnected. $\mathbb{N}_k$ and $\mathbb{N}_\infty$ both have cardinality $2^{\aleph}$.

In $\mathbb{N}_k$ the embedded image of $\mathbb{N}$ is the set of point $\mathcal{Z}_k$-ultrafilters. Each non-point $\mathcal{Z}_k$-ultrafilter has a unique associated sequence. Inductive properties of these sequences are described. Convergence of sequences of point $\mathcal{Z}_k$-ultrafilters is characterized.

For positive integers $k, j \geq 2$ with least common multiple $[k, j]$, $\mathbb{N}_{[k,j]}$ represents the compactification class of $\mathbb{N}_k \vee \mathbb{N}_j$. By identifying $\mathbb{N}_1$ as the one-point compactification, $\mathbb{N}_{(k,j)}$ represents the compactification class of $\mathbb{N}_k \wedge \mathbb{N}_j$.

There are also some equivalence results: $\mathbb{N}_k$ is not equivalent to $\mathbb{N}_\infty$. $\mathbb{N}_\infty$ is equivalent to $\vee\{ \mathbb{N}_p : p \text{ is prime}\}$. For primes $p \neq q$, $\mathbb{N}_p$ is not equivalent to $\mathbb{N}_q$.

R11: The Magill-Glasenapp Theorem This section contains an exposition of a published result, which was used in a previous section. The result: Let X be a zero-dimensional $T_{3\frac{1}{2}}$ space. The supremum of any non-empty collection of zero-dimensional T_2 compactifications of X is zero-dimensional.

R12: Extending Arithmetic Operations By using the uniform space approach, it is shown that addition and multiplication on $\mathbb{N}$ have unique continuous extensions to $\mathbb{N}_\infty$ and $\mathbb{N}_k$ for all $k \geq 2$. The extensions have the commutative and associative properties. Multiplication is distributive over addition and there is a multiplicative identity.

For each $k \geq 2$, the set of non-point $(\mathcal{Z}_k-)$ultrafilters is closed under addition and multiplication. The operations can be described as follows: Let $\mathcal{F}, \mathcal{G}$ be non-point ultafilters associated with $\{x_n\}$ and $\{y_n\}$ respectively. Let $\mathcal{F} + \mathcal{G}$ be associated with $\{z_n\}$ and $\mathcal{F} \cdot \mathcal{G}$ with $\{w_n\}$. For every n, $z_n \equiv x_n + y_n \bmod k^n$ and $w_n \equiv x_n y_n \bmod k^n$. There is a non-point ultrafilter $\mathcal{O}_k$, which is an additive identity for the non-point ultrafilters. $\mathcal{O}_k$ is not an additive identity on $\mathbb{N}_k$.

The operations on $\mathbb{N}_\infty$ can be described by reference to the operations on $\mathbb{N}_k$ for all k. They are binary operations on the set of non-point ultrafilters on $\mathbb{N}_\infty$, and there is an additive identity for that subset.

For $k \geq 2$ $\mathbf{R}_k$ is defined as the set of all non-point ultrafilters in $\mathbb{N}_k$. Similarly $\mathbf{R}_\infty$ is the set of all non-point ultrafilters in $\mathbb{N}_\infty$. $\mathbf{R}_k$ and $\mathbf{R}_\infty$ are compact topological rings, which are subsequently referred to as the remnant rings because they are remainder spaces.

Let $\mathcal{O}_k$ and $\mathcal{O}_\infty$ denote the additive identities in $\mathbf{R}_k$ and $\mathbf{R}_\infty$ respectively. For $j \in \mathbb{N}$ let $\hat{j}$ denote the point ultrafilter of j in either context. The set $\{\hat{j} + \mathcal{O}_k : j \in \mathbb{N}\}$ is dense and non-discrete in $\mathbf{R}_k$. It is also isomorphic to $\mathbb{N}$ by the map $j \mapsto \hat{j} + \mathcal{O}_k$. The set $\{\hat{j} + \mathcal{O}_\infty : j \in \mathbb{N}\}$ is dense and non-discrete in $\mathbf{R}_\infty$. It is also isomorphic to $\mathbb{N}$ by the map $j \mapsto \hat{j} + \mathcal{O}_\infty$.

R13: Mixed Suprema It noted that the usual definition of ordering of compactification generalizes to the case of compactifications arising from different $T_{3\frac{1}{2}}$ topologies on a fixed set. The basic development of this notion is presented.

For a given set X, the set of all separated, totally bounded uniformites on X, TBS(X) is a complete upper semi-lattice. Each element $\mathcal{U}$ in TBS(X) corresponds to a unique T_2 compactification class for $(X, \tau(\mathcal{U}))$. With TBS(X) ordered by containment, the correspondence is an order isomorphism from TBS(X) to any set image of the compactifications. As a result, any collection of such compactifications has a supremum, which can be represented as an inverse limit. In contrast to the usual case, TBS(X) is a lattice if and only if X is finite.

R14: Uniformities and Normal Bases Given a normal basis for a $T_{3\frac{1}{2}}$ space X, the separated, totally bounded uniformity corresponding to the Wallman compactification is described. An approach to the converse is discussed, but it cannot succeed because the Frink question is undecidable.

R15: S-Maps For a $T_{3\frac{1}{2}}$ topological space (X, τ), $\mathcal{TB}(X, \tau)$ denotes the set of totally bounded uniformities on X that generate τ, each of which must be separated. Given a totally bounded uniformity $\mathcal{V}$ for X, not necessarily in $\mathcal{TB}(X, \tau)$ or separated, the S-map $S_{\mathcal{V}}$ from $\mathcal{TB}(X, \tau)$ to $\mathcal{TB}(X, \tau \vee \tau(\mathcal{V}))$ is defined by $\mathcal{U} \mapsto \mathcal{U} \vee \mathcal{V}$.

Such a map preserves suprema and induces an order and suprema preserving map between set-images of the compactification classes of the two underlying spaces.

Simple examples show that, depending on the choice of $\mathcal{V}$, $S_{\mathcal{V}}$ may be one-to-one and onto but need not be either.

When $\mathcal{V}$ is the uniformity generated by an equivalence relation on X with finitely many equivalence classes, the compactification corresponding to $S_{\mathcal{V}}(\mathcal{U})$ can be calculated from the compactification corresponding to $\mathcal{U}$. It is the disjoint union of compactifications determined by the images of the infinite equivalence classes. This leads to a description of the compactifications of the simple extension of (X, τ) by a closed set.

For certain examples of $\mathcal{V}$, the question of when $S_{\mathcal{V}}(\mathcal{U})$ corresponds to a finite-point or zero-dimensional compactification is examined.

R16: The Remnant Rings as Compactifications For $k \geq 2$ and $m \in \mathbf{Z}$, $f_k(m)$ is the non-point ultrafilter with associated sequence $\{x_n\}$, where $x_n \equiv m \bmod k^n$ for all n. Then $(\mathbf{R}_k, f_k)$ is a T_2-compactification of $(\mathbf{Z}, \tau_k)$, where τ_k is the topology with a clopen basis consisting of the equivalence classes of $m \bmod k^n$ over all $m \in \mathbf{Z}$ and $n \in \mathbb{N}$. A map f_∞ from $\mathbf{Z}$ to $\mathbf{R}_\infty$ is described so that $(\mathbf{R}_\infty, f_\infty)$ is a T_2-compactification of $(\mathbf{Z}, \tau_\infty)$. It represents the compactification class of the suprema of the classes of $(\mathbf{R}_k, f_k)$ for all $k \geq 2$ and $\tau_\infty = \vee\{\tau_k : k \geq 2\}$.

R17: Algebraic Structure of the Remnant Rings For each $k \geq 2$, $\mathbf{R}_k$ is a communtative ring with unity. For $\mathcal{F} \in \mathbf{R}_k$ associated with $\{x_n\}$, $\mathcal{F}$ is invertible if and only if x_1 is invertible mod k. If k is prime, then $\mathbf{R}_k$ is a local ring and the quotient ring by the unique maximal ideal is the field of integers mod k. If k is prime, a valuation making $\mathbf{R}_k$ a Euclidean ring is described and the ideals of $\mathbf{R}_k$ are identified. If k is prime, by using the universal property of the inverse limit, it is shown that $\mathbf{R}_k$ is topologically isomorphic to the k-adic numbers.

For any $k, j \geq 2$, $(\mathbf{R}_{[k,j]}, f_{[k,j]})$ is a representative of the compactification class of the supremum of $(\mathbf{R}_k, f_k)$ and $(\mathbf{R}_j, f_j)$. $(\mathbf{R}_k, f_k)$ and $(\mathbf{R}_j, f_j)$ have an infimum if and only if $(k, j) \geq 2$. If $(k, j) \geq 2$, $(\mathbf{R}_{(k,j)}, f_{(k,j)})$ is less than or equal that infimum. These statements have to be interpreted in terms of the mixed compactification ordering of R13.

If the distinct prime factors of $k \geq 2$ are $p_1, \ldots, p_t$, then $\mathbf{R}_k$ is topologically isomorphic to $\Pi_{i=1}^t \mathbf{R}_{p_i}$. $\mathbf{R}_\infty$, also a commutative ring with unity, is topologically isomorphic to the direct product $\Pi\{\mathbf{R}_p : p \text{ is prime}\}$.

R18: Metric Compactifications Because βX is metrizable if and only if X is compact and metrizable, every non-compact $T_{3\frac{1}{2}}$ space has a non-metrizable compactification. Because $\mathbb{N}_k$ and $\mathbf{R}_k$ are metrizable, the question arises of which spaces can have a metrizable compactification. The answer: A $T_{3\frac{1}{2}}$ space X has at least one metrizable compactification if and only if X is second countable. For any second countable, locally compact T_2 space, every finite-point compactification is metrizable. It is also shown that, for (X, τ) $T_{3\frac{1}{2}}$, (X, τ) is second countable if and only if its Stone-Čech compactification is the supremum of a non-empty family of metrizable compactifications of (X, τ).

R19: Ordering the Remnant Rings For each $k \geq 2$, a linear order $<_k$ is defined inductively on $\mathbf{R}_k$. In an added subsection an equivalent and simpler fomulation is described: For $\mathcal{F} \neq \mathcal{G}$ in $\mathbf{R}_k$ associated with $\{x_n\}$ and $\{y_n\}$ respectively, $\mathcal{F} <_k \mathcal{G}$ if and only if x_M is smaller than y_M in the natural order for $\mathbb{N}$, where M is the first subscript such that $x_M \neq y_M$. The order topology generated by $<_k$ is the same as the topology of $\mathbf{R}_k$. The consecutive points of $\mathbf{R}_k$ under $<_k$ are completely described.

R20: *p*-adic Tools for the Remnant Rings When p is prime, $\mathbf{R}_p$ was shown in R17 to be topologically isomorphic to the p-adic integers by using a universal property. That approach is non-constructive. In this section a possible constructive approach, which is derived from the associated sequences of the ultrafilters, is presented. The word 'possible' is used because the result of R17 is only suggested and not re-proven in detail. Here the focus is to generate several known results about the p-adic numbers in the remnant ring context. Some of these results also apply generally in $\mathbf{R}_k$, i.e., not just for primes.

A metric that generates the topology of $\mathbf{R}_k$ is defined. This metric is also an ultrametric and invariant under addition or multiplication by an invertible. The ϵ-balls are all clopen.

Equivalence mod k^n is extended to $\mathbf{R}_k$ for all n.

The two principles of Hensel's Philosophy are presented. The first says that a polynomial in several variables with coefficients in $\mathbf{Z}$ has a solution in $\mathbf{R}_k$ if and only if it has a solution in $\mathbf{Z} / < k^t >$ for every t. The second principle is a technique for finding roots in $\mathbf{R}_p$, p prime, of polynomials with coefficients in $\mathbf{Z}$. The method involves finding a root as a limit of a sequence, constructed analogously to Newton's method.

R21: Order Compactifications Given a linearly ordered set $(X, <)$, a separated totally bounded uniformity $\mathcal{U}(<)$ for X is constructed. This uniformity generates the order topology $\tau(<)$. When $(X, \tau(<))$ is connected and non-compact, the compactification class corresponding to $\mathcal{U}(<)$ is either the one-point compactification or a two-point compactification.

For an integer $k \geq 2$, the unique uniformity on $\mathbf{R}_k$ must be $\mathcal{U}(<_k)$, where $<_k$ is the ordering on $\mathbf{R}_k$ defined in R19. $<_k$ induces a linear order $\prec_k$ on $\mathbb{N}$. $\tau(\mathcal{U}(\prec_k))$ is shown to be strictly smaller than $\tilde{\tau}_k$, the topology on $\mathbb{N}$ induced by the subspace uniformity from

$\mathcal{U}(<_k)$ on $f_k[\mathbb{N}]$. A non-T_2 topology τ^* is described so that $\tilde{\tau}_k = \tau(\mathcal{U}(\prec_k)) \vee \tau^*$.

In an added subsection, a compactification in the class corresponding to $\mathcal{U}(<)$ in general is constructed by a technique related to the Dedekind cut method of constructing the reals from the rationals. This compactification is the smallest order-generated compactification of $(X, \tau(<))$.

R22: Extensions and Compactifications In this section and the next the word 'extension' does not refer to continuous functions but rather the enlargement of a uniformity by taking its supremum with a uniformity generated by equivalence relations.

Let $\mathcal{U}$ be a uniformity on X and let $\mathcal{A}$ be a non-empty collection of subsets of X. For each $A \in \mathcal{A}$, the partition $\{A, X - A\}$ determines an equivalence relation. $\mathcal{U}_e(\mathcal{A})$ is the supremum of $\mathcal{U}$ and the uniformity with subbasis consisting of all those equivalence relations.

Two problems are considered. First, if (Y, f) is in the compactification class corresponding to $\mathcal{U}$, can a T_2-compactification in the class corresponding to $\mathcal{U}_e(\mathcal{A})$ be described? Secondly, if (Z, g) is a T_2-compactification in the class corresponding to $\mathcal{U}_e(\mathcal{A})$, can a compactification in the class corresponding to $\mathcal{U}$ be described?

The first question is answered in the case that $\mathcal{A}$ is finite. In that case the desired compactification is a (finite) disjoint union of compactifications determined from Y and certain sets generated by $\mathcal{A}$. There are also partial results for the second question in two cases, when $\mathcal{A}$ is finite and when it is a countable increasing chain. In both cases the desired compactification is a quotient space of Z by an equivalence relation determined by the given collection $\mathcal{A}$.

R23: Special Cases of Extensions The main aim of this section is to consider a conjecture mentioned in R19. Let $<_k$ be the linear ordering for $\mathbf{R}_k$ that generates its topology and unique uniformity. Let $\tilde{\mathcal{V}}_k$ be the subspace uniformity from the uniformity for $\mathbf{Z}$ corresponding to $\mathbf{R}_k$. Let $\prec_k$ be the ordering for $\mathbb{N}$ induced by $<_k$ and let $\mathcal{A}$ be the collection of closed right $\prec_k$-rays. It is shown that $(\mathcal{U}(\prec_k))_e(\mathcal{A})) = \tilde{\mathcal{V}}_k$. $\mathcal{A}$ is a countable chain but no indexing makes it monotone and the result of R22 does not apply. However, the arguments of R22 can be extended to define an equivalence relation so that the quotient space of $\mathbf{R}_k$ by that relation represents the class corresponding to $\mathcal{U}(\prec_k)$.

The section also contains some examples and results about other special cases: Stone-Čech compactifications, finite-point compactifications, and zero-dimensional compactifications. Of course, $\mathcal{U} \subseteq \mathcal{U}_e(\mathcal{A})$ but the topologies generated are distinct unless every $A \in \mathcal{A}$ is $\tau(\mathcal{U})$-clopen.

For the Stone-Čech case, there are only examples showing that the four logically possible cases all can occur. Both uniformities (or neither, of course) may correspond to a Stone-Čech compactification, or the smaller may but not the larger, or the larger but not the smaller.

For finite-point compactifications, if $\mathcal{U}_e(\mathcal{A})$ corresponds to a finite-point compactification, then so does $\mathcal{U}$. The converse is false in general but does hold with certain conditions on $\mathcal{A}$.

Lastly, let X be discrete, let $\mathcal{U}_m$ correspond to the class of the one-point compactification of X, and let (Y, f) be a compactification of X. Then Y is zero-dimensional if and only if $[(Y, f)]$ corresponds to $(\mathcal{U}_m)_e(\mathcal{A})$ for some collection $\mathcal{A}$.

R24: Disjoint Unions of Uniform Spaces This section contains an exposition of the disjoint union or co-product of uniform spaces, including the universal property characterizing it. Some uniformities constructed earlier in connection with extensions of uniform spaces, S-maps, and finite point compactifications are recognized as disjoint unions. For a finite set of separated, totally bounded uniform spaces, the disjoint union corresponds to the disjoint union of the corresponding compactifications. For a finite disjoint union of compactifications, each of which is a supremum, the dijoint union and supremum operators commute.

R25: Compactification and Hyperspaces Here 'hyperspace' means a subset of the power set of some given set. Let $(X,\mathcal{U})$ be a uniform space. A uniformity on the power set, $\hat{\mathcal{U}}$, is constructed from $\mathcal{U}$, in such a way that, for a pseudo-metric ρ on X which generates the usual uniformity $\mathcal{U}_\rho$, $\widehat{\mathcal{U}_\rho} = \mathcal{U}_{\hat{\rho}}$, where $\hat{\rho}$ is the Hausdorff pseudo-metric. For a hyperspace $\mathcal{S}$, $\hat{\mathcal{U}}(\mathcal{S})$, the subspace uniformity on $\mathcal{S}$, is totally bounded if and only if $\mathcal{U}$ is. A necessary and sufficient condition based on $\mathcal{S}$ for $\hat{\mathcal{U}}(\mathcal{S})$ to be separated is presented. Although completeness of $\hat{\mathcal{U}}$ does not in general follow from completeness of $\mathcal{U}$, it does when $\tau(\mathcal{U})$ is compact, the case needed here.

Now suppose $(X,\mathcal{U})$ is separated and totally bounded, and let $\mathcal{U}$ correspond to the compactification class of (Y, f). Let $\mathcal{S}$ be a hyperspace such that $\hat{\mathcal{U}}(\mathcal{S})$ is separated. A compactification in the class corresponding to $\hat{\mathcal{U}}(\mathcal{S})$ is described as a quotient space.

The section concludes with examples showing that suprema, $\mathcal{U}$ corresponding to a finite-point compactification, and $\mathcal{U}$ corresponding to the Stone-Čech compactification need not carry over to the hyperspace.

R26: The Remnant Rings are Homeomorphic For $k, j \geq 2$, the topological spaces $\mathbf{R}_k$ and $\mathbf{R}_j$ are shown to be homeomorphic. In addition, each is is homeomorphic to the Cantor set. The proof is a detailed development of a brief suggestion in a book about p-adic numbers, but here k, j need not be prime. In an added subsection the technique is modified to show that $(\mathbf{R}_k\ , <_k)$ and $(\mathbf{R}_j\ , <_j)$ are order isomorphic. This implies that $\mathbb{N}_k$ and $\mathbb{N}_j$ are also homeomorphic, even in cases when they are not equivalent compactifications.

R27: Normal Bases for the Remnant Rings For $k \geq 2$ let τ_k be the topology for $\mathbf{Z}$ such that $\mathbf{R}_k$ is a T_2-compactification of $(\mathbf{Z}\ , \tau_k)$. A normal basis for $(\mathbf{Z}\ , \tau_k)$ is described so that the Wallman compactification generated from the basis is equivalent to $\mathbf{R}_k$. A similar construction is carried out for $\mathbf{R}_\infty$.

It is also shown that $\mathbf{R}_k$ is a T_2-compactification of $(\mathbb{N}\ , \sigma_k)$, where σ_k is the subspace topology on $\mathbb{N}$ from τ_k. The embedding is the restriction to $\mathbb{N}$ of the embedding for $\mathbf{Z}$. The normal basis for τ_k induces a normal basis for σ_k, the Wallman compactification of which is equivalent to $\mathbf{R}_k$. Again, similar results are obtained for $\mathbf{R}_\infty$.

R28: Order-Reversing Involutions for the Remnant Rings For $k \geq 2$ the linear ordering $<_k$ makes $\mathbf{R}_k$ a complete, completely distributive lattice. Although $<_k$ is not in general compatible with the algebraic operations in $\mathbf{R}_k$, it is shown that the map $\mathcal{F} \mapsto f_k(1) - \mathcal{F}$ is an order-reversing involution on $\mathbf{R}_k$. When k is odd, there is a unique self-involute element. When k is even, there is no self-involute element. The order-reversing involution on $\mathbf{R}_k$ is not unique.

R29: Point Spaces of the Remnant Rings Let **Sp** denote the category of topological spaces with continuous maps and **Loc** the category of locales with locale morphisms. Most of this section is an exposition of basic definitions and facts, with an emphasis on two functors, $\Omega : \mathbf{Sp} \rightarrow \mathbf{Loc}$ and $H : \mathbf{Loc} \rightarrow \mathbf{Sp}$. Ω maps a space to its collection of open sets, which can be treated as a locale, and a continuous map to its inverse operator, which is a locale morphism. The object-image under H is the point space of the locale. It is shown that H is right adjoint to Ω.

As noted in R28, for any $k \geq 2$, $\mathbf{R}_k$ with $<_k$ is a complete bounded chain, which can be treated as a locale. The point space of any complete bounded chain has its open sets form a chain. If the complete bounded chain has cardinality at least 3, its point space is sober but not T_1.

R30: Realcompactness and Uniformity The realcompactification of a $T_{3\frac{1}{2}}$ space can be defined using its Stone-Čech compactification and extensions of real-valued continuous maps as maps into $\mathbb{R}^+$, the one-point compactification of $\mathbb{R}$. The maps used can be described as the uniformly continuous maps from $(X, \mathcal{U}_M)$ into $\mathbb{R}^+$with its unique uniformity, where $\mathcal{U}_M$ is the uniformity corresponding to the class of the Stone-Čech uniformity.

This formulation allows definition of a generalized realcompactness based any separated totally bounded uniformity on X and its corresponding compactification class. Some properties of this generalization and related examples are presented.

One result is a sufficient condition for standard realcompactness of a locally compact T_2 space X. Let $\mathcal{U}_m$ be the separated totally bounded uniformity on X corresponding to the compactification class of X^+, the one-point compactification of X. If X is RK$\mathcal{U}_m$ (generalized realcompactness based on $\mathcal{U}_m$), then X is realcompact in the usual sense.

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